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Measurements of Nondegenerate Observables

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1. Introduction

Every measuring apparatus inputs the state ρ of the measured system and outputs the classical output $\mathbf{x}$ and the state $\rho_{\{\mathbf{x}=x\}}$ of the measured system conditional upon the outcome $\mathbf{x}=x$. In the conventional approach, the probability distribution of the classical output $\mathbf{x}$ and the output state $\rho_{\{\mathbf{x}=x\}}$ are determined by the spectral projections of the measured observable by the Born statistical formula and the projection postulate, respectively. This description has been a very familiar principle in quantum mechanics, but is much more restrictive than what quantum mechanics allows. In the modern measurement theory, the problem has been investigated as to what is the most general description of measurement allowed by quantum mechanics. This paper investigates the problem of the determination of all the possible measurements of observables with nondegenerate spectrum and shows that the following conditions are equivalent for measurements of nondegenerate observables: (i) The joint probability distribution of the outcomes of successive measurements depends affinely on the initial state. (ii) The apparatus has an indirect measurement model. (iii) The state change is described by a positive superoperator valued measure. (iv) The state change is described by a completely positive superoperator valued measure. (v) The family of output states is a Borel family of density operators independent of the input state and can be arbitrarily chosen by the choice of the apparatus.

2. Measurement schemes

Let $\mathcal{H}$ be a separable Hilbert space. A *density operator* on $\mathcal{H}$ is a positive operator on $\mathcal{H}$ with unit trace. The *state space* of $\mathcal{H}$ is the set $\mathcal{S}(\mathcal{H})$ of density operators on $\mathcal{H}$. In what follows, we shall give a general mathematical formulation for the statistical properties of measuring apparatuses. For heuristics, we shall consider a measuring apparatus which measures the quantum system described by the Hilbert space $\mathcal{H}$. Every measuring apparatus has the output variable that gives the outcome on each measurement. We assume that the output variable takes values in a standard Borel space which is specified by each measuring apparatus. We shall denote by $\mathbf{A}(\mathbf{x})$ the measuring apparatus with the output variable $\mathbf{x}$ taking values in a standard Borel space Λ with the Borel σ-field $\mathcal{B}(\Lambda)$. The statistical property of the apparatus $\mathbf{A}(\mathbf{x})$

consists of the output distribution $\Pr\{\mathbf{x} \in \Delta \| \rho\}$ and the state reduction $\rho \mapsto \rho_{\{\mathbf{x}=x\}}$. The output distribution $\Pr\{\mathbf{x} \in \Delta \| \rho\}$ describes the probability distribution of the output variable $\mathbf{x}$ when the input state is $\rho \in \mathcal{S}(\mathcal{H})$, where $\Delta \in \mathcal{B}(\Lambda)$. The state reduction $\rho \mapsto \rho_{\{bx=x\}}$ describes the state change from the input state ρ to the output state $\rho_{\{\mathbf{x}=x\}}$, when the measurement leads to the output $\mathbf{x} = x$. Since the output variable can have a continuous probability distribution, the output state $\rho_{\{\mathbf{x}=x\}}$ is determined up to probability one with respect to the output distribution. The state reduction determines the collective state reduction $\rho \mapsto \rho_{\{\mathbf{x}\in\Delta\}}$ that describes the output state $\rho_{\{\mathbf{x}\in\Delta\}}$ given that the output of the measurement is in a Borel set Δ. The collective state reduction is naturally related to the state reduction by the integral formula

$$\rho_{\{\mathbf{x}\in\Delta\}} = \frac{1}{\Pr\{\mathbf{x} \in \Delta \| \rho\}} \int_{\Delta} \rho_{\{\mathbf{x}=x\}} \Pr\{\mathbf{x} \in dx \| \rho\}.$$

In order to justify the integral we need to require that the function $\rho \mapsto \rho_{\{\mathbf{x}=x\}}$ is a density operator valued Borel function on Λ. Formal description of the statistical properties of measuring apparatuses will be given as follows.

Let Λ be a standard Borel space [1] with σ-field $\mathcal{B}(\Lambda)$. A *Borel family of states* for $(\mathcal{H}, \Lambda)$ is a family $\{\rho_x\}_{x\in\Lambda}$ of density operators on $\mathcal{H}$ such that the function $x \mapsto \mathrm{Tr}[A\rho_x]$ is a Borel function on Λ for every $A \in \mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ stands for the space of bounded operators on $\mathcal{H}$. Since $\mathcal{H}$ is separable, every Borel family of states is Bochner integrable with respect to every probability measure on Λ [2]. Denote by $B(\Lambda, \mathcal{S}(\mathcal{H}))$ the space of Borel families of states for $(\mathcal{H}, \Lambda)$. The *state space* of Λ is the set $\mathcal{S}(\Lambda)$ of probability measures on $\mathcal{B}(\Lambda)$. A *measurement scheme* for $(\mathcal{H}, \Lambda)$ is the pair $(\mathcal{P}, \mathcal{Q})$ of a function $\mathcal{P}$ from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\Lambda)$ and a function $\mathcal{Q}$ from $\mathcal{S}(\mathcal{H})$ to $B(\Lambda, \mathcal{S}(\mathcal{H}))$. The function $\mathcal{P}$ is called the *output probability scheme*, and $\mathcal{Q}$ is called the *state reduction scheme*. Two measurement schemes $(\mathcal{P}, \mathcal{Q})$ and $(\mathcal{P}', \mathcal{Q}')$ are said to be *equivalent*, in symbols $(\mathcal{P}, \mathcal{Q}) \cong (\mathcal{P}', \mathcal{Q}')$, if $\mathcal{P} = \mathcal{P}'$ and they differ only on a null set of the probability measure $\mathcal{P}\rho$, i.e.,

$$(\mathcal{P}\rho)\{x \in \Lambda |\ (\mathcal{Q}\rho)(x) \neq (\mathcal{Q}'\rho)(x)\} = 0,$$

for all $\rho \in \mathcal{S}(\mathcal{H})$.

The set of equivalence classes of measurement schemes for $(\mathcal{H}, \Lambda)$ is denoted by $\mathcal{M}(\mathcal{H}, \Lambda)$ and we define $\mathcal{M}(\mathcal{H}) = \bigcup_{\Lambda} \mathcal{M}(\mathcal{H}, \Lambda)$. A *measurement theory* for $\mathcal{H}$ is a pair $(\mathcal{A}, \mathbf{M})$ consisting of a nonempty set $\mathcal{A}$ and a function $\mathbf{M}$ from $\mathcal{A}$ to $\mathcal{M}(\mathcal{H})$. An element of $\mathcal{A}$ is called an *apparatus*. Every apparatus has its distinctive *output variable*. We denote the apparatuses with output variables $\mathbf{x}, \mathbf{y}, \ldots$ by $\mathbf{A}(\mathbf{x}), \mathbf{A}(\mathbf{y}), \ldots$, respectively. We assume that $\mathbf{x} = \mathbf{y}$ if and only if $\mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{y})$. The image of $\mathbf{A}(\mathbf{x})$ by $\mathbf{M}$ is denoted by $\mathbf{M}(\mathbf{x})$ instead of $\mathbf{M}(\mathbf{A}(\mathbf{x}))$ for simplicity, so that $\mathbf{M}(\mathbf{x})$ denotes the equivalence class of the measurement scheme corresponding to the apparatus $\mathbf{A}(\mathbf{x})$. The output variable $\mathbf{x}$ or the apparatus $\mathbf{A}(\mathbf{x})$ is called Λ*-valued* if $\mathbf{M}(\mathbf{x}) \in \mathcal{M}(\mathcal{H}, \Lambda)$. For any $\mathbf{A}(\mathbf{x}) \in \mathcal{A}$, the equivalence class $\mathbf{M}(\mathbf{x})$ of the measurement scheme is called the *statistical property of* $\mathbf{A}(\mathbf{x})$. We shall denote by $(\mathcal{P}_{\mathbf{x}}, \mathcal{Q}_{\mathbf{x}})$

a representative of $\mathbf{M}(\mathbf{x})$; in this case, we shall also write $\mathbf{M}(\mathbf{x}) = [\mathcal{P}_{\mathbf{x}}, \mathcal{Q}_{\mathbf{x}}]$. The function $\mathcal{P}_{\mathbf{x}}$ is called the *output probability* of $\mathbf{A}(\mathbf{x})$ and $\mathcal{Q}_{\mathbf{x}}$ the *state reduction* of $\mathbf{A}(\mathbf{x})$ which is determined uniquely up to the output probability one. Two apparatuses $\mathbf{A}(\mathbf{x})$ and $\mathbf{A}(\mathbf{y})$ are said to be *statistically equivalent*, in symbols $\mathbf{A}(\mathbf{x}) \cong \mathbf{A}(\mathbf{y})$, if they have the same statistical property, i.e., $\mathbf{M}(\mathbf{x}) = \mathbf{M}(\mathbf{y})$. Two measurement theories $(\mathcal{A}, \mathbf{M})$ and $(\mathcal{A}', \mathbf{M}')$ are said to be *consistent* if $\mathbf{M}(\mathbf{x}) = \mathbf{M}'(\mathbf{x})$ for all $\mathbf{A}(\mathbf{x}) \in \mathcal{A} \cap \mathcal{A}'$. Two consistent measurement theories $(\mathcal{A}, \mathbf{M})$ and $(\mathcal{A}', \mathbf{M}')$ are said to be *statistically equivalent*, in symbols $(\mathcal{A}, \mathbf{M}) \cong (\mathcal{A}', \mathbf{M}')$, if they have the same set of the statistical equivalence classes of apparatuses, i.e., $\mathbf{M}(\mathcal{A}) = \mathbf{M}'(\mathcal{A}')$.

Suppose that a measurement theory $(\mathcal{A}, \mathbf{M})$ is given. Let $\mathbf{A}(\mathbf{x}) \in \mathcal{A}$ and $\mathbf{M}(\mathbf{x}) = [\mathcal{P}_{\mathbf{x}}, \mathcal{Q}_{\mathbf{x}}]$. The *probability distribution* of the output variable $\mathbf{x}$ in the state ρ is defined by

$$\Pr\{\mathbf{x} \in \Delta \| \rho\} = (\mathcal{P}_{\mathbf{x}} \rho)(\Delta)$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $\Delta \in \mathcal{B}(\Lambda)$. This probability distribution is called the *output distribution* of $\mathbf{A}(\mathbf{x})$ in ρ. The *output states* $\{\rho_{\{\mathbf{x}=x\}}\}_{x \in \Lambda}$ of $\mathbf{A}(\mathbf{x})$ in ρ is defined by

$$\rho_{\{\mathbf{x}=x\}} = (\mathcal{Q}_{\mathbf{x}} \rho)(x)$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $x \in \Lambda$. This is defined uniquely up to the output probability one.

Let $\Lambda_1, \dots, \Lambda_n$ be standard Borel spaces. For $j = 1, \dots, n$, let $\mathbf{A}(\mathbf{x}_j)$ be a Λ_j-valued apparatus. A *successive measurement in the input state* ρ is a sequence of measurements using $\mathbf{A}(\mathbf{x}_1), \dots, \mathbf{A}(\mathbf{x}_n)$ such that the input state of the apparatus $\mathbf{A}(\mathbf{x}_1)$ is ρ and the input state of the apparatus $\mathbf{A}(\mathbf{x}_{j+1})$ is the output state of the apparatus $\mathbf{A}(\mathbf{x}_j)$ for $j = 1, \dots, n-1$. The joint probability distribution of the outcomes of the successive measurements using $\mathbf{A}(\mathbf{x}_1), \dots, \mathbf{A}(\mathbf{x}_n)$ in the input state ρ is naturally defined recursively by

$$\begin{aligned} &\Pr\{\mathbf{x}_1 \in \Delta_1, \dots, \mathbf{x}_n \in \Delta_n \| \rho\} \\ &\quad = \int_{\Delta_1} \Pr\{\mathbf{x}_2 \in \Delta_2, \dots, \mathbf{x}_n \in \Delta_n \| \rho_{\{\mathbf{x}_1 = x_1\}}\} \Pr\{\mathbf{x}_1 \in dx_1 \| \rho\} \end{aligned} \tag{2.1}$$

for $\Delta_1 \in \mathcal{B}(\Lambda_1), \dots, \Delta_n \in \mathcal{B}(\Lambda_n)$.

Now, we consider the following requirement for a measurement theory $(\mathcal{A}, \mathbf{M})$:

Mixing law of the n-joint probability distributions (nMLPD): *For any sequence $\mathbf{A}(\mathbf{x}_1), \dots, \mathbf{A}(\mathbf{x}_n)$ of apparatuses with values in $\Lambda_1, \dots, \Lambda_n$, respectively, if the input state ρ is the mixture of ρ_1 and ρ_2 such that $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$ with $0 < \alpha < 1$ then we have*

$$\begin{aligned} &\Pr\{\mathbf{x}_1 \in \Delta_1, \dots, \mathbf{x}_n \in \Delta_n \| \rho\} \\ &\quad = \alpha \Pr\{\mathbf{x}_1 \in \Delta_1, \dots, \mathbf{x}_n \in \Delta_n \| \rho_1\} \\ &\qquad + (1-\alpha) \Pr\{\mathbf{x}_1 \in \Delta_1, \dots, \mathbf{x}_n \in \Delta_n \| \rho_2\} \end{aligned} \tag{2.2}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $\Delta_1 \in \mathcal{B}(\Lambda_1), \dots, \Delta_n \in \mathcal{B}(\Lambda_n)$.

This requirement is justified as follows. If the system is in the state ρ_1 with probability α and in the state ρ_2 with probability $1-\alpha$ then the joint probability should be the mixture of the joint probabilities in the right hand side. On the other hand, the state of the system is described by the density operator ρ and the joint probability is also expressed as in the left hand side, so that the above equality should hold.

It can be seen by induction that the relation

$$\Pr\{\mathbf{x}_1 \in \Delta_1, \ldots, \mathbf{x}_{n-1} \in \Delta_{n-1}, \mathbf{x}_n \in \Lambda_n \| \rho\} = \Pr\{\mathbf{x}_1 \in \Delta_1, \ldots, \mathbf{x}_{n-1} \in \Delta_{n-1} \| \rho\}$$

holds generally, so that the nMLPD implies the $(n-1)$MLPD.

An *observable* of $\mathcal{H}$ is a self-adjoint operator (densely defined) on $\mathcal{H}$. We denote by E^A the spectral measure of an observable A. According to the Born statistical formula, we say that an $\mathbf{R}$-valued apparatus $\mathbf{A}(\mathbf{x})$ *measures an observable* A if

$$\Pr\{\mathbf{x} \in \Delta \| \rho\} = \mathrm{Tr}[E^A(\Delta)\rho] \tag{2.3}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $\Delta \in \mathcal{B}(\mathbf{R})$, where $\mathbf{R}$ stands for the real number field. A measurement theory $(\mathcal{A}, \mathbf{M})$ is called *nonsuperselective* if for any observable A there is at least one apparatus measuring A.

A Λ-*valued observable* of $\mathcal{H}$ is a projection valued measure E from $\mathcal{B}(\Lambda)$ to $\mathcal{L}(\mathcal{H})$ such that $E(\Lambda) = I$; Λ-valued observables are also called spectral measures, for which we refer the reader to [3]. The Born statistical formula is generalized as follows. We say that a Λ-valued apparatus $\mathbf{A}(\mathbf{x})$ *measures a* Λ-*valued observable* E if

$$\Pr\{\mathbf{x} \in \Delta \| \rho\} = \mathrm{Tr}[E(\Delta)\rho] \tag{2.4}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $\Delta \in \mathcal{B}(\Lambda)$. According to spectral theory, the observables A of $\mathcal{H}$ are in one-to-one correspondence with the $\mathbf{R}$-valued observables E^A for $(\mathcal{H}, \mathbf{R})$. From (2.3) and (2.4), an $\mathbf{R}$-valued apparatus measures an observable A if and only if it measures $\mathbf{R}$-valued observable E^A. Given a finite sequence $(A_1, \cdots, A_n)$ of mutually commuting observables, the simultaneous measurement of those observables is considered as a measurement of $\mathbf{R}^n$-valued observable $E^{(A_1,\ldots,A_n)}$ such that

$$E^{(A_1,\ldots,A_n)}(\Delta_1 \times \cdots \times \Delta_n) = E^{A_1}(\Delta) \cdots E^{A_n}(\Delta_n) \tag{2.5}$$

for any $\Delta_1, \ldots, \Delta_n \in \mathcal{B}(\mathbf{R})$.

A *probability operator valued measure (POVM)* for $(\mathcal{H}, \Lambda)$ is a positive operator valued measure F from $\mathcal{B}(\Lambda)$ to $\mathcal{L}(\mathcal{H})$ such that $F(\Lambda) = I$; for general theory we refer the reader to [4]. Since the correspondence $\Delta \mapsto \mathrm{Tr}[F(\Delta)\rho]$ is a probability measure, the Born statistical formula can be generalized formally to POVMs as follows. We say that a Λ-valued apparatus $\mathbf{A}(\mathbf{x})$ *measures a POVM* F for $(\mathcal{H}, \Lambda)$ if

$$\Pr\{\mathbf{x} \in \Delta \| \rho\} = \mathrm{Tr}[F(\Delta)\rho] \tag{2.6}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $\Delta \in \mathcal{B}(\Lambda)$. Conventional measurement theory is devoted to measurements of observables but modern theory extends the notion of measurements to measurements of POVMs [5, 6, 7, 8, 9]. We shall describe in the following requirement the essential feature of the modern approach.

Existence of probability operator valued measures (EPOVM): *For any apparatus* $\mathbf{A}(\mathbf{x})$*, there exists a POVM* $F_{\mathbf{x}}$ *uniquely such that* $\mathbf{A}(\mathbf{x})$ *measures* $F_{\mathbf{x}}$.

The EPOVM is justified by the following theorem proved in [7].

Theorem 2.1. *For any measurement theory* $(\mathcal{A}, \mathbf{M})$*, the EPOVM is equivalent to the 1MLPD.*

3. Collective measurement schemes

In order to provide an alternative definition of measurement schemes, we call a pair $(\mathcal{P}, \mathcal{R})$ as a *collective measurement scheme* if $\mathcal{P}$ is a function from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\Lambda)$ and $\mathcal{R}$ is a function from $\mathcal{B}(\Lambda) \times \mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\mathcal{H})$ satisfying

$$\sum_n (\mathcal{P}\rho)(\Delta_n)\mathcal{R}(\Delta_n, \rho) = \mathcal{R}(\Lambda, \rho) \tag{3.1}$$

for any countable Borel partition $\{\Delta_1, \Delta_2, \ldots\}$ of Λ and $\rho \in \mathcal{S}(\mathcal{H})$, where the sum is convergent in the trace norm. The function $\mathcal{R}$ is called the *collective reduction scheme*. Two collective measurement schemes $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{P}', \mathcal{R}')$ are said to be *equivalent*, in symbols $(\mathcal{P}, \mathcal{R}) \cong (\mathcal{P}', \mathcal{R}')$, if $\mathcal{P} = \mathcal{P}'$ and $\mathcal{R}(\Delta, \rho) = \mathcal{R}'(\Delta, \rho)$ for all $\Delta \in \mathcal{B}(\Lambda)$ with $(\mathcal{P}\rho)(\Delta) > 0$.

Theorem 3.1. *The relation*

$$(\mathcal{P}\rho)(\Delta)\mathcal{R}(\Delta, \rho) = \int_\Delta (\mathcal{Q}\rho)(x)\, d(\mathcal{P}\rho)(x) \tag{3.2}$$

where $\Delta \in \mathcal{B}(\Lambda)$ *and* $\rho \in \mathcal{S}(\mathcal{H})$*, sets up a one-to-one correspondence between the equivalence classes of measurement schemes* $(\mathcal{P}, \mathcal{Q})$ *and the equivalence classes of collective measurement schemes* $(\mathcal{P}, \mathcal{R})$.

Let $(\mathcal{P}, \mathcal{Q})$ be a measurement scheme for $(\mathcal{H}, \Lambda)$. The collective measurement scheme $(\mathcal{P}, \mathcal{R})$ defined by (3.2) up to equivalence is called the *collective measurement scheme induced by* $(\mathcal{P}, \mathcal{Q})$ and the function $\mathcal{R}$ is called the *collective reduction scheme induced by* $(\mathcal{P}, \mathcal{Q})$.

Let $(\mathcal{A}, \mathbf{M})$ be a measurement theory satisfying the 1MLPD. For any $\mathbf{A}(\mathbf{x}) \in \mathcal{A}$, define $\mathcal{R}_{\mathbf{x}}$ to be the collective reduction scheme induced by $(\mathcal{P}_{\mathbf{x}}, \mathcal{Q}_{\mathbf{x}})$. The functions $\mathcal{R}_{\mathbf{x}}$ is called the *collective reduction* of the apparatus $\mathbf{A}(\mathbf{x})$. The *collective output states* $\{\rho_{\{\mathbf{x}\in\Delta\}}\}_{\Delta\in\mathcal{B}(\Lambda)}$ of $\mathbf{A}(\mathbf{x})$ in ρ is defined by

$$\rho_{\{\mathbf{x}\in\Delta\}} = \mathcal{R}_{\mathbf{x}}(\Delta, \rho)$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $\Delta \in \mathcal{B}(\Lambda)$.

From the 1MLPD, for any $\Delta, \Delta' \in \mathcal{B}(\Lambda)$ and $\rho \in \mathcal{S}(\mathcal{H})$, we have

$$\begin{aligned}\Pr\{\mathbf{x} \in \Delta, \mathbf{y} \in \Delta' \| \rho\} &= \int_{\Delta} \mathrm{Tr}[F_{\mathbf{y}}(\Delta')\rho_{\{\mathbf{x}=x\}}] \Pr\{\mathbf{x} \in dx \| \rho\} \\ &= \mathrm{Tr}\left[F_{\mathbf{y}}(\Delta') \int_{\Delta} \rho_{\{\mathbf{x}=x\}} \Pr\{\mathbf{x} \in dx \| \rho\}\right] \\ &= \Pr\{\mathbf{y} \in \Delta' \| \rho_{\{\mathbf{x}\in\Delta\}}\} \Pr\{\mathbf{x} \in \Delta \| \rho\}.\end{aligned}$$

Thus, the collective output state is characterized by

$$\Pr\{\mathbf{x} \in \Delta, \mathbf{y} \in \Delta' \| \rho\} = \Pr\{\mathbf{y} \in \Delta' \| \rho_{\{\mathbf{x}\in\Delta\}}\} \Pr\{\mathbf{x} \in \Delta \| \rho\}, \tag{3.3}$$

or

$$\Pr\{\mathbf{y} \in \Delta' \| \rho_{\{\mathbf{x}\in\Delta\}}\} = \Pr\{\mathbf{y} \in \Delta' | \mathbf{x} \in \Delta \| \rho\}, \tag{3.4}$$

where the right hand side is the conditional distribution defined by

$$\Pr\{\mathbf{y} \in \Delta' | \mathbf{x} \in \Delta \| \rho\} = \frac{\Pr\{\mathbf{x} \in \Delta, \mathbf{y} \in \Delta' \| \rho\}}{\Pr\{\mathbf{x} \in \Delta \| \rho\}}. \tag{3.5}$$

Similarly, in the measurement theory satisfying $(n-1)$MLPD we have the following relation

$$\begin{aligned}&\Pr\{\mathbf{x}_1 \in \Delta, \ldots, \mathbf{x}_n \in \Delta_n \| \rho\} \\ &\quad = \Pr\{\mathbf{x}_n \in \Delta_n \| \rho_{\{\mathbf{x}_1\in\Delta_1,\ldots,\mathbf{x}_{n-1}\in\Delta_{n-1}\}}\} \times \cdots \qquad (3.6) \\ &\quad\quad \times \Pr\{\mathbf{x}_2 \in \Delta_2 \| \rho_{\{\mathbf{x}_1\in\Delta_1\}}\} \Pr\{\mathbf{x}_1 \in \Delta_1 \| \rho\} \qquad (3.7)\end{aligned}$$

for all $\Delta_j \in \mathcal{B}(\Lambda_j)$ and $j = 1, \cdots, n$, where

$$\rho_{\{\mathbf{x}_1\in\Delta_1,\ldots,\mathbf{x}_{n-1}\in\Delta_{n-1}\}} = (\cdots(\rho_{\{\mathbf{x}_1\in\Delta_1\}})\cdots)_{\{\mathbf{x}_{n-1}\in\Delta_{n-1}\}}.$$

4. Davies-Lewis postulate

In what follows, we shall introduce some mathematical terminology independent of particular measurement theory. A *superoperator* for $\mathcal{H}$ is a bounded linear transformation on the space $\tau c(\mathcal{H})$ of trace class operators on $\mathcal{H}$. A *dual superoperator* for $\mathcal{H}$ is an ultraweakly continuous linear transformation on the space $\mathcal{L}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. The *dual* of a superoperator L is the dual superoperator L^* defined by $\langle L^*A, \rho\rangle = \langle A, L\rho\rangle$ for all $A \in \mathcal{L}(\mathcal{H})$ and $\rho \in \tau c(\mathcal{H})$, where $\langle\cdot,\cdot\rangle$ stands for the duality pairing defined by $\langle A, \rho\rangle = \mathrm{Tr}[A\rho]$ for all $A \in \mathcal{L}(\mathcal{H})$ and $\rho \in \tau c(\mathcal{H})$. The correspondence $L \mapsto L^*$ is a one-to-one correspondence between the superoperators and the dual superoperators. A superoperator or a dual superoperator is called *positive* if it maps positive operators to positive operators. We denote by $\mathcal{L}(\tau c(\mathcal{H}))$ the space of superoperators and $\mathcal{P}(\tau c(\mathcal{H}))$ the space of positive superoperators. Positive contractive superoperators are called *operations* [10]. From Corollary 3.2.6 of [11], a positive superoperator T is an operation if and only if $0 \leq T^*(I) \leq I$, or equivalently $0 \leq \mathrm{Tr}[T(\rho)] \leq 1$ for all $\rho \in \mathcal{S}(\mathcal{H})$ [6, Lemma 2.2.2].

A *positive superoperator valued (PSV) measure* is a mapping $\mathcal{I}$ from $\mathcal{B}(\Lambda)$ to $\mathcal{P}(\tau c(\mathcal{H}))$ such that if $\{\Delta_1, \Delta_2, \ldots\}$ is a countable Borel partition of Λ, then we have

$$\mathcal{I}(\Lambda)\rho = \sum_n \mathcal{I}(\Delta_n)\rho$$

for any $\rho \in \tau c(\mathcal{H})$, where the sum is convergent in the trace norm. The PSV measure $\mathcal{I}$ is said to be *normalized* if it satisfies the further condition

$$\mathrm{Tr}[\mathcal{I}(\Lambda)\rho] = \mathrm{Tr}[\rho]$$

for any $\rho \in \tau c(\mathcal{H})$. Normalized PSV measures are called *instruments* [12, 6] for short. Accordingly, we will not use "instrument" for a synonym for "apparatus" but eventually it will turn out that an "instrument" corresponds to the statistical equivalence class of an "apparatus".

Let $\mathcal{I}$ be an instrument for $(\Lambda, \mathcal{H})$. Then, the function $\Delta \mapsto \mathrm{Tr}[\mathcal{I}(\Delta)]$ is a probability measure on $\mathcal{B}(\Lambda)$ for all $\rho \in \mathcal{S}(\mathcal{H})$. The relation

$$X(\Delta) = \mathcal{I}(\Delta)^* I \tag{4.1}$$

for all $\Delta \in \mathcal{B}(\Lambda)$, defines a POVM for $(\mathcal{H}, \Lambda)$, called the *POVM* of $\mathcal{I}$. The relation $T = \mathcal{I}(\Lambda)$ defines a trace preserving operation, called the *total operation* of $\mathcal{I}$.

A measurement theory $(\mathcal{A}, \mathbf{M})$ is said to satisfy the *Davies-Lewis postulate* if it satisfies the follows postulate.

Davies-Lewis postulate: *For any apparatus* $\mathbf{A}(\mathbf{x})$*, there is a normalized PSV measure* $\mathcal{I}_{\mathbf{x}}$ *satisfying the following relations for any* $\rho \in \mathcal{S}(\mathcal{H})$ *and Borel set* $\Delta \in \mathcal{B}(\Lambda)$*:*

(DL1) $\Pr\{\mathbf{x} \in \Delta \| \rho\} = \mathrm{Tr}[\mathcal{I}_{\mathbf{x}}(\Delta)\rho]$.

(DL2) $\rho_{\{\mathbf{x} \in \Delta\}} = \dfrac{\mathcal{I}_{\mathbf{x}}(\Delta)\rho}{\mathrm{Tr}[\mathcal{I}_{\mathbf{x}}(\Delta)\rho]}$.

It should be noted that the original Davies-Lewis formulation [12] does not mention the existence of state reduction $\mathcal{Q}_{\mathbf{x}}$ so that their theory is formulated for the apparatuses associated with the collective measurement scheme. However, we have proved in Theorem 3.1 the formulation based on the measurement scheme as the primitive notion is equivalent to the formulation based on the collective measurement scheme as the primitive notion. Thus, the above presentation of Davies-Lewis postulate is fully equivalent to their original formulation.

From Theorem 3.1, the normalized PSV measure $\mathcal{I}_{\mathbf{x}}$ determines the output state $\rho_{\{\mathbf{x}=x\}}$ uniquely up to equivalence by

(DL3) $\int_\Delta \rho_{\{\mathbf{x}=x\}} \operatorname{Tr}[d\mathcal{I}_{\mathbf{x}}(x)\rho] = \mathcal{I}_{\mathbf{x}}(\Delta)\rho$.

From (DL1), the Davies-Lewis postulate implies 1MLPD. Although the Davies-Lewis description of measurement is quite general, it is not clear by itself whether it is general enough to allow all the possible measurements. The following theorem shows indeed it is the case.

Theorem 4.1. *For any nonsuperselective measurement theory, the Davies-Lewis postulate is equivalent to the 2MLPD.*

A measurement theory $(\mathcal{A}, \mathbf{M})$ is called a *statistical measurement theory* if it is nonsuperselective and satisfies 2MLPD.

Theorem 4.2. *Every statistical measurement theory satisfies nMLPD for all positive integer n.*

In the rest of this paper, we shall take it for granted that every measurement theory consistent with quantum mechanics should be a statistical measurement theory.

5. Measurements of observables

In the preceding section, we have shown that in any statistical measurement theory the statistical equivalence class of an apparatus is represented by an instrument which determines the statistical property of the apparatus, although it depends on the particular measurement theory whether a given instrument (in mathematics) has the corresponding apparatus (in physics).

In what follows, we shall explore mathematical properties of instruments (normalized PSV measures) independent of particular measurement theory. An instrument $\mathcal{I}$ for $(\Lambda, \mathcal{H})$ is said to be *decomposable* if $\mathcal{I}(\Delta)^*A = X(\Delta)T^*(A)$ for all $\Delta \in \mathcal{B}(\Lambda)$ and $A \in \mathcal{L}(\mathcal{H})$, where X is the POVM of $\mathcal{I}$ and T the total operation. For a given Λ-valued observable E for $(\Lambda, \mathcal{H})$, an instrument $\mathcal{I}$ is said to be *E-compatible* if the POVM of $\mathcal{I}$ is E. i.e., $\mathcal{I}^*(\Delta)I = E(\Delta)$ for all $\Delta \in \mathcal{B}(\Lambda)$; such an instrument is also called *observable measuring.* For an observable A, an instrument is called *A-compatible* if it is E^A-compatible. The following theorem shows in particular that every observable measuring instrument is decomposable (see [9, Proposition 4.3] for the case of completely positive instruments).

Theorem 5.1. *Let E be a Λ-valued observable of $\mathcal{H}$. Let $\mathcal{I}$ be an E-compatible instrument and T its total operation. Then, we have the following statements.*

(i) *For any $\Delta \in \mathcal{B}(\Lambda)$ and $\rho \in \tau c(\mathcal{H})$, we have*

$$\mathcal{I}(\Delta)\rho = T[E(\Delta)\rho] = T[\rho E(\Delta)] = T[E(\Delta)\rho E(\Delta)]. \tag{5.1}$$

(ii) *For any $\Delta \in \mathcal{B}(\Lambda)$ and $B \in \mathcal{L}(\mathcal{H})$, we have*

$$\mathcal{I}(\Delta)^*B = E(\Delta)T^*(B) = T^*(B)E(\Delta) = E(\Delta)T^*(B)E(\Delta). \tag{5.2}$$

It follows from (5.2) that the range of T^* is included in the commutant of the range of E, i.e., $T^*[\mathcal{L}(\mathcal{H})] \subseteq E[\mathcal{B}(\Lambda)]'$. An operation L is called *E-compatible* if $L^*[\mathcal{L}(\mathcal{H})] \subseteq E[\mathcal{B}(\Lambda)]'$. For an observable A, the operation L is called *A-compatible* if it is E^A-compatible.

All the E-compatible instruments are determined as follows.

Theorem 5.2. *Let E be an Λ-valued observable of $\mathcal{H}$. The relation*

$$\mathcal{I}(\Delta)\rho = T[E(\Delta)\rho] \tag{5.3}$$

for all $\Delta \in \mathcal{B}(\Lambda)$ and $\rho \in \tau c(\mathcal{H})$ sets up a one-to-one correspondence between the E-compatible instruments $\mathcal{I}$ and the E-compatible trace preserving operations T.

From the above theorem, in every statistical measurement theory we have the following: *For any apparatus* $\mathbf{A}(\mathbf{x})$ *measuring a Λ-valued observable E, there is an E-compatible trace preserving operation T such that the statistical property of* $\mathbf{A}(\mathbf{x})$ *is represented as follows.*

$$\text{output distribution:} \qquad \Pr\{\mathrm{x} \in \Delta \| \rho\} = \mathrm{Tr}[E(\Delta)\rho] \tag{5.4}$$

$$\text{collective output state:} \qquad \rho_{\{\mathrm{x}\in\Delta\}} = \frac{T[E(\Delta)\rho]}{\mathrm{Tr}[E(\Delta)\rho]} \tag{5.5}$$

6. Measurements of nondegenerate observables

Let E be a Λ-valued observable for $\mathcal{H}$. We say that E is *nondegenerate* if the commutant of E is abelian. The spectral measure E^A of an observable A of $\mathcal{H}$ is nondegenerate if and only if A has nondegenerate spectral. Two Borel families $\{\varrho_x\}_{x\in\Lambda}$ and $\{\varrho'_x\}_{x\in\Lambda}$ of density operators are said to be *E-equivalent* if they differ only on an E-null set, i.e.,

$$E\{x \in \Lambda | \ \varrho_x \neq \varrho'_x\} = 0.$$

Theorem 6.1. *Let E be a nondegenerate Λ-valued observable of $\mathcal{H}$. The Bochner integral formula*

$$T\rho = \int_\Lambda \varrho_x \,\mathrm{Tr}[\rho \, dE(x)] \tag{6.1}$$

for all $\rho \in \tau c(\mathcal{H})$ sets up a one-to-one correspondence between the E-compatible trace preserving operations T and the E-equivalence classes of the Borel families $\{\varrho_x\}_{x\in\Lambda}$ of density operators indexed by Λ.

Let us consider a statistical measurement theory $(\mathcal{A}, \mathbf{M}(\mathbf{x}))$. Let $\rho_{\{\mathrm{x}=x\}}$ be the output state for the input state ρ of an apparatus $\mathbf{A}(\mathbf{x})$ measuring a Λ-valued observable E. Then, the apparatus $\mathbf{A}(\mathbf{x})$ has the instrument $\mathcal{I}_{\mathrm{x}}$ and the E-compatible operation T satisfying (5.3) and (5.5). By Theorem 6.1 there is a Borel family $\{\varrho_x\}_{x\in\Lambda}$ of density operators satisfying (6.1), so that we have

$$\int_\Delta \rho_{\{\mathbf{x}=x\}}\mathrm{Tr}[d\mathcal{I}_{\mathbf{x}}(x)\rho] = \int_\Delta \varrho_x \mathrm{Tr}[d\mathcal{I}_{\mathbf{x}}(x)\rho]$$

for all $\Delta \in \mathcal{B}(\Lambda)$. It follows that the output state for the output $\mathbf{x} = x$ is given by

$$\rho_{\{\mathbf{x}=x\}} = \varrho_x \tag{6.2}$$

up to the output probability one.

From the above theorem, in any statistical measurement theory we conclude the following: *For any apparatus* $\mathbf{A}(\mathbf{x})$ *measuring an* Λ*-valued observable* E*, there is a Borel family* $\{\varrho_x\}_{x\in\Lambda}$ *of density operators uniquely up to* E*-equivalence such that the statistical property of* $\mathbf{A}(\mathbf{x})$ *is represented as follows.*

$$\text{output distribution:} \quad \Pr\{\mathbf{x} \in \Delta \| \rho\} = \mathrm{Tr}[E(\Delta)\rho] \tag{6.3}$$

$$\text{output state:} \quad \rho_{\{\mathbf{x}=x\}} = \varrho_x \tag{6.4}$$

It follows that the problem of determining all the possible quantum state reductions arising in the measurement of a nondegenerate Λ-valued observable E is reduced to the problem as to what Borel family $\{\varrho_x\}$ of states can be obtained from the measurement of E. In order to obtain the answer to this question, in the next section we shall consider indirect measurement models and ask what families of states can be obtained from those models.

7. Indirect measurement models

An *indirect measurement model* for $(\Lambda, \mathcal{H})$ is defined to be a 4-tuple $(\mathcal{K}, \sigma, U, E)$ consisting of a separable Hilbert space $\mathcal{K}$, a density operator σ on $\mathcal{K}$, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$, and a Λ-valued observable E of $\mathcal{K}$. For a given indirect measurement model $(\mathcal{K}, \sigma, U, E)$ the relation

$$\mathcal{I}(\Delta)\rho = \mathrm{Tr}_{\mathcal{K}}[(I \otimes E(\Delta))U(\rho \otimes \sigma)U^*], \tag{7.1}$$

where $\Delta \in \mathcal{B}(\Lambda)$ and $\rho \in \tau c(\mathcal{H})$, defines an instrument $\mathcal{I}$ for $(\Lambda, \mathcal{H})$, which is called the *instrument determined by* $(\mathcal{K}, \sigma, U, E)$.

Let $\mathbf{S}$ be the quantum system described by the Hilbert space $\mathcal{H}$. If an apparatus $\mathbf{A}(\mathbf{x})$ for $(\mathcal{H}, \Lambda)$ has the the indirect measurement model $(\mathcal{K}, \sigma, U, E)$, the process of measurement on $\mathbf{S}$ using the apparatus $\mathbf{A}(\mathbf{x})$ is described as follows [13, 14]. The measurement is carried out by the interaction between the measured system $\mathbf{S}$ and the apparatus $\mathbf{A}(\mathbf{x})$ for the finite time interval from the time t to the time $t + \Delta t$. We assume that the measured system is free from the apparatus $\mathbf{A}(\mathbf{x})$ after the time $t + \Delta t$. The time t is called the time of measurement. The time $t + \Delta t$ is called the time just after measurement. We define the probe system of the apparatus $\mathbf{A}(\mathbf{x})$ to be the smallest subsystem $\mathbf{P}$ of the apparatus $\mathbf{A}(\mathbf{x})$ such that the composite system $\mathbf{S}+\mathbf{P}$ is isolated from t to $t+\Delta t$. We assume that the probe $\mathbf{P}$ is a quantum system described by the Hilbert space $\mathcal{K}$.

At the time t, the system $\mathbf{S}$ is supposed to be in the input state ρ and the probe $\mathbf{P}$ is in the fixed state σ. The time evolution of the composite system $\mathbf{S}+\mathbf{P}$ from t to $t+\Delta t$ is described by the unitary operator U on $\mathcal{H}\otimes\mathcal{K}$. Hence, at the time $t+\Delta t$ the composite system $\mathbf{S}+\mathbf{P}$ is in the state $U(\rho\otimes\sigma)U^*$. The outcome of this measurement is obtained by the measurement of the Λ-valued observable E of $\mathbf{P}$ at the time $t+\Delta t$ locally at the system $\mathbf{P}$. Thus, the output distribution of the apparatus $\mathbf{A}(\mathbf{x})$ is given by

$$\Pr\{\mathbf{x}\in\Delta\|\rho\} = \mathrm{Tr}[(I\otimes E(\Delta))U(\rho\otimes\sigma)U^*] \tag{7.2}$$

for all $\Delta\in\mathcal{B}(\Lambda)$. The apparatus $\mathbf{A}(\mathbf{e})$ to measure E is supposed to be a part of the apparatus $\mathbf{A}(\mathbf{x})$ comprising of the subsequent stages later than $\mathbf{P}$, so that we may assume $\mathbf{A}(\mathbf{x})=\mathbf{P}+\mathbf{A}(\mathbf{e})$ and that the output variables $\mathbf{x}$ and $\mathbf{e}$ might correspond to the same macroscopic variable, though we need to distinguish them by different symbols.

In order to determine the output state of $\mathbf{A}(\mathbf{x})$, suppose that at the time $t+\Delta t$, the above measurement on $\mathbf{S}$ using $\mathbf{A}(\mathbf{x})$ is followed immediately by the measurement of an observable B of $\mathbf{S}$ using another apparatus $\mathbf{A}(\mathbf{b})$. Then, the joint probability distribution of the outcomes $\mathbf{x}$ and $\mathbf{b}$ of this successive measurement coincides with the joint probability distribution of the outcomes $\mathbf{e}$ and $\mathbf{b}$ of the E-measurement on $\mathbf{P}$ and the B-measurement on $\mathbf{S}$ in $U(\rho\otimes\sigma)U^*$, i.e.,

$$\Pr\{\mathbf{x}\in\Delta,\mathbf{b}\in\Delta'\|\rho\} = \Pr\{\mathbf{e}\in\Delta,\mathbf{b}\in\Delta'\|U(\rho\otimes\sigma)U^*\}. \tag{7.3}$$

Since the E-measurement is carried out locally at the system $\mathbf{P}$, the joint probability distribution of their outcomes is given by

$$\begin{aligned}
&\Pr\{\mathbf{e}\in\Delta,\mathbf{b}\in\Delta'\|U(\rho\otimes\sigma)U^*\}\\
&\quad= \mathrm{Tr}[(E^B(\Delta')\otimes E(\Delta))U(\rho\otimes\sigma)U^*]\\
&\quad= \mathrm{Tr}[E^B(\Delta')\mathrm{Tr}_{\mathcal{K}}[(I\otimes E(\Delta))U(\rho\otimes\sigma)U^*]],
\end{aligned} \tag{7.4}$$

where $\mathrm{Tr}_{\mathcal{K}}$ stands for the partial trace over $\mathcal{K}$. For the detailed justification of the above formula, we refer the reader to [15, 16, 17, 14]. From (3.3), the (collective) output state $\rho_{\{\mathbf{x}\in\Delta\}}$ satisfies

$$\Pr\{\mathbf{x}\in\Delta,\mathbf{b}\in\Delta'\|\rho\} = \mathrm{Tr}[E^B(\Delta')\Pr\{\mathbf{x}\in\Delta\|\rho\}\rho_{\{\mathbf{x}\in\Delta\}}]. \tag{7.5}$$

From (7.2)–(7.5), we determine the output state as

$$\rho_{\{\mathbf{x}\in\Delta\}} = \frac{\mathrm{Tr}_{\mathcal{K}}[(I\otimes E(\Delta))U(\rho\otimes\sigma)U^*]}{\mathrm{Tr}[(I\otimes E(\Delta))U(\rho\otimes\sigma)U^*]}. \tag{7.6}$$

We have, thus, demonstrated that if the apparatus $\mathbf{A}(\mathbf{x})$ has the indirect measurement model $(\mathcal{K}, \sigma, U, E)$, then $\mathbf{A}(\mathbf{x})$ has the instrument $\mathcal{I}_{\mathbf{x}}$ defined by

$$\mathcal{I}_{\mathbf{x}}(\Delta)\rho = \mathrm{Tr}_{\mathcal{K}}[(I \otimes E(\Delta))U(\rho \otimes \sigma)U^*], \qquad (7.7)$$

where $\Delta \in \mathcal{B}(\Lambda)$ and $\rho \in \tau c(\mathcal{H})$, so that the statistical property of $\mathbf{A}(\mathbf{x})$ is described by $\mathcal{I}_{\mathbf{x}}$ with relations (DL1) and (DL2). The above instrument $\mathcal{I}_{\mathbf{x}}$ is called the *instrument of* $\mathbf{A}(\mathbf{x})$ and is actually the instrument $\mathcal{I}$ of $(\mathcal{K}, \sigma, U, E)$ defined by (7.1).

Now, we consider the following hypothesis.

Indirect measurability hypothesis: *For any indirect measurement model $(\mathcal{K}, \sigma, U, E)$, there is an apparatus $\mathbf{A}(\mathbf{x})$ with the instrument $\mathcal{I}_{\mathbf{x}}$ defined by (7.7).*

In general, an instrument $\mathcal{I}$ is said to be *realized* by an indirect measurement model $(\mathcal{K}, \sigma, U, E)$ if (7.1) holds for any $\rho \in \tau c(\mathcal{H})$. In this case, the instrument $\mathcal{I}$ is called *unitarily realizable.* Under the indirect measurability hypothesis, every unitarily realizable instrument represents the statistical property of an apparatus.

In the sequel, a statistical measurement theory $(\mathcal{A}, \mathbf{M})$ is called a *standard measurement theory*, if it satisfies the indirect measurability hypothesis. It is natural to consider that any standard measurement theory is consistent with the standard formulation of quantum mechanics by the following argument. In the standard formulation of quantum mechanics, the system $\mathbf{P}$ can be prepared in any σ with given finite accuracy and the unitary operator U can be realized as a time evolution with given finite accuracy in principle, otherwise a certain superselection rule would be required. Thus, it suffices to show that the Λ-valued observable can be measured by an apparatus. We have already argue that the statistical measurement theory is consistent with the standard formulation of quantum mechanics. Thus, for any observable A we have an apparatus $\mathbf{A}(\mathbf{a})$ to measure A. Since Λ is a standard Borel space, there is a Borel subset Λ' of $\mathbf{R}$ Borel isomorphic with Λ. It follows that there is a Borel function $\Lambda \to \mathbf{R}$ and an observable A such that $E(\Delta) = E^A(f^{-1}(\Delta))$ for all $\Delta \in \mathcal{B}(\Lambda)$. Thus, the apparatus $\mathbf{A}(\mathbf{z})$ measuring E is constructed by an apparatus $\mathbf{A}(\mathbf{x})$ measuring A with output processing $\mathbf{z} = f(\mathbf{x})$.

8. Complete positivity

Let $\mathcal{D} = \tau c(\mathcal{H})$ or $\mathcal{D} = \mathcal{L}(\mathcal{H})$. A linear transformation L on $\mathcal{D}$ is called *completely positive (CP)* iff for any finite sequences of operators $A_1, \ldots, A_n \in \mathcal{D}$ and vectors $\xi_1, \ldots, \xi_n \in \mathcal{H}$ we have

$$\sum_{ij} \langle \xi_i | L(A_i^* A_j) | \xi_j \rangle \geq 0.$$

The above condition is equivalent to that $L \otimes I$ maps positive operators in the algebraic tensor product $\mathcal{D} \otimes \mathcal{L}(\mathcal{K})$ to positive operators in $\mathcal{D} \otimes \mathcal{L}(\mathcal{K})$ for any Hilbert space $\mathcal{K}$. Obviously, every CP superoperators are positive. A superoperator is CP if and only if its dual superoperator is CP. An instrument $\mathcal{I}$ is called *completely positive (CP)* if every operation $\mathcal{I}(\Delta)$ is CP. It can be seen easily from (7.1) that unitarily realizable instruments are CP. Conversely, the following theorem, proved in [18, 9], asserts that every CP instrument is unitarily realizable.

Theorem 8.1. *For any CP instrument $\mathcal{I}$ for $(\Lambda, \mathcal{H})$, there is a separable Hilbert space $\mathcal{K}$, a unit vector Φ in $\mathcal{K}$, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$, and a Λ-valued observable E of $\mathcal{K}$ satisfying the relation*

$$\mathcal{I}(\Delta)\rho = \mathrm{Tr}_{\mathcal{K}}[(I \otimes E(\Delta))U(\rho \otimes |\Phi\rangle\langle\Phi|)U^*]$$

for all $\Delta \in \mathcal{B}(\Lambda)$ and $\rho \in \tau c(\mathcal{H})$.

The following theorem shows that the complete positivity of observable measuring instruments is determined by their total operations.

Theorem 8.2. *Let E be a Λ-valued observable. Then, an E-compatible instrument is CP if and only if its total operation is CP.*

From the above theorem, in any standard measurement theory we conclude the following statement [9]: *The statistical equivalence classes of apparatuses* $\mathbf{A}(\mathbf{x})$ *measuring a Λ-valued observable E with indirect measurement models are in one-to-one correspondence with the E-compatible trace preserving CP operations, where the statistical property is represented by (5.4) and (5.5).*

For the case of nondegenerate observables, we have the following simple characterizations.

Theorem 8.3. *Let E be a nondegenerate Λ-valued observable. Then, every E-compatible operation is completely positive. Every E-compatible instrument is completely positive.*

From the above theorem and Theorem 8.1, in the statistical measurement theory we conclude: *Every apparatus measuring a nondegenerate (Λ-valued) observable is statistically equivalent to the one having an indirect measurement model.*

Every Borel family $\{\varrho_x\}$ of density operators indexed by Λ defines an E-compatible trace preserving operation by Theorem 6.1, and it is automatically completely positive so that it is realized by an indirect measurement model. Thus, we conclude the following: *The statistical equivalence classes of apparatuses* $\mathbf{A}(\mathbf{x})$ *measuring a nondegenerate Λ-valued observable E are in one-to-one correspondence with the Borel family $\{\varrho_x\}$ of density operators indexed by Λ, where the statistical property is represented by (6.3) and (6.4).*

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