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Quasi M-convex Functions and Minimization Algorithms

Kazuo MUROTA[1] and Akiyoshi SHIOURA[2]

[1]Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
murota@kurims.kyoto-u.ac.jp

[2] Department of Mechanical Engineering
Sophia University
7-1 Kioi-cho, Chiyoda-ku
Tokyo 102-8554, Japan
shioura@me.sophia.ac.jp

Abstract: We introduce a class of discrete quasiconvex functions, called quasi M-convex functions, by generalizing the concept of M-convexity due to Murota (1996). We investigate the structure of quasi M-convex functions with respect to level sets, and show that various greedy algorithms work for the minimization of quasi M-convex functions.

Keywords: quasiconvex function, discrete optimization, matroid, base polyhedron.

1 Introduction

The concept of convexity for sets and functions plays a central role in continuous optimization (or nonlinear programming with continuous variable), and has various applications in the areas of mathematical economics, engineering, operations research, etc. [2, 12, 15]. The importance of convexity relies on the fact that a local minimum of a convex function is also a global minimum. Due to this property, we can find a global minimum of a convex function by iteratively moving in descent directions, i.e., so-called descent algorithms work for the convex function minimization. Therefore, convexity for a function is a sufficient condition for the success of descent methods. Most of descent methods, however, work for a fairly larger class of functions called quasiconvex functions.

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be defined over a nonempty convex set, i.e., $\mathrm{dom} f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\}$ is a nonempty convex set. A function f is said to be quasiconvex if it satisfies

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in \mathrm{dom} f$ and $0 < \alpha < 1$, and semistrictly quasiconvex if it satisfies

$$f(\alpha x + (1-\alpha)y) < \max\{f(x), f(y)\}$$

for all $x, y \in \mathrm{dom} f$ with $f(x) \neq f(y)$ and $0 < \alpha < 1$. It is easy to see that convexity implies semistrict quasiconvexity, and semistrict quasiconvexity implies quasiconvexity under the assumption of lower semicontinuity. Although (semistrict) quasiconvexity is a weaker property than convexity, it still has nice properties as follows:

- strict local minimality leads to global minimality for quasiconvex functions,
- local minimality leads to global minimality for semistrictly quasiconvex functions,
- level sets of quasiconvex functions are convex sets.

Due to these properties, quasiconvexity also plays an important role in continuous optimization. See [1] for more accounts on quasiconvexity.

In the area of discrete optimization, on the other hand, discrete analogues of convexity, or "discrete convexity" for short, have been considered, with a view to identifying the discrete structure that guarantees the success of descent methods, i.e., so-called "greedy algorithms." Examples of discrete convexity are "discretely-convex functions" by Miller [7], "integrally-convex functions" by Favati–Tardella [3], and "M-convex/L-convex functions" by Murota [8, 9, 10].

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called M-convex if $\mathrm{dom} f \neq \emptyset$ and f satisfies the following property:

(M-EXC) $\forall x, y \in \mathrm{dom} f$, $\forall u \in \mathrm{supp}^+(x-y)$,

$\exists v \in \mathrm{supp}^-(x-y)$:

$$f(x)+f(y) \geq f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v),$$

where

$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\},$$
$$\mathrm{supp}^+(x-y) = \{w \in V \mid x(w) > y(w)\},$$
$$\mathrm{supp}^-(x-y) = \{w \in V \mid x(w) < y(w)\},$$

and $\chi_w \in \{0,1\}^V$ is the characteristic vector of $w \in V$. M-convex functions have various desirable properties as discrete convexity:

(i) local minimality leads to global minimality for M-convex functions,
(ii) M-convex functions can be extended to ordinary convex functions,
(iii) various duality theorems hold,
(iv) M-convex functions are conjugate to L-convex functions.

In particular, the property (i) shows that greedy algorithms work for the M-convex function minimization. However, we see from results in continuous optimization that strong properties such as M-convexity are not required for the success of greedy algorithms, and that some property like "quasi M-convexity" will suffice.

The main aim of this paper is to introduce the concept of quasi M-convex functions by generalizing the concept of M-convexity. To define quasi M-convexity, we use the following weaker properties than (M-EXC):

(QM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
$f(x-\chi_u+\chi_v) \leq f(x)$ or $f(y+\chi_u-\chi_v) \leq f(y)$.

(SSQM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
(i) $f(x-\chi_u+\chi_v) \geq f(x)$
$\Longrightarrow \quad f(y+\chi_u-\chi_v) \leq f(y)$, and
(ii) $f(y+\chi_u-\chi_v) \geq f(y)$
$\Longrightarrow \quad f(x-\chi_u+\chi_v) \leq f(x)$.

We define a quasi M-convex (resp. semistrictly quasi M-convex) function as a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ satisfying (QM) (resp. (SSQM)). We show that various nice properties hold for (semistrictly) quasi M-convex functions, which justifies the definitions of quasi M-convexity above.

We first review some fundamental results on M-convex functions in Section 2. Then, we show some properties for level sets of quasi M-convex functions, and prove that the class of quasi M-convex functions is closed under various fundamental operations in Section 3. Finally, we show that greedy algorithms work for the minimization of (semistrictly) quasi M-convex functions in Section 4. We also show a proximity theorem on (semistrictly) quasi M-convex functions, which guarantee that the so-called "scaling technique" is applicable to the quasi M-convex function minimization.

2 Review of Fundamental Results on M-convex Functions

We denote by $\mathbf{R}$ the set of reals, and by $\mathbf{Z}$ the set of integers. Also, we denote by $\mathbf{R}_{++}$ the set of positive reals. Throughout this paper, we assume that V is a nonempty finite set of cardinality n (> 0). For $w \in V$, we denote by $\chi_w \in \{0,1\}^V$ the characteristic vector of w.

Let $x \in \mathbf{R}^V$. For $S \subseteq V$, we define $x(S) = \sum_{v \in S} x(v)$. We also define

$$\mathrm{supp}^+(x) = \{w \in V \mid x(w) > 0\},$$
$$\mathrm{supp}^-(x) = \{w \in V \mid x(w) < 0\},$$
$$\|x\|_1 = \sum_{w \in V} |x(w)|.$$

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. The *effective domain* $\mathrm{dom}\, f$ of f is defined by

$$\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}.$$

We denote by $\arg\min f$ the set of the minimizers of f, i.e.,

$$\arg\min f = \{x \in \mathbf{Z}^V \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^V)\}.$$

For any $\alpha \in \mathbf{R} \cup \{+\infty\}$, the *level set* $\mathrm{L}(f, \alpha)$ is defined by

$$\mathrm{L}(f, \alpha) = \{x \in \mathbf{Z}^V \mid f(x) \leq \alpha\}.$$

Note that $\arg\min f = \mathrm{L}(f, \inf f)$ and $\mathrm{dom}\, f = \mathrm{L}(f, +\infty)$ are special cases of level sets. For any $x \in \mathrm{dom}\, f$ and $u, v \in V$, we denote the *directional difference* of f at x w.r.t. u and v by

$$\Delta f(x; u, v) = f(x+\chi_u-\chi_v) - f(x).$$

For a set $S \subseteq \mathbf{Z}^V$, the function $\delta_S : \mathbf{Z}^V \to \{0, +\infty\}$ given by

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S) \end{cases}$$

is called the *indicator function* of S.

Let $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$. A function φ is called *quasiconvex* if it satisfies

$$\varphi(\beta) \leq \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} \quad (\forall \alpha_1, \alpha_2, \beta \in \mathbf{Z} \text{ with } \alpha_1 < \beta < \alpha_2).$$

Similarly, φ is called *semistrictly quasiconvex* if it is a quasiconvex function and satisfies

$$\varphi(\beta) < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\} \ (\forall \alpha_1, \alpha_2, \beta \in \mathbf{Z} \text{ with } \alpha_1 < \beta < \alpha_2,\ \varphi(\alpha_1) \neq \varphi(\alpha_2)). \tag{2.1}$$

Remark 2.1. For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, semistrict quasiconvexity implies quasi convexity under the assumption of lower semicontinuity [1, 2]. For a function $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$, on the other hand, the property (2.1) alone does not imply the quasiconvexity in general. For convenience, we assume quasiconvexity in the definition of semistrict quasiconvexity for φ. □

Theorem 2.2. *Let* $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$.
(i) *φ is quasiconvex if and only if for all $\alpha_1, \alpha_2 \in \operatorname{dom} \varphi$ with $\alpha_1 < \alpha_2$ we have $\varphi(\alpha_1 + 1) \leq \varphi(\alpha_1)$ or $\varphi(\alpha_2 - 1) \leq \varphi(\alpha_2)$.*
(ii) *φ is semistrictly quasiconvex if and only if for all $\alpha_1, \alpha_2 \in \operatorname{dom} \varphi$ with $\alpha_1 < \alpha_2$ we have both*

$$\varphi(\alpha_1 + 1) \geq \varphi(\alpha_1) \Longrightarrow \varphi(\alpha_2 - 1) \leq \varphi(\alpha_2), \text{ and}$$
$$\varphi(\alpha_2 - 1) \geq \varphi(\alpha_2) \Longrightarrow \varphi(\alpha_1 + 1) \leq \varphi(\alpha_1).$$

A function $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ is said to be *nondecreasing* if $\varphi(\alpha) \leq \varphi(\beta)$ holds for all $\alpha, \beta \in \mathbf{R}$ with $\alpha < \beta$, and *strictly increasing* if for all $\alpha, \beta \in \mathbf{R}$ with $\alpha < \beta$ we have either $\varphi(\alpha) < \varphi(\beta)$ or $\varphi(\alpha) = \varphi(\beta) = +\infty$.

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M-convex* if $\operatorname{dom} f \neq \emptyset$ and f satisfies the following property:

(M-EXC) $\forall x, y \in \operatorname{dom} f$, $\forall u \in \operatorname{supp}^+(x - y)$, $\exists v \in \operatorname{supp}^-(x - y)$:

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v). \tag{2.2}$$

Note that the inequality (2.2) can be rewritten as follows in terms of directional differences:

$$\Delta f(x; v, u) + \Delta f(y; u, v) \leq 0. \tag{2.3}$$

M-convex functions can be characterized by the following (seemingly) weaker property:

(M-EXC$_\mathbf{w}$) $\forall x, y \in \operatorname{dom} f$ with $x \neq y$, $\exists u \in \operatorname{supp}^+(x - y)$, $\exists v \in \operatorname{supp}^-(x - y)$ satisfying (2.2).

Theorem 2.3 ([9, Th. 3.1]). *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we have* (M-EXC) $\Longleftrightarrow$ (M-EXC$_\mathrm{w}$).

We also define the set version of M-convexity as follows. A set $B \subseteq \mathbf{Z}^V$ is called *M-convex* if $B \neq \emptyset$ and it satisfies

(B-EXC) $\forall x, y \in B$, $\forall u \in \operatorname{supp}^+(x - y)$, $\exists v \in \operatorname{supp}^-(x - y)$:
$x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

Note that an M-convex set is nothing but (the set of integral vectors in) an integral base polyhedron [4]. For $x \in B$ and $u, v \in V$, the *exchange capacity* associated with x, v and u is defined as

$$\tilde{c}_B(x, v, u) = \max\{\alpha \in \mathbf{R} \mid x + \alpha(\chi_v - \chi_u) \in B\}.$$

M-convex sets can be characterized also by the following (seemingly) weaker property:

(B-EXC$_\mathbf{w}$) $\forall x, y \in B$ with $x \neq y$, $\exists u \in \operatorname{supp}^+(x - y)$, $\exists v \in \operatorname{supp}^-(x - y)$:
$x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

Theorem 2.4 ([16]). *For $B \subseteq \mathbf{Z}^V$, we have* (B-EXC) $\Longleftrightarrow$ (B-EXC$_\mathrm{w}$).

3 Quasi M-convex Functions

3.1 Definitions

To extend the concept of M-convexity to quasi M-convexity, we relax the condition (2.3) while keeping the possible sign patterns of values $\Delta f(x; v, u)$ and $\Delta f(y; u, v)$ in mind. Table 1 shows the possible sign patterns of those values.

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function. Then, we call f a *quasi M-convex function* if $\operatorname{dom} f \neq \emptyset$ and it satisfies the following property:

(QM) $\forall x, y \in \operatorname{dom} f$, $\forall u \in \operatorname{supp}^+(x - y)$, $\exists v \in \operatorname{supp}^-(x - y)$:
$\Delta f(x; v, u) \leq 0$ or $\Delta f(y; u, v) \leq 0$.

Table 1: Possible sign patterns of $\alpha = \Delta f(x; v, u)$ and $\beta = \Delta f(y; u, v)$ in (M-EXC)

sign(α) \ sign(β)	−	0	+
−	○	○	○
0	○	○	×
+	○	×	×

○··· possible, ×··· impossible

Similarly, we call f a *semistrictly quasi M-convex function* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies the following property:

(SSQM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
(i) $\Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0$, and
(ii) $\Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0$.

Note that (SSQM) can be rewritten as follows:

$\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying at least one of the following:
(i) $\Delta f(x; v, u) < 0$, (ii) $\Delta f(y; u, v) < 0$,
(iii) $\Delta f(x; v, u) = \Delta f(y; u, v) = 0$.

We also consider weaker properties than (QM) and (SSQM):

($\mathbf{QM_w}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
$\Delta f(x; v, u) \leq 0$ or $\Delta f(y; u, v) \leq 0$.

($\mathbf{SSQM_w}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
(i) $\Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0$, and
(ii) $\Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0$.

The set version of quasi M-convexity can be obtained by translating the properties (QM) and ($\mathrm{QM_w}$) for the indicator function $\delta_B : \mathbf{Z}^V \to \{0, +\infty\}$ of a set $B \subseteq \mathbf{Z}^V$ in terms of B.

(Q-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
$x - \chi_u + \chi_v \in B$ or $y + \chi_u - \chi_v \in B$.

($\mathbf{Q\text{-}EXC_w}$) $\forall x, y \in B$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
$x - \chi_u + \chi_v \in B$ or $y + \chi_u - \chi_v \in B$.

Note that the properties (Q-EXC) and ($\mathrm{Q\text{-}EXC_w}$) are the same as (EXC) and ($\mathrm{EXC_w}$) discussed in [14], respectively.

Theorem 3.1. *Let* $B \subseteq \mathbf{Z}^V$.
(i) (Q-EXC) *for* $B \iff$ (QM) *for* δ_B.
(ii) ($\mathrm{Q\text{-}EXC_w}$) *for* $B \iff$ ($\mathrm{QM_w}$) *for* δ_B.
(iii) (B-EXC) *for* $B \iff$ (SSQM) *for* $\delta_B \iff$ ($\mathrm{SSQM_w}$) *for* δ_B.

We show some examples of quasi M-convex functions below.

Example 3.2. Let $\psi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$. We define $f : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ by

$$f(x_1, x_2) = \begin{cases} \psi(x_1) & (x_1 + x_2 = 0), \\ +\infty & (x_1 + x_2 \neq 0). \end{cases} \tag{3.1}$$

By Theorem 2.2, f satisfies (QM) (or ($\mathrm{QM_w}$)) if and only if ψ is quasiconvex, and f satisfies (SSQM) (or ($\mathrm{SSQM_w}$)) if and only if ψ is semistrictly quasiconvex. □

Example 3.3. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function, and $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ be a nondecreasing function. We define a function $\tilde{f} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{f}(x) = \begin{cases} \varphi(f(x)) & (x \in \mathrm{dom}\, f), \\ +\infty & (x \notin \mathrm{dom}\, f). \end{cases} \tag{3.2}$$

Then, $\tilde{f}$ satisfies (QM). Furthermore, if φ is strictly increasing, then $\tilde{f}$ satisfies (SSQM). □

Example 3.4. Let $B \subseteq \mathbf{Z}^V$ be an M-convex set, $w \in \mathbf{R}^V$, and $\alpha \in \mathbf{R}$. Then, the set $S = \{x \in B \mid \langle p, x \rangle \leq \alpha\}$ satisfies (Q-EXC). Moreover, the function $f : S \to \mathbf{R}$ defined by $f(x) = \langle p, x \rangle$ ($x \in S$) satisfies (SSQM). □

Remark 3.5. The concept of (semistrict) quasi M-convexity can be naturally extended to functions $f : S \to T$ with $S \subseteq \mathbf{Z}^V$ and a totally ordered set T with total order $\prec$. For example, the property (SSQM) is rewritten for such functions as follows:

$\forall x, y \in S$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
(i) if either $x - \chi_u + \chi_v \notin S$, or $x - \chi_u + \chi_v \in S$ and $f(x - \chi_u + \chi_v) \succeq f(x)$, then $y + \chi_u - \chi_v \in S$ and $f(y + \chi_u - \chi_v) \preceq f(y)$, and
(ii) if either $y + \chi_u - \chi_v \notin S$, or $y + \chi_u - \chi_v \in S$ and $f(y + \chi_u - \chi_v) \succeq f(y)$, then $x - \chi_u + \chi_v \in S$ and $f(x - \chi_u + \chi_v) \preceq f(x)$,

where for $p, q \in T$ the notation $p \preceq q$ means $p \prec q$ or $p = q$. It is easy to see that the properties of (semistrictly) quasi M-convex functions shown in this paper still holds true. For simplicity and convenience, we assume, in this paper, that the codomain of a function is $\mathbf{R} \cup \{+\infty\}$. □

Example 3.6. Suppose that $V = \{1, 2, \cdots, n\}$ $(n \geq 1)$. Let $a : V \to \mathbf{Z} \cup \{-\infty\}$, $b : V \to \mathbf{Z} \cup \{+\infty\}$, and $\alpha \in \mathbf{Z}$ satisfy $a(v) \leq b(v)$ $(v \in V)$ and $\sum_{i \in V} a(i) \leq \alpha \leq \sum_{i \in V} b(i)$. For $i \in V$, let $f_i : [a(i), b(i)] \to \mathbf{R}$ be a semistrictly quasiconvex function. We define $B \subseteq \mathbf{Z}^V$ and $f : B \to \mathbf{R}^V$ by

$$B = \{x \in \mathbf{Z}^V \mid x(V) = \alpha,\ a \leq x \leq b\},$$
$$f(x) = (f_i(x(i)) \mid i \in V) \quad (x \in B),$$

where the total order $\prec$ on the codomain $\mathbf{R}^V$ of f is given by the lexicographic order, i.e., for each $p, q \in \mathbf{R}^V$, $p \prec q$ holds if there exists some k $(1 \leq k \leq n)$ such that $p_i = q_i$ for $i = 1, \cdots, k-1$ and $p_i < q_i$. Then, f satisfies (SSQM) in the extended sense (see Remark 3.5).

Proof. Let $x, y \in B$ be distinct vectors. Also, let $u \in \mathrm{supp}^+(x-y)$, $v \in \mathrm{supp}^-(x-y)$ be any elements, and w.l.o.g. assume that $u < v$. Then, we have $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$. If $f_u(x(u)-1) < f_u(x(u))$ or $f_u(y(u)+1) < f_u(y(u))$ holds, then we have $f(x - \chi_u + \chi_v) \prec f(x)$ or $f(y + \chi_u - \chi_v) \prec f(y)$. Otherwise, we have $f_u(x(u)-1) = f_u(x(u))$ and $f_u(y(u)+1) = f_u(y(u))$ by Theorem 2.2. If $f_v(x(v)+1) < f_v(x(v))$ or $f_v(y(v)-1) < f_v(y(v))$ holds, then we have $f(x - \chi_u + \chi_v) \prec f(x)$ or $f(y + \chi_u - \chi_v) \prec f(y)$. Otherwise, we have $f_v(x(v)+1) = f_v(x(v))$ and $f_v(y(v)-1) = f_v(y(v))$, from which follows $f(x - \chi_u + \chi_v) = f(x)$ and $f(y + \chi_u - \chi_v) = f(y)$. □

The relationship among various properties for sets and functions is summarized as follows. Note that the claim (i) of Theorem 3.7 is already shown in [14, Remark 11].

Theorem 3.7. (i) *For $S \subseteq \mathbf{Z}^V$, we have*

$$\begin{array}{ccc} \text{(B-EXC)} & \Longrightarrow & \text{(Q-EXC)} \\ \Updownarrow & & \Downarrow \\ (\text{B-EXC}_\text{w}) & \Longrightarrow & (\text{Q-EXC}_\text{w}). \end{array}$$

(ii) *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, we have*

$$\begin{array}{ccccc} \text{(M-EXC)} & \Longrightarrow & \text{(SSQM)} & \Longrightarrow & \text{(QM)} \\ \Updownarrow & & \Downarrow & & \Downarrow \\ (\text{M-EXC}_\text{w}) & \Longrightarrow & (\text{SSQM}_\text{w}) & \Longrightarrow & (\text{QM}_\text{w}). \end{array}$$

3.2 Level Sets

We show various properties for level sets of quasi M-convex functions.

The following two theorems claim that level sets of quasi M-convex functions have quasi M-convexity. Furthermore, the weaker version of quasi M-convexity (QM_w) for functions can be characterized by quasi M-convexity (Q-EXC_w) of level sets.

Lemma 3.8 ([14]). *Let $B \subseteq \mathbf{Z}^V$.*
(i) *If B satisfies* (Q-EXC_w), *then $x(V) = y(V)$ for all $x, y \in \mathrm{dom}\, f$.*
(ii) (Q-EXC_w) $\Longleftrightarrow$ ($\text{Q-EXC}_{\text{w}+}$):

($\text{Q-EXC}_{\text{w}+}$) $\forall x, y \in B$, $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$: $x - \chi_u + \chi_v \in B$.

Theorem 3.9. *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies* (QM_w) *if and only if the level set $\mathrm{L}(f, \alpha)$ satisfies* (Q-EXC_w) *for all $\alpha \in \mathbf{R} \cup \{+\infty\}$. In particular, if f satisfies* (QM_w), *then $\mathrm{dom}\, f$ and $\arg\min f$ satisfy* (Q-EXC_w).

Proof. [$\Longrightarrow$] Let $\alpha \in \mathbf{R} \cup \{+\infty\}$, and $x, y \in \mathrm{L}(f, \alpha)$ be vectors with $x \neq y$. Applying (QM_w) to x and y, we have $\Delta f(x; v, u) \leq 0$ or $\Delta f(y; u, v) \leq 0$ for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$. Therefore, we have $x - \chi_u + \chi_v \in \mathrm{L}(f, \alpha)$ or $y + \chi_u - \chi_v \in \mathrm{L}(f, \alpha)$.

[$\Longleftarrow$] Let $x, y \in \mathrm{dom}\, f$, and we may assume that $f(x) \geq f(y)$. By Lemma 3.8 (ii), the level set $\mathrm{L}(f, f(x))$ satisfies ($\text{Q-EXC}_{\text{w}+}$), from which follows $x - \chi_u + \chi_v \in \mathrm{L}(f, f(x))$ for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$. This implies that $f(x - \chi_u + \chi_v) \leq f(x)$, which yields ($\text{QM}_\text{w}$) for $\mathrm{L}(f, f(x))$. □

Theorem 3.10. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function satisfying* (QM). *Then, the level set $\mathrm{L}(f, \alpha)$ satisfies* (Q-EXC) *for all $\alpha \in \mathbf{R} \cup \{+\infty\}$. In particular, $\mathrm{dom}\, f$ and $\arg\min f$ satisfy* (Q-EXC).

Proof. The proof is similar to that for the "only if" part of Theorem 3.9. □

Theorem 3.11. *Suppose $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies* (SSQM_w). *Then $\arg\min f$ satisfies* (B-EXC), *i.e., $\arg\min f$ is an M-convex set if it is nonempty.*

An M-convex function can be characterized also by quasi M-convexity for level sets of a function perturbed by linear functions. For any function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and any vector $p \in \mathbf{R}^V$, the function $f[p] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is given by

$$f[p](x) = f(x) + \sum_{v \in V} p(v)x(v) \quad (x \in \mathbf{Z}^V).$$

Theorem 3.12 ([14, Th. 1]). *A function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (M-EXC) *if and only if* $\mathrm{L}(f[p], \alpha)$ *satisfies* (Q-EXC$_\mathrm{w}$) *for all* $p \in \mathbf{R}^V$ *and* $\alpha \in \mathbf{R} \cup \{+\infty\}$.

Combining Theorems 3.9 and 3.12, we have the following property.

Corollary 3.13. *A function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (M-EXC) *if and only if* $f[p]$ *satisfies* (QM$_\mathrm{w}$) *for all* $p \in \mathbf{R}^V$.

3.3 Operations

The classes of (semistrictly) quasi M-convex functions are closed under several fundamental operations.

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. For any subset $U \subseteq V$, define $f_U : \mathbf{Z}^U \to \mathbf{R} \cup \{+\infty\}$ by

$$f_U(y) = f(y, \mathbf{0}_{V \setminus U}) \qquad (y \in \mathbf{Z}^U),$$

where $\mathbf{0}_{V\setminus U} \in \mathbf{R}^{V \setminus U}$ denotes the vector with each component equal to zero. For any functions $a : V \to \mathbf{Z} \cup \{-\infty\}$ and $b : V \to \mathbf{Z} \cup \{+\infty\}$, define $f_a^b : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$f_a^b(x) = \begin{cases} f(x) & (a \le x \le b), \\ +\infty & (\text{otherwise}). \end{cases}$$

Theorem 3.14. *Let* (*QM$_*$) *denote one of* (QM), (QM$_\mathrm{w}$), (SSQM), *or* (SSQM$_\mathrm{w}$), *and* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function with* (*QM$_*$).
(i) *For any* $a \in \mathbf{Z}^V$ *and* $\nu > 0$, *the functions* $\nu \cdot f(a - x)$ *and* $\nu \cdot f(a + x)$ *satisfy* (*QM$_*$) *as functions in* x.
(ii) *For any* $U \subseteq V$, *the function* $f_U : \mathbf{Z}^U \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (*QM$_*$).
(iii) *For any* $a : V \to \mathbf{Z} \cup \{-\infty\}$ *and* $b : V \to \mathbf{Z} \cup \{+\infty\}$ *with* $a \le b$, *the function* $f_a^b : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (*QM$_*$).
(iv) *Let* $f_i : \mathbf{Z}^{V_i} \to \mathbf{R}_{++} \cup \{+\infty\}$ $(i = 1, 2)$ *be functions with* (*QM$_*$). *Then, the function* $f : \mathbf{Z}^{V_1} \times \mathbf{Z}^{V_2} \to \mathbf{R}_{++} \cup \{+\infty\}$ *defined by*

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \quad ((x_1, x_2) \in \mathbf{Z}^{V_1} \times \mathbf{Z}^{V_2})$$

satisfies (*QM$_*$).

Proof. We prove (iv) only. We consider the case when (*QM$_*$) = (SSQM). Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathrm{dom}\, f_1 \times \mathrm{dom}\, f_2$, and let $u \in \mathrm{supp}^+(x - y)$, where $u \in \mathrm{supp}^+(x_1 - y_1)$ w.l.o.g. Then, there exists $v \in \mathrm{supp}^-(x_1 - y_1)$ such that

$$\Delta f_1(x_1; v, u) \ge 0 \Longrightarrow \Delta f_1(y_1; u, v) \le 0, \text{ and}$$
$$\Delta f_1(y_1; u, v) \ge 0 \Longrightarrow \Delta f_1(x_1; v, u) \le 0.$$

This implies that

$$\Delta f(x; v, u) \ge 0 \Longrightarrow \Delta f(y; u, v) \le 0, \text{ and}$$
$$\Delta f(y; u, v) \ge 0 \Longrightarrow \Delta f(x; v, u) \le 0.$$

Hence, (SSQM) holds for f. □

Remark 3.15. The class of (semistrictly) quasi M-convex functions is not closed under addition; in particular, it is not closed under addition of a linear function. □

Theorem 3.16. *For* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$, *define* $\tilde{f} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *by (3.2).*
(i) *If* f *satisfies* (QM) *(resp.* (QM$_\mathrm{w}$)*) and* φ *is nondecreasing, then* $\tilde{f}$ *satisfies* (QM) *(resp.* (QM$_\mathrm{w}$)*).*
(ii) *If* f *satisfies* (SSQM) *(resp.* (SSQM$_\mathrm{w}$)*) and* φ *is strictly increasing, then* $\tilde{f}$ *satisfies* (SSQM) *(resp.* (SSQM$_\mathrm{w}$)*).*

Remark 3.17. A quasi M-convex function $\tilde{f} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is not necessarily given as the form (3.2). As an example, let $\tilde{f} : \mathbf{Z}^3 \to \mathbf{R} \cup \{+\infty\}$ be a function given by

$$\mathrm{dom}\, \tilde{f} = \{(0,0,0), (1,0,-1), (2,0,-2), (2,1,-3), (2,2,-4)\},$$
$$\tilde{f}(x_1, x_2, x_3) = -x_1 + x_2 \quad (x \in \mathrm{dom}\, \tilde{f}).$$

Although $\tilde{f}$ satisfies (SSQM), it cannot be represented in the form (3.2) with an M-convex function $f : \mathbf{Z}^3 \to \mathbf{R} \cup \{+\infty\}$ and a nondecreasing function $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$. □

Theorem 3.18. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *be functions such that* $g(x) > 0$ *for all* $x \in \mathrm{dom}\, f$. *Suppose that the function* $f(\cdot) - \alpha g(\cdot)$ *satisfies* (QM$_\mathrm{w}$) *for all* $\alpha \in \mathbf{R} \cup \{+\infty\}$. *Then, the function* $r : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *given by*

$$r(x) = \begin{cases} f(x)/g(x) & (x \in \mathrm{dom}\, f), \\ +\infty & (x \notin \mathrm{dom}\, f), \end{cases}$$

satisfies (QM$_\mathrm{w}$).

Proof. Clear from Theorem 3.9. □

Remark 3.19. The statement of Theorem 3.18 cannot be strengthened by replacing ($\mathrm{QM_w}$) with (QM), even if f and g are linear functions. □

3.4 Characterization by Local Exchange Properties

An M-convex function is characterized by a localized version of the property (M-EXC):

(M-EXC-loc) $\forall x, y \in \mathrm{dom}\, f$ with $\|x-y\|_1 = 4$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying (2.2).

Theorem 3.20 ([9, Th. 3.1], [14, Th. 2]). *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that* $\mathrm{dom}\, f$ *is a nonempty set with* ($\mathrm{Q\text{-}EXC_w}$)*. Then,* (M-EXC) $\Longleftrightarrow$ (M-EXC-loc).

We show that (semistrict) quasi M-convexity can be characterized also by the localized version of (SSQM) and (QM).

(SSQM-loc) $\forall x, y \in \mathrm{dom}\, f$ with $\|x-y\|_1 = 4$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
(i) $\Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0$, and
(ii) $\Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0$.

($\mathbf{SSQM_w}$-loc) $\forall x, y \in \mathrm{dom}\, f$ with $\|x-y\|_1 = 4$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:
(i) $\Delta f(x; v, u) \geq 0 \Longrightarrow \Delta f(y; u, v) \leq 0$, and
(ii) $\Delta f(y; u, v) \geq 0 \Longrightarrow \Delta f(x; v, u) \leq 0$.

Theorem 3.21. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that* $\mathrm{dom}\, f$ *satisfies* ($\mathrm{Q\text{-}EXC_w}$)*. Then,*
(i) (SSQM) $\Longleftrightarrow$ (SSQM-loc).
(ii) ($\mathrm{SSQM_w}$) $\Longleftrightarrow$ ($\mathrm{SSQM_w}$-loc).

Proof. See [11]. □

Remark 3.22. The localized version of (QM) does not characterize (QM) in general. Let $f : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ be a function such that

$$\mathrm{dom}\, f = \{(0,0), (1,-1), (2,-2), (3,-3)\},$$
$$f(0,0) = f(3,-3) = 0, \ f(1,-1) = f(2,-2) = 1.$$

Then, $\mathrm{dom}\, f$ satisfies (Q-EXC), and (QM) holds for any $x, y \in \mathrm{dom}\, f$ with $\|x-y\|_1 = 4$. However, (QM) does not hold for $x = (0,0)$ and $y = (3,-3)$. □

4 Minimization of Quasi M-convex Functions

4.1 Theorems

Global minimality of quasi M-convex functions is characterized by local minimality.

Theorem 4.1. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$.*
(i) *Assume* ($\mathrm{QM_w}$) *for f. Then, $\Delta f(x; v, u) > 0$ $(\forall u, v \in V,\ u \neq v) \Longleftrightarrow f(x) < f(y)$ $(\forall y \in \mathbf{Z}^V \setminus \{x\})$.*
(ii) *Assume* ($\mathrm{SSQM_w}$) *for f. Then, $\Delta f(x; v, u) \geq 0$ $(\forall u, v \in V) \Longleftrightarrow f(x) \leq f(y)$ $(\forall y \in \mathbf{Z}^V)$.*

Proof. We show the sufficiency of (ii) only. Assume, to the contrary, that there exists some $y \in \mathrm{dom}\, f$ such that $f(y) < f(x)$. We further assume that y minimizes the value $\|y-x\|_1$ among all such vectors. By ($\mathrm{SSQM_w}$), there exist some $u' \in \mathrm{supp}^+(x-y)$ and $v' \in \mathrm{supp}^-(x-y)$ such that if $\Delta f(x; v', u') \geq 0$ then $\Delta f(y; u', v') \leq 0$. Since $\Delta f(x; v', u') \geq 0$ holds true, we have $f(y + \chi_{u'} - \chi_{v'}) \leq f(y) < f(x)$ and $\|(y + \chi_{u'} - \chi_{v'}) - x\|_1 < \|y - x\|_1$, a contradiction to the choice of y. □

If f satisfies (SSQM), then any vector in $\mathrm{dom}\, f$ can be easily separated from some minimizer of f (cf. [13, Th. 2.2, Cor. 2.3]).

Theorem 4.2. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function with* (SSQM)*. Assume* $\arg\min f \neq \emptyset$*.*
(i) *For $x \in \mathrm{dom}\, f$ and $v \in V$, let $u \in V$ satisfy*

$$f(x - \chi_u + \chi_v) = \min_{s \in V} f(x - \chi_s + \chi_v).$$

Then, there exists $x_* \in \arg\min f$ with $x_*(u) \leq x(u) - 1 + \chi_v(u)$.
(ii) *For $x \in \mathrm{dom}\, f$ and $u \in V$, let $v \in V$ satisfy*

$$f(x - \chi_u + \chi_v) = \min_{t \in V} f(x - \chi_u + \chi_t).$$

Then, there exists $x_* \in \arg\min f$ with $x_*(v) \geq x(v) - \chi_u(v) + 1$.

Proof. We prove (i) only. Put $x' = x - \chi_u + \chi_v$. Assume, to the contrary, that there is no $x \in \arg\min f$ with $x(u) \leq x'(u)$. Let $x_* \in \arg\min f$ with minimum $x_*(u)$. Then, we have $x_*(u) > x'(u)$. By applying (SSQM) to x_*, x', and u, we have some $w \in \mathrm{supp}^-(x_* - x')$ such that if

$\Delta f(x_*; w, u) > 0$ then $\Delta f(x'; u, w) < 0$. Due to the choice of x_*, we have $\Delta f(x_*; w, u) > 0$. Hence, it holds that

$$f(x') > f(x' + \chi_u - \chi_w) = f(x - \chi_w + \chi_v),$$

a contradiction to the definition of $u \in V$. □

Corollary 4.3. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function with* (SSQM). *Also, let $x \in \operatorname{dom} f \setminus \arg\min f$, and $u, v \in V$ satisfy*

$$f(x - \chi_u + \chi_v) = \min_{s,t \in V} f(x - \chi_s + \chi_t).$$

Then, there exists $x_* \in \arg\min f$ with $x_*(u) \leq x(u) - 1$ and $x_*(v) \geq x(v) + 1$.

Remark 4.4. The statements in Theorem 4.2 do not hold even if f satisfies the property (SSQM$_w$) (and not (SSQM)). □

The following theorem shows that a global minimizer of a semistrictly quasi M-convex function exists in the neighborhood of a Δ-local minimum. This generalizes [6, Th. 4.1].

Theorem 4.5. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function with* (SSQM), *and Δ be any positive integer. Suppose that $x_\Delta \in \operatorname{dom} f$ satisfies $f(x_\Delta) \leq f(x_\Delta + \Delta(\chi_v - \chi_u))$ for all $u, v \in V$. Then, there exists some $x_* \in \arg\min f$ such that*

$$|x_\Delta(v) - x_*(v)| \leq (n-1)(\Delta - 1) \quad (v \in V). \tag{4.1}$$

Proof. It suffices to show that for all $\varepsilon > 0$ there exists some $x_* \in \operatorname{dom} f$ satisfying $f(x_*) \leq \inf f + \varepsilon$ and (4.1).

Let $x_* \in \operatorname{dom} f$ satisfy $f(x_*) \leq \inf f + \varepsilon$, and suppose that x_* minimizes the value $||x_* - x_\Delta||_1$ among all such vectors. In the following, we fix $v \in V$ and prove $x_\Delta(v) - x_*(v) \leq (n-1)(\Delta - 1)$. The inequality $x_*(v) - x_\Delta(v) \leq (n-1)(\Delta - 1)$ can be shown similarly.

We may assume $x_\Delta(v) > x_*(v)$. We first prove the following two claims.

Claim 1 There exist $w_1, w_2, \cdots, w_k \in V \setminus \{v\}$ and $y_0 (= x_\Delta), y_1, \cdots, y_k \in \operatorname{dom} f$ with $k = x_\Delta(v) - x_*(v)$ such that

$$y_i = y_{i-1} - \chi_v + \chi_{w_i},$$
$$f(y_i) < f(y_{i-1}) \quad (i = 1, \cdots, k).$$

[Proof of Claim 1] We show the claim by induction on i. If $i - 1 < k$, then $v \in \operatorname{supp}^+(y_{i-1} - x_*)$. By (SSQM) applied to y_{i-1}, x_*, and v, we have some $w_i \in \operatorname{supp}^-(y_{i-1} - x_*) \subseteq \operatorname{supp}^-(x_\Delta - x_*) \subseteq V \setminus \{v\}$ such that if $\Delta f(x_*; v, w_i) > 0$ then $\Delta f(y_{i-1}; w_i, v) < 0$. By the choice of x_*, we have $\Delta f(x_*; v, w_i) > 0$ since $||(x_* + \chi_v - \chi_{w_i}) - x_\Delta||_1 < ||x_* - x_\Delta||_1$. Therefore, $f(y_i) = f(y_{i-1} - \chi_v + \chi_{w_i}) < f(y_{i-1})$.

[End of Proof for Claim 1]

Claim 2 For any $w \in V \setminus \{v\}$ with $y_k(w) > x_\Delta(w)$ and $\alpha \in [0, y_k(w) - x_\Delta(w) - 1]$, we have

$$f(x_\Delta - (\alpha + 1)(\chi_v - \chi_w)) < f(x_\Delta - \alpha(\chi_v - \chi_w)). \tag{4.2}$$

[Proof of Claim 2] We prove (4.2) by induction on α. Put $x' = x_\Delta - \alpha(\chi_v - \chi_w)$ for $\alpha \in [0, y_k(w) - x_\Delta(w) - 1]$, and suppose $x' \in \operatorname{dom} f$. Let j_* $(1 \leq j_* \leq k)$ be the largest index such that $w_{j_*} = w$. Then, $y_{j_*}(w) = y_k(w) > x'(w)$ and $\operatorname{supp}^-(y_{j_*} - x') = \{v\}$. (SSQM) implies that if $\Delta f(y_{j_*}; v, w) > 0$ then $\Delta f(x'; w, v) < 0$. By Claim 1, we have $\Delta f(y_{j_*}; v, w) > 0$. Hence, (4.2) follows.

[End of Proof for Claim 2]

The Δ-local minimality of x_Δ implies $f(x_\Delta - \Delta(\chi_v - \chi_w)) \geq f(x_\Delta)$, which, combined with Claim 2, implies $y_k(w) - x_\Delta(w) \leq \Delta - 1$. Thus,

$$\begin{aligned} x_\Delta(v) - x_*(v) &= x_\Delta(v) - y_k(v) \\ &= \sum_{w \in V \setminus \{v\}} \{y_k(w) - x_\Delta(w)\} \\ &\leq (n-1)(\Delta - 1), \end{aligned}$$

where the second equality is by Lemma 3.8 (i). □

4.2 Algorithms

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that $\operatorname{dom} f$ is a nonempty bounded set, and put

$$L = \max\{||x - y||_\infty \mid x, y \in \operatorname{dom} f\}.$$

Assume (SSQM$_w$) for f. Then, Theorem 4.1 immediately leads to the following algorithm.

Algorithm Descent
Step 0: Let x be any vector in $\operatorname{dom} f$.
Step 1: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop.
[x is a minimizer of f.]

Step 2: Find $u, v \in V$ with $f(x-\chi_u+\chi_v) < f(x)$.
Step 3: Set $x := x - \chi_u + \chi_v$. Go to Step 1. □

Algorithm DESCENT terminates in at most $|\mathrm{dom}\, f| \le (L+1)^{n-1}$ iterations since it generates distinct x in each iteration.

To the end of this section we assume (SSQM) for f. Based on Theorem 4.5, we apply the scaling technique to Algorithm DESCENT to obtain a faster algorithm.

Algorithm SCALING_DESCENT
Step 0: Let x be any vector in $\mathrm{dom}\, f$. Put $\Delta := \lceil L/4n \rceil$, $B := \mathrm{dom}\, f$.
Step 1: [Δ-scaling phase]
Step 1-1: If

$$f(x) = \min\{f(x - \Delta(\chi_s - \chi_t)) \mid s,t \in V,\ x - \Delta(\chi_s - \chi_t) \in B\}$$

then go to Step 2.
Step 1-2: Find $u, v \in V$ with $x - \Delta(\chi_u - \chi_v) \in B$ satisfying $f(x - \Delta(\chi_u - \chi_v)) < f(x)$.
Step 1-3: Set $x := x - \Delta(\chi_u - \chi_v)$. Go to Step 1-1.
Step 2: If $\Delta = 1$ then stop. [x is a minimizer of f.]
Step 3: Put

$$B := B \cap \{y \in \mathbf{Z}^V \mid |y(v) - x(v)| \le (n-1)(\Delta - 1)\ (v \in V)\}$$

and $\Delta := \lceil \Delta/2 \rceil$. Go to Step 1. □

The number of scaling phases is $\lceil \log L \rceil$, and each scaling phase terminates in $(4n)^{n-1}$ iterations. Therefore, Algorithm SCALING_DESCENT runs in $(4n)^{n-1}\lceil \log L \rceil$ iterations.

We then propose another elaboration of Algorithm DESCENT by exploiting Corollary 4.3

Algorithm STEEPEST_DESCENT
Step 0: Let x be any vector in $\mathrm{dom}\, f$. Set $B := \mathrm{dom}\, f$.
Step 1: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. [x is a minimizer of f.]
Step 2: Find $u, v \in V$ with $x - \chi_u + \chi_v \in B$ satisfying

$$f(x - \chi_u + \chi_v) = \min\{f(x - \chi_s + \chi_t) \mid s,t \in V,\ x - \chi_s + \chi_t \in B\}. \tag{4.3}$$

Step 3: Set $x := x - \chi_u + \chi_v$ and

$$B := B \cap \{y \in \mathbf{Z}^V \mid y(u) \le x(u) - 1,\ y(v) \ge x(v) + 1\}. \tag{4.4}$$

Go to Step 1. □

By Corollary 4.3, the set B always contains a minimizer of f. Hence, Algorithm STEEPEST_DESCENT finds a minimizer of f. To analyze the number of iterations, we consider the value

$$\sum_{w \in V} \{\max_{y \in B} y(w) - \min_{y \in B} y(w)\}.$$

This value is bounded by nL and decreases at least by two in each iteration. Therefore, STEEPEST_DESCENT terminates in $\mathrm{O}(nL)$ iterations. In particular, if $\mathrm{dom}\, f \subseteq \{0,1\}^V$ then the number of iterations is $\mathrm{O}(n^2)$.

It is shown in [13] that the minimization of an M-convex function can be done in polynomial time by the domain reduction method explained below. We show that the domain reduction method also works for the minimization of a function with (SSQM) if its effective domain is a bounded M-convex set.

Given a bounded M-convex set B, we define the set $N_B \subseteq B$ as follows. For $w \in V$, define

$$l_B(w) = \min_{y \in B} y(w), \quad u_B(w) = \max_{y \in B} y(w),$$
$$l'_B(w) = \left\lfloor \left(1 - \frac{1}{n}\right) l_B(w) + \frac{1}{n} u_B(w) \right\rfloor,$$
$$u'_B(w) = \left\lceil \frac{1}{n} l_B(w) + \left(1 - \frac{1}{n}\right) u_B(w) \right\rceil.$$

Then, N_B is defined as

$$N_B = \{y \in B \mid l'_B \le y \le u'_B\}.$$

Theorem 4.6 ([13, Th. 2.4]). *N_B is a (nonempty) M-convex set.*

The next algorithm maintains a set B ($\subseteq \mathrm{dom}\, f$) which is an M-convex set containing a minimizer of f. It reduces B iteratively by exploiting Corollary 4.3 and finally finds a minimizer.

Algorithm DOMAIN_REDUCTION
Step 0: Set $B := \mathrm{dom}\, f$.
Step 1: Find a vector $x \in N_B$.
Step 2: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. [x is a minimizer of f.]
Step 3: Find $u, v \in V$ with $x - \chi_u + \chi_v \in B$ satisfying (4.3).
Step 4: Set B by (4.4). Go to Step 1. □

We analyze the number of iterations of DOMAIN_REDUCTION. Denote by B_i the set B in the i-th iteration, and let $l_i(w) = l_{B_i}(w)$, $u_i(w) = u_{B_i}(w)$ $(w \in V)$. It is clear that $u_i(w) - l_i(w)$ is nonincreasing w.r.t. i. Furthermore, we have the following property:

Lemma 4.7 ([13, Lemma 3.1]). $u_{i+1}(w) - l_{i+1}(w) < (1 - 1/n)\{u_i(w) - l_i(w)\}$ *for* $w \in \{u, v\}$*, where* $u, v \in V$ *are the elements found in Step 3.*

This lemma implies that Algorithm DOMAIN_REDUCTION terminates in $\mathrm{O}(n^2 \log L)$ iterations.

We then consider the time complexity of each step. Steps 2, 3, and 4 can be done in $\mathrm{O}(n^2)$ time. In Step 1, we use the exchange capacity to compute the values $l_B(w)$ and $u_B(w)$ and to find a vector in N_B. For any $w \in V$, the values $l_B(w)$ and $u_B(w)$ can be computed by evaluating the exchange capacity at most n times, provided that a vector in B is given [4, Th. 3.27]. A vector in N_B can be found by evaluating the exchange capacity at most n^2 times, provided that a vector in B is given [13, Th. 2.5]. The exchange capacity can be computed in $\mathrm{O}(\log L)$ time by binary search. Hence, Step 1 requires $\mathrm{O}(n^2 \log L)$ time.

Theorem 4.8. *Suppose that* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (SSQM) *and that* $\mathrm{dom}\, f$ *is a bounded M-convex set. If a vector in* $\mathrm{dom}\, f$ *is given, Algorithm* DOMAIN_REDUCTION *finds a minimizer of* f *in* $\mathrm{O}(n^4 (\log L)^2)$ *time.*

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