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# Quantum Logical Gate Based on Fock Space

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**Abstracts:**

In usual computer, there exists a restriction of computational speed because of irreversibility of logical gate. In order to avoid this demerit, Fredkin and Toffoli [3] proposed a conservative logical gate. Based on their work, Milburn [4] introduced a physical model of reversible quantum logical gate using beam splittings and a Kerr medium. This model is called FTM (Fredkin - Toffoli - Milburn gate). FTM gate was described by the quantum channel and the efficiency of information transmission of the FTM gate was discussed in [10]. FTM gate is using a photon number state as an input state for control gate. The photon number state might be difficult to realize physically. In this paper, we introduced a new unitary operator related to the Kerr device on symmetric Fock space in order to avoid this difficulty.

**Key words:** quantum logical gate, channels, beam splittings, FTM gate, Fock space

## 1. Quantum channels

Let $(\mathbf{B}(\mathcal{H}_1), \mathfrak{S}(\mathcal{H}_1))$ and $(\mathbf{B}(\mathcal{H}_2), \mathfrak{S}(\mathcal{H}_2))$ be input and output systems, respectively, where $\mathbf{B}(\mathcal{H}_k)$ is the set of all bounded linear operators on a separable Hilbert space $\mathcal{H}_k$ and $\mathfrak{S}(\mathcal{H}_k)$ is the set of all density operators on $\mathcal{H}_k$ $(k = 1, 2)$. Quantum channel $\Lambda^*$ is a mapping from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$. $\Lambda^*$ is linear if $\Lambda^*(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda\Lambda^*(\rho_1) + (1-\lambda)\Lambda^*(\rho_2)$ holds for any $\rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H}_1)$

and any $\lambda \in [0,1]$. $\Lambda^*$ is completely positive (C.P.) if $\Lambda^*$ is linear and its dual $\Lambda : \mathbf{B}(\mathcal{H}_2) \to \mathbf{B}(\mathcal{H}_1)$ satisfies

$$\sum_{i,j=1}^{n} A_i^* \Lambda(\overline{A}_i^* \overline{A}_j) A_j \geq 0$$

for any $n \in \mathbf{N}$, any $\{\overline{A}_i\} \subset \mathbf{B}(\mathcal{H}_2)$ and any $\{A_i\} \subset \mathbf{B}(\mathcal{H}_1)$, where the dual map $\Lambda$ of $\Lambda^*$ is defined by

$$tr\Lambda^*(\rho)B = tr\rho\Lambda(B), \quad \forall\rho \in \mathfrak{S}(\mathcal{H}_1), \quad \forall B \in \mathbf{B}(\mathcal{H}_2). \tag{1.1}$$

Almost all physical transformation can be described by the CP channel [5], [7], [8]

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Hilbert spaces expressing noise and loss systems, respectively. Quantum communication process including the influence of noise and loss is denoted by the following scheme [6]: Let $\rho$ be an input state in $\mathfrak{S}(\mathcal{H}_1)$, $\xi$ be a noise state in $\mathfrak{S}(\mathcal{K}_1)$.

$$\begin{array}{ccc} \mathfrak{S}(\mathcal{H}_1) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}_2) \\ \gamma^* \downarrow & & \uparrow a^* \\ \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1) & \xrightarrow[\Pi^*]{} & \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2) \end{array}$$

The above maps $\gamma^*$, $a^*$ are given as

$$\gamma^*(\rho) = \rho \otimes \xi, \quad \rho \in \mathfrak{S}(\mathcal{H}_1), \tag{1.2}$$

$$a^*(\sigma) = tr_{\mathcal{K}_2}\sigma, \quad \sigma \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2). \tag{1.3}$$

The map $\Pi^*$ is a channel from $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$to $\mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ determined by physical properties of the device transmitting information. Hence the channel for the above process is given by

$$\Lambda^*(\rho) \equiv tr_{\mathcal{K}_2}\Pi^*(\rho \otimes \xi) = (a^* \circ \Pi^* \circ \gamma^*)(\rho) \tag{1.4}$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$. Based on this scheme, the noisy quantum channel [9] are constructed as follows:

Noisy quantum channel $\Lambda^*$ with a noise state $\xi$ is defined by

$$\Lambda^*(\rho) \equiv tr_{\mathcal{K}_2}\Pi^*(\rho \otimes \xi) = tr_{\mathcal{K}_2}V(\rho \otimes \xi)V^*, \tag{1.5}$$

where $\xi = |m_1\rangle\langle m_1|$ is the $m_1$ photon number state in $\mathfrak{S}(\mathcal{K}_1)$ and $V$ is a mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ denoted by

$$V(|n_1\rangle \otimes |m_1\rangle) = \sum_{j}^{n_1+m_1} C_j^{n_1,m_1} |j\rangle \otimes |n_1+m_1-j\rangle,$$

$$C_j^{n_1,m_1} = \sum_{r=L}^{K} (-1)^{n_1+j-r} \frac{\sqrt{n_1!m_1!j!(n_1+m_1-j)!}}{r!(n_1-j)!(j-r)!(m_1-j+r)!} \alpha^{m_1-j+2r} \left(-\bar{\beta}\right)^{n_1+j-2r} \tag{1.6}$$

$K$ and $L$ are constants given by $K = \min\{n_1, j\}$, $L = \max\{m_1 - j, 0\}$. In particular for the coherent input state $\rho = |\theta\rangle\langle\theta| \otimes |\kappa\rangle\langle\kappa| \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$, we obtain the output state of $\Pi^*$ by

$$\Pi^*\left(|\theta\rangle\langle\theta| \otimes |\kappa\rangle\langle\kappa|\right) = |\alpha\theta + \beta\kappa\rangle\langle\alpha\theta + \beta\kappa| \otimes \left|-\bar{\beta}\theta + \alpha\kappa\right\rangle\left\langle -\bar{\beta}\theta + \alpha\kappa\right|,$$

where $\Pi^*$ is called a generalized beam splitting. When the noise $\xi_0 = |0\rangle\langle 0|$ is given by the vacuum state, $\Lambda_0^*$ is called an attenuation channel [5] and $\mathcal{E}_0^*$ (or $\Pi_0^*$) is called a beam splitting. Based on liftings, the beam splitting was studied by Accardi - Ohya [1] and Fichtner - Freudenberg - Libsher [2].

## 2. Quantum logical gate on symmetric Fock space

Recently, we reformulate a quantum channel for the FTM gate and we rigorously study the conservation of information for FTM gate [10]. However, it might be difficult to realize the photon number state $|n\rangle\langle n|$ for the input of the Kerr medium physically.

In this section, we reformulate beam splittings on symmetric Fock space and we introduce a new operator on this space instead of the Kerr medium. We discuss the mathematical formulation of quantum logical gate by means of beam splittings and the new operator.

Let $G$ be a complete separable metric space and $\mathcal{G}$ be a Borel $\sigma$-algebra of $G$. $v$ is called a locally finite diffuse measure on the measurable space $(G, \mathcal{G})$ if $v$ satisfies the conditions (1) $v(K) < \infty$ for bounded $K \in \mathcal{G}$ and (2) $v(\{x\}) = 0$ for any $x \in G$. We denote the set of all finite integer - valued measures $\varphi$ on $(G, \mathcal{G})$ by $M$. For a set $K \in \mathcal{G}$ and a nutural number $n \in \mathbb{N}$, we put the set of $\varphi$ satisfying $\varphi(K) = n$ as

$$M_{K,n} \equiv \{\varphi \in M; \varphi(K) = n\}.$$

Let $\mathfrak{M}$ be a $\sigma$-algebra generated by $M_{K,n}$. $F$ is the $\sigma$-finite measure on $(M, \mathfrak{M})$ defined by

$$F(Y) \equiv 1_Y(\varphi_0) + \sum_{n=1} \frac{1}{n!} \int_M 1_Y \left( \sum_{j=1}^{n} \delta_{x_j} \right) v^n (dx_1 \cdots dx_n),$$

where $1_Y$ is the characteristic function of a set $Y$, $\varphi_0$ is an empty configulation in $M$ and $\delta_{x_j}$ is a Dirac measure in $x_j$. $\mathcal{M} \equiv L^2(M, \mathfrak{M}, F)$ is called a (symmetric) Fock space. We define an exponetal vector $\exp_g : M \to \mathbb{C}$ generated by a given function $g : G \to \mathbb{C}$ such that

$$\exp_g(\varphi) \equiv \begin{cases} 1 & (\varphi = \varphi_0), \\ \prod_{x \in \varphi} g(x) & (\varphi \neq \varphi_0), \end{cases} \quad (\varphi \in M).$$

## 2.1. Generalized beam splittings on Fock space

Let $\alpha, \beta$ be measurable mappings from $G$ to $\mathbb{C}$ satisfying $\bar{\alpha}$

$$|\alpha(x)|^2 + |\beta(x)|^2 = 1, \quad x \in G.$$

We intoduce an unitary operator $V_{\alpha,\beta} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ defined b

$$\begin{aligned}(V_{\alpha,\beta}\Phi)(\varphi_1, \varphi_2) \equiv & \sum_{\hat{\varphi}_1 \leq \varphi_1} \sum_{\hat{\varphi}_2 \leq \varphi_2} \exp_\alpha(\hat{\varphi}_1) \exp_\beta(\varphi_1 - \hat{\varphi}_1) \exp_{-\bar{\beta}}(\hat{\varphi}_2) \exp_{\bar{\alpha}}(\varphi_2 - \hat{\varphi}_2) \\ & \times \Phi(\hat{\varphi}_1 + \hat{\varphi}_2, \varphi_1 + \varphi_2 - \hat{\varphi}_1 - \hat{\varphi}_2)\end{aligned}$$

for $\Phi \in \mathcal{M} \otimes \mathcal{M}$ and $\varphi_1, \varphi_2 \in M$. Let $\mathcal{A} \equiv \mathbb{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{M}$ and $\mathfrak{S}(\mathcal{A})$ be the set of all normal states on $\mathcal{A}$. $\mathcal{E}_{\alpha,\beta} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by

$$\mathcal{E}_{\alpha,\beta}(C) \equiv V_{\alpha,\beta}^* C V_{\alpha,\beta}, \quad \forall C \in \mathcal{A} \otimes \mathcal{A}$$

is the lifting in the sense of Accardi and Ohya [1] and the dual map $\mathcal{E}_{\alpha,\beta}^*$ of $\mathcal{E}_{\alpha,\beta}$ given by

$$\mathcal{E}_{\alpha,\beta}^*(\omega)(\bullet) \equiv \omega(\mathcal{E}_{\alpha,\beta}(\bullet)), \quad \forall \omega \in \mathfrak{S}(\mathcal{A} \otimes \mathcal{A})$$

is the CP channel from $\mathfrak{S}(\mathcal{A} \otimes \mathcal{A})$ to $\mathfrak{S}(\mathcal{A} \otimes \mathcal{A})$. Using the exponetial vectors, one can denote a coherent state $\theta^f$ ßby

$$\theta^f(A) \equiv \langle \exp_f, A \exp_f \rangle e^{-\|f\|^2}, \quad \forall f \in L^2(G, \nu), \forall A \in \mathcal{A}.$$

In particular, for the input coherent states $\eta_0 \otimes \omega_0 = \theta^f \otimes \theta^g$, two output states $\omega_1(\bullet) \equiv \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha,\beta}((\bullet) \otimes I))$ and $\eta_1(\bullet) \equiv \eta_0 \otimes \omega_0 (\mathcal{E}_{\alpha,\beta}(I \otimes (\bullet)))$ are obtained by

$$\omega_1 = \theta^{\alpha f + \beta g}, \quad \eta_1 = \theta^{-\bar{\beta} f + \bar{\alpha} g}.$$

$\mathcal{E}^*_{\alpha,\beta}$ is called a generalized beam splitting on Fock space because it also hold the same properties satisfied by the generated beam splitting $\Pi^*$in Section 1.

Now we introduce a self-adjoint unitary operator $\tilde{U}$, which denotes a new device instead of the Kerr medium, defined by

$$\tilde{U}(\Phi)(\varphi_1, \varphi_2) \equiv (-1)^{|\varphi_1||\varphi_2|} \Phi(\varphi_1, \varphi_2)$$

for $\Phi \in \mathcal{M} \otimes \mathcal{M}$ and $\varphi_1, \varphi_2 \in G$, where $|\varphi_k| \equiv \varphi_k(G) \quad (k = 1,2)$. For the input state $\omega_1 \otimes \kappa \equiv \theta^f \otimes \frac{1}{\|\psi\|^2} \langle \psi, \bullet\psi \rangle$, the output state $\omega_2$ of new device is

$$\omega_2(A) \equiv \omega_1 \otimes \kappa \left( \tilde{U}(A \otimes I) \tilde{U} \right) = \frac{1}{\|\psi\|^2} \int_M F(d\varphi) |\psi(\varphi)|^2 \theta^{(-1)^{|\varphi|^2} f}(A)$$

for any $A \in \mathcal{A}$, $\psi \in \mathcal{M}$ $(\psi \neq 0)$ and $f \in L^2(G, \nu)$. If $\kappa$ is given by the vacuum state $\theta^0$, then the output state $\omega_2$ is equals to $\omega_1$ and if $\kappa$ is given by one particle state, that is, $\kappa = \frac{1}{\|\psi\|^2} \langle \psi, \bullet\psi \rangle$ with $\psi \restriction_{M_1^c}$(where $M_1$ is the set of one-particle states), then $\omega_2$ is obtained by $\theta^{-f}$. Let $M_o$ (resp. $M_e$) be the set of $\varphi \in M$ which satisfies that $|\varphi|$ is odd (resp. even) and $M$ be the union of $M_o$ and $M_e$. The output states $\omega_2$ of the new device is written by

$$\omega_2(A) = \lambda_1 \theta^{-f}(A) + \lambda_2 \theta^f(A) \quad \forall A \in \mathcal{A},$$

where $\lambda_1$ and $\lambda_2$ are given by

$$\begin{cases} \lambda_1 = \frac{1}{\|\psi\|^2} \int_{M_o} F(d\varphi) |\psi(\varphi)|^2, \\ \lambda_2 = \frac{1}{\|\psi\|^2} \int_{M_e} F(d\varphi) |\psi(\varphi)|^2. \end{cases}$$

Two output states $\omega_3(\bullet) \equiv \omega_2 \otimes \eta_2 \left( \mathcal{E}_{\alpha_2,\beta_2}((\bullet) \otimes I) \right)$ and $\eta_3(\bullet) \equiv \omega_2 \otimes \eta_2 \left( \mathcal{E}_{\alpha_2,\beta_2}(I \otimes (\bullet)) \right)$ of the total logical gate including two beam splittings $\mathcal{E}^*_{\alpha_k,\beta_k}$ with $\left( |\alpha_k|^2 + |\beta_k|^2 = 1 \right)$ $(k = 1.2)$ and the new device instead of Kerr medium are obtained by

$$\begin{aligned} \omega_3 &= \lambda_1 \theta^{\alpha_2(-(\alpha_1 f + \beta_1 g)) + \beta_2\left(-\bar{\beta}_1 f + \bar{\alpha}_1 g\right)} + \lambda_2 \theta^{\alpha_2(\alpha_1 f + \beta_1 g) + \beta_2\left(-\bar{\beta}_1 f + \bar{\alpha}_1 g\right)}, \\ \eta_3 &= \lambda_1 \theta^{-\bar{\beta}_2(-(\alpha_1 f + \beta_1 g)) + \bar{\alpha}_2\left(-\bar{\beta}_1 f + \bar{\alpha}_1 g\right)} + \lambda_2 \theta^{-\bar{\beta}_2(\alpha_1 f + \beta_1 g) + \bar{\alpha}_2\left(-\bar{\beta}_1 f + \bar{\alpha}_1 g\right)}, \end{aligned}$$

where $\omega_2 = \lambda_1\theta^{-(\alpha_1 f+\beta_1 g)} + \lambda_2\theta^{\alpha_1 f+\beta_1 g}$ and $\eta_2 = \eta_1 = \theta^{-\bar{\beta}_1 f+\bar{\alpha}_1 g}$.

Based on the above settings, we could show that new logical gate performs the complete truth table. The furtherdevelopment of our study will be appear in [11].

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