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Towards "Monotonic Probability"

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§0. Introduction

This note is a survey of our recent preprints [6], [7]. In previous works [3],[4], we investigated a noncommutative "de Moivre-Laplace theorem" and a noncommutative "Brownian motion" based on the baby and adult *monotone Fock spaces*, respectively. In these cases, we obtained the normalized arcsine law

$$p(x) \;=\; \frac{1}{\pi\sqrt{2-x^2}} \qquad (-\sqrt{2} < x < \sqrt{2})$$

as the central limit distribution. Also in [2] Lu discussed the essentially same structure as monotone Fock space under the name of *chronological Fock space*, independently from the author. But we did not catch the complete characterization of the independence structure which must have been hidden in our Fock space. The purpose of this note is to explain this independence structure which we call the *monotonic independence* and to give a monotonic analogue of some probabilistic results obtained from the independence argument, which contains the followings:

> *monotonic central limit theorem, monotonic law of small numbers,*
> *monotonic convolution for probability measures on the real line, and*
> *monotonic Lévy-Hinčin formula.*

§1. Monotonic Independence

Let $(\mathcal{A}, \phi)$ be a C^*-probability space consisting of a unital C^*-algebra $\mathcal{A}$ and a state ϕ over $\mathcal{A}$. Each element $X \in \mathcal{A}$ is interpreted as a (bounded) random variable on a C^*-probability space $(\mathcal{A}, \phi)$.

Definition 1 (monotonic independence). [6] A family $\{X_i\}_{i\in I} \subset \mathcal{A}$ of random variables on $(\mathcal{A}, \phi)$ with totally ordered index set I is said to be *monotonically independent* w.r.t. a state ϕ if the following two conditions are satisfied.

$$(a) \qquad X_i X_j^p X_k = \phi(X_j^p) X_i X_k \quad \text{whenever} \quad i < j > k.$$

$$\begin{aligned}(b) \qquad & \phi(X_{i_m}^{p_m} \cdots X_{i_2}^{p_2} X_{i_1}^{p_1} X_i^p X_{j_1}^{q_1} X_{j_2}^{q_2} \cdots X_{j_n}^{q_n}) \\ &= \phi(X_{i_m}^{p_m}) \cdots \phi(X_{i_2}^{p_2}) \phi(X_{i_1}^{p_1}) \phi(X_i^p) \phi(X_{j_1}^{q_1}) \phi(X_{j_2}^{q_2}) \cdots \phi(X_{j_n}^{q_n}) \\ &\qquad \text{whenever} \quad i_m > \cdots > i_2 > i_1 > i < j_1 < j_2 < \cdots < j_n.\end{aligned}$$

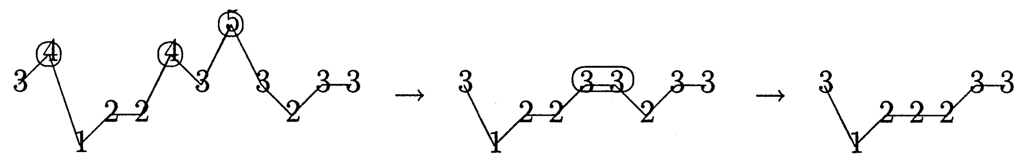
Here p's and q's are arbitrary natural numbers (≥ 0). The notation $i < j > k$ is understood as $i < j$ and $j > k$ (there is no assumption on the order relation between i and k). The notation $i_m > \cdots > i_2 > i_1 > i < j_1 < j_2 < \cdots < j_n$ is understood as $i_m > \cdots > i_2 > i_1 > i$ and $i < j_1 < j_2 < \cdots < j_n$. Of course, the case of $m = 0$ (resp. $n = 0$) in the condition (b) is understood in the natural way.

The above two conditions (a) and (b) can be viewed as the decomposition rules for expectations $\phi(X_{i_r} \cdots X_{i_2} X_{i_1})$ of monomials $X_{i_r} \cdots X_{i_2} X_{i_1}$ in X's, as explained as follows.

Denote by $\langle i_r \cdots i_2 i_1 \rangle$ the expectation $\phi(X_{i_r} \cdots X_{i_2} X_{i_1})$ for short. Then the expectation $\langle i_r \cdots i_2 i_1 \rangle$ can be uniquely decomposed based on the following procedure. As an example, take a configuration $(i_r \cdots i_2 i_1) = (341224353233)$. At first, by the repeated use of rule (a), we have

$$\begin{aligned}\langle\, 341224353233 \,\rangle &= \langle 4 \rangle \langle 4 \rangle \langle 5 \rangle \langle\, 312233233 \,\rangle \\ &= \langle 4 \rangle \langle 4 \rangle \langle 5 \rangle \langle 33 \rangle \langle\, 3122233 \,\rangle.\end{aligned}$$

This process can be visualized as



We see that once use of rule (a) means to take a "top" off the "mountains." After the maximal use of rule (a), we get a factor $\langle$ 3122233 $\rangle$ which has a form of "valley." But this final factor can be decomposed further by the use of rule (b). After all we obtain the final decomposition:

$$\langle\, 341224353233 \,\rangle = \langle 4 \rangle \langle 4 \rangle \langle 5 \rangle \langle 33 \rangle \langle 3 \rangle \langle 1 \rangle \langle 222 \rangle \langle 33 \rangle.$$

Of course this procedure works well for general configurations $(i_r \cdots i_2 i_1)$, and it uniquely defines the natural decomposition of $\langle X_{i_r} \cdots X_{i_2} X_{i_1} \rangle$.

Monotonically independent random variables naturally arise on the *monotone Fock space* [6]. Also monotonically independent random variables with prescribed probability distributions can be naturally constructed with the help of the *monotone product* of C^*-probability spaces [7].

§2. Central Limit Theorem and Law of Small Numbers

Now let us give a monotonic analogue of central limit theorem. Denote by χ_I the indicator function of an interval I.

Theorem 2 (monotonic central limit theorem). *[6] Let $(\mathcal{A}, \phi)$ be a C^*-probability space, and let $\{X_n \mid n = 1, 2, 3, \cdots\} \subset \mathcal{A}$ be self-adjoint, monotonically independent and identically distributed random variables with mean* 0 *and variance* 1. *Then the probability distribution of*

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}$$

converges in the weak topology as $n \to \infty$ to the normalized arcsine distribution μ which is given by the probability density function*

$$p(x) \;=\; \chi_{(-\sqrt{2},\sqrt{2})}(x)\;\frac{1}{\pi\sqrt{2-x^2}}.$$

Next let us give a monotonic analogue of law of small numbers. Denote by E_n^{-1} the nth branch of the product log function E^{-1} (= the inverse analytic function of an entire function $E(z) = z\,e^z$.) See [6] for the details. We denote E_{-1}^{-1} by E^{-1} for short. $\mathbf{N}^*$ denotes the set of all nonzero natural numbers.

Theorem 3 (monotonic law of small numbers). *[6] Let $(\mathcal{A}, \phi)$ be a C^*-probability space, and let $\{x_i^{(n)} \mid 1 \le i \le n;\; n \in \mathbf{N}^*\}$ be a family of elements in $\mathcal{A}$, satisfying the following conditions:*

(a) $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$ are self-adjoint, monotonically independent and identically distributed random variables for each fixed n;
(b) There exists a constant $\lambda > 0$ such that $\lim_{n\to\infty} n\,\phi((x_i^{(n)})^p) = \lambda$ for all $p \in \mathbf{N}^*$.

Then the probabilty distribution of the sum of $x_1^{(n)} + x_2^{(n)} + \cdots + x_n^{(n)}$ converges in the weak topology as $n \to \infty$ to a unique probability measure ν. The measure ν is the sum $\nu_1 + \nu_2$ of its absolutely continuous part ν_1 and the atomic part ν_2. The absolutely continuous part ν_1 is given by the density function*

$$p(x) \;=\; \chi_{(a,b)}(x)\cdot\frac{1}{\pi}\,\mathrm{Im}\frac{1}{E^{-1}(e^{\lambda}E(-x))},$$

where the support of $p(x)$ is $\mathrm{supp}(p) = [a, b]$. *The atomic part is $\nu_2 = c\,\delta_0$, where δ_0 is the Dirac measure at the origin $x = 0$. Here the constants a, b, c are given by*

$$a = -E_0^{-1}\left(-\frac{1}{e^{1+\lambda}}\right), \quad b = -E_{-1}^{-1}\left(-\frac{1}{e^{1+\lambda}}\right), \quad c = \frac{1}{e^{\lambda}},$$

respectively.

§3. Monotonic Convolution

Let μ be a probability measure on the real line $\mathbf{R}$. Then the *Cauchy transform* $G_\mu(z)$ of μ is defined by

$$G_\mu(z) \;=\; \int_{-\infty}^{+\infty}\frac{1}{z-x}\,d\mu(x), \qquad z \in \mathbf{C}^+.$$

Here $\mathbf{C}^+$ denotes the upper half plane of complex numbers. The *reciprocal Cauchy transform* $H_\mu(z)$ of μ is defined by

$$H_\mu(z) = \frac{1}{G_\mu(z)}, \qquad z \in \mathbf{C}^+.$$

This $H_\mu(z)$ satisfies $H_\mu(\mathbf{C}^+) \subset \mathbf{C}^+$.

The following theorem saids that the reciprocal Cauchy transform $H_\mu(z)$ plays in "monotonic probability" a role analogous to that played by the Fourier transform in "classical probability" and also to that played by the Voiculescu R-transform in "free probability" [9].

Theorem 4. *[7] Let $X_1, X_2, \cdots, X_n \in \mathcal{A}$ be monotonically independent self-adjoint random variables on a C^*-probability space $(\mathcal{A}, \phi)$, in the natural order of $\{1, 2, \cdots, n\}$. Then*

$$H_{X_1+X_2+\cdots+X_n}(z) = H_{X_1}(H_{X_2}(\cdots(H_{X_n}(z))\cdots)).$$

Here $H_X(z)$ denotes the reciprocal Cauchy transform of the probability distribution μ_X of X under ϕ.

This result motivates us to give the following definition.

Definition 5 (monotonic convolution). [7] For a pair of probability measures μ, ν on $\mathbf{R}$, the unique probability measure λ satisfying $H_\lambda(z) = H_\mu(H_\nu(z))$, $z \in \mathbf{C}^+$, is called the *monotonic convolution* of μ and ν, and denoted by $\lambda = \mu \rhd \nu$.

The unique existence of such measure λ is assured based on the theory of Pick-Nevanlinna functions on the upper half plane $\mathbf{C}^+$. The monotonic convolution $\mu \rhd \nu$ satisfies the following properties.

Properties of monotonic convolution.

(1) $\delta_0 \rhd \mu = \mu \rhd \delta_0 = \mu$;
(2) $(\lambda \rhd \mu) \rhd \nu = \lambda \rhd (\mu \rhd \nu)$;
(3) the map $\mu \mapsto \mu \rhd \nu$ is affine;
(4) the map $\mu \mapsto \mu \rhd \nu$ (resp. $\nu \mapsto \mu \rhd \nu$) is weak* continuous.

Here δ_0 denotes the point measure at the origin $x = 0$. Note that the monotonic convolution is not commutative in general: $\mu \rhd \nu \neq \nu \rhd \mu$.

§4. Monotonic Lévy-Hinčin formula

Now let us formulate, in the sense of "monotonic probability," the following three objects:

(A) infinitely divisible distribution ;
(B) continuous one-parameter convolution semigroup ;
(C) (certain) integral representation (= "Lévy measure").

We wish to establish the equivalence beteween among three objects (A), (B) and (C). This should be the content of "monotonic Lévy-Hinčin formula." The equivalence between (B) and (C) will be established in Theorem 10 in the general setting. On the other hand, the equivalence between (A) and (B) is established in Theorem 12, but in the restricted class of compactly supported probability measures

Let us give the definitions of notions concerning the "infinite divisibility."

Definition 6. A probability measure μ on $\mathbf{R}$ is said to be $\triangleright$-*infinitely divisible* if, for each $n \in \mathbf{N}^*$, there exists some probability measure ν on $\mathbf{R}$ such that

$$\mu \;=\; \overbrace{\nu \triangleright \nu \triangleright \cdots \triangleright \nu}^{n}.$$

Definition 7. A one-parameter family $\{\mu_t\}_{t\geq 0}$ of probability measures on $\mathbf{R}$ is said to be a *weak* continuous one-parameter* $\triangleright$*-semigroup* if the following conditions are satisfied: (1) $\mu_0 = \delta_0$; (2) $\mu_{s+t} = \mu_s \triangleright \mu_t$; (3) the map $t \mapsto \mu_t$ is weak* continuous.

Definition 8. A one-parameter family $\{H_t(z)\}_{t\geq 0}$ of reciprocal Cauchy transforms of probability measures on $\mathbf{R}$ is said to be a *continuous one-parameter semigroup of reciprocal Cauchy transforms* if the following conditions are satisfied: (1) $H_0(z) = z$; (2) $H_{s+t}(z) = H_s(H_t(z))$; (3) the map $t \mapsto H_t(z)$ is continuous for each fixed $z \in \mathbf{C}^+$.

There is the natural bijective correspondence beteween the above two kinds of continuous one-parameter semigroups $\{\mu_t\}_{t\geq 0}$ and $\{H_t(z)\}_{t\geq 0}$. Besides there is the natural correspondence from the set of all weak* continuous one-parameter $\triangleright$-semigroups $\{\mu_t\}_{t\geq 0}$ to the set of all $\triangleright$-infinitely divisible distributions μ given by the specialization (t:=1) : $\{\mu_t\}_{t\geq 0} \mapsto \mu_1$. (In Theorem 12, we show a partial converse $\mu \mapsto \{\mu_t\}_{t\geq 0}$ for the class of $\triangleright$-infinitely divisible distributions with compact supports.)

Let us give some examples of continuous one-parameter semigroups $\{H_t(z)\}_{t\geq 0}$ and its associated $\triangleright$-infinitely divisible distributions $\mu = \mu_1$. Denote by μ_{ac} (resp. μ_s) the absolutely continuous part (resp. the singular part) of μ w.r.t. the Lebesgue measure dx. Here E_*^{-1} denotes an appropriate branch of E^{-1} composed from E_0^{-1} and E_{-1}^{-1}.

Example 9.

(a) Arcsine distribution (= monotonic Gaussian distribution) [3]:

$$H_t(z) = \sqrt{z^2 - 2t}, \quad d\mu(x) = \chi_{(-\sqrt{2},\sqrt{2})}(x) \cdot \frac{1}{\pi\sqrt{2-x^2}}\, dx.$$

(b) Monotonic Poisson distribution [6]:

$$\begin{aligned} &H_t(z) = -E_*^{-1}(e^{\lambda t}\, E(-z)),\\ &d\mu_{ac}(x) = \chi_{(a,b)}(x) \cdot \frac{1}{\pi}\,\mathrm{Im}\frac{1}{E^{-1}(e^{\lambda}E(-x))}\, dx, \quad \mu_s = c\,\delta_0,\\ &a = -E_0^{-1}(-\frac{1}{e^{1+\lambda}}), \quad b = -E_{-1}^{-1}(-\frac{1}{e^{1+\lambda}}), \quad c = \frac{1}{e^{\lambda}}, \quad (\lambda > 0). \end{aligned}$$

(c) Cauchy distribution:

$$H_t(z) \;=\; z + ibt, \qquad d\mu(x) \;=\; \frac{1}{\pi}\frac{b}{x^2+b^2}\, dx \qquad (b > 0).$$

These examples reveal the following two features of "monotonic probability."

- It is often that important probability distributions may have the reciprocal form: $\frac{1}{\text{some function}}$. (Of course this is an immediate effect of the reciprocal Cauchy transform.) It can be said that, in a sense, "monotonic probability" is a "reciprocal probability."

- It is often that the reciprocal Cauchy transform $H_\mu(z)$ of a $\triangleright$-infinitely divisible distribution μ includes a pair consisting of some function f and its inverse function f^{-1}. In fact, this is a general phenomenon as shown in the following.

Theorem 10 (monotonic Lévy-Hinčin formula in terms of semigroups). *[7] Let $\{\mu_t\}_{t\geq 0}$ be a one-parameter familiy of probability measures on $\mathbf{R}$. Assume that $\mu_t \neq \delta_0$ for all $t > 0$. Then the following two conditions are equivalent.*

(1) $\{\mu_t\}_{t\geq 0}$ is a weak continuous one-parameter $\triangleright$-semigroup.*
(2) There exists a pair (α, γ) $(\neq (0,0))$ of a real number α and a finite positive measure γ on $\mathbf{R}$ such that the reciprocal Cauchy transform $H_t(z)$ of μ_t is given by

$$w \ = \ H_t(z) \quad \Longleftrightarrow \quad {}^{\exists 1}w \in \mathbf{C}^+ \ s.t. \ \int_z^w \frac{dz}{A(z)} \ = \ t, \qquad (\star)$$

where the function $A(z)$ is defined by

$$A(z) \ = \ \alpha + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\gamma(x). \qquad (\star\star)$$

If the above conditions hold, (α, γ) and $A(z)$ are unique.

Remark. Put $F(z) = \displaystyle\int_i^z \frac{dz}{A(z)}$, then the condition $(\star)$ can be rewritten as follows:

$$w \ = \ H_t(z) \quad \Longleftrightarrow \quad \text{unique } w \text{ s.t. } F(w) - F(z) \ = \ t. \qquad (\star')$$

Hence $H_t(z)$ has the representation

$$H_t(z) \ = \ F^{-1}(F(z) + t). \qquad (\star\star\star)$$

Note that, for any weak* continuous one-parameter $\triangleright$-semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures, it is hold that either i) $\mu_t \neq \delta_0$ for all $t > 0$, or, ii) $\mu_t = \delta_0$ for all $t \geq 0$. The case ii) corresponds to $(\alpha, \gamma) = (0, 0)$. The pair (α, γ) is called the *Lévy measure* for short although it is not a measure but a pair of a number and a measure. For each semigroup $\{H_t(z)\}_{t\geq 0}$ in Example 9, let us give its generator $A(z)$ and the Lévy measure (α, γ) in the standard form $(\star\star)$.

Example 11.

(a) Arcsine distribution : $A(z) = -\dfrac{1}{z}$, $(\alpha, \gamma) = (0, \delta_0)$.

(b) Monotonic Poisson distribution: $A(z) = \dfrac{\lambda z}{1-z}$, $(\alpha, \gamma) = \left(-\dfrac{\lambda}{2}, \dfrac{\lambda}{2}\delta_1\right)$.

(c) Cauchy distribution:

$$A(z) = ib = \frac{b}{\pi}\int_{-\infty}^{+\infty} \frac{1+xz}{x-z}\frac{dx}{1+x^2}, \quad \alpha = 0, \quad d\gamma(x) = \frac{b}{\pi}\frac{dx}{1+x^2}.$$

Now, let us establish the equivalence among the three objects (A), (B) and (C), but in the restricted class of compactly supported probability measures. Denote by $\mathcal{P}_c$ the set of all probability meassures on $\mathbf{R}$ which are compactly supported. Then

Theorem 12 (monotonic Lévy-Hinčin formula for class $\mathcal{P}_c$). *[7] Let μ be a compactly supported probability measure on* $\mathbf{R}$. *Assume that $\mu \neq \delta_0$. Then the following three conditions are equivalent.*

(1) μ is $\triangleright$-infinitely divisible.
(2) There exists a weak continuous one-parameter $\triangleright$-semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures on* $\mathbf{R}$ *such that $\mu_1 = \mu$.*
(3) There exists a pair (a, ρ) ($\neq (0,0)$) of a real number a and a compactly supported finite positive measure ρ on $\mathbf{R}$ *such that the Pick function*

$$A(z) = a + \int_{-\infty}^{+\infty} \frac{1}{x-z} d\rho(x) \tag{5.1}$$

generates $H_\mu(z)$ as

$$w = H_\mu(z) \iff {}^{\exists 1} w \in \mathbf{C}^+ \ s.t. \int_z^w \frac{dz}{A(z)} = 1.$$

If the above conditions hold, then $\{\mu_t\}_{t\geq 0}$, (a, ρ) and $A(z)$ are unique, and $\mu_t \in \mathcal{P}_c$ for all $t \geq 0$.

An example of compactly supported $\triangleright$-infinitely divisible distributions is given by a *compound monotonic Poisson distribution* μ_τ which is defined by its generator

$$A(z) = \int_{-\infty}^{+\infty} \frac{xz}{x-z} d\tau(x),$$

where τ is a compactly supported finite positive measure on $\mathbf{R}$. This compound monotonic Poisson distribution μ_τ satisfies a generalization of monotonic law of small numbers [7]. Note that the case $\tau = \lambda\delta_1$ corresponds to the monotonic Poisson distribution.

§5. Conclusion

Finally, as a conclusion of this note, we summarize our results in the following table. We see that several probabilistic concepts can be built based on our "monotonic independence," in an analogous way as in "classical probability" and also as in "free probability."

Classical Probability (commutative prob.)	Free Probability (D. Voiculescu)	Monotonic Probability (N.M.)
commuting independence	free independence	monotonic independence
tensor product	free product	monotone product
Fourier transform	R-transform	reciprocal Cauchy transform
convolution $\mu * \nu$	free convolution $\mu \boxplus \nu$	monotonic convolution $\mu \triangleright \nu$
Gaussian distribution	Wigner semi-circle dist.	arcsine distribution
Poisson distribution	Marčenko-Pastur dist.	monotonic Poisson distribution
Lévy-Hinčin formula	free Lévy-Hinčin formula	monotonic Lévy-Hinčin formula
symmetric Fock space	full Fock space	monotone Fock space
Brwonian motion	free Brownian motion	monotonic Brownian motion

Some aspects concerning noncommutative stochastic processes on the monotone Fock space were treated in [1], [5], [8].

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