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# ON PROPERTIES OF RELATIVE METACOMPACTNESS AND PARACOMPACTNESS TYPE

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ABSTRACT. We study several natural relative properties of metacompactness and paracompactness types and the relationships among them. We characterize several topological properties by relative paracompactness and metacompactness.

## 1. INTRODUCTION

Arhangel'skiĭ and Genedi showed in 1989 that for any topological property $\mathcal{P}$ one can associate a relative property characterized in terms of a subspace $Y$ of a space $X$, in such a way that this relative property coincides with $\mathcal{P}$ whenever $Y = X$.

One would expect that for a given topological property there should be a variety of relative topological properties associated with it. We study several natural relative properties of paracompactness and metacompactness type and the relationships among them.

Throughout this paper all spaces are $T_1$. Ordinals will have the order topology and subsets of topological spaces will have the subspace topology. For any collection $\mathcal{A}$ of subsets of a set $X$, any $C \subset X$ and any $x \in X$, $(\mathcal{A})_C = \{A \in \mathcal{A} : C \cap A \neq \phi\}$, $(\mathcal{A})_x = \{A \in \mathcal{A} : x \in A\}$ and $st(x, \mathcal{A}) = \cup(\mathcal{A})_x$. If $X$ is a set, $\mathcal{H}$ a collection of subsets of $X$ and $C \subset X$ then $\mathcal{H}$ is said to be $\underline{(locally)\ point\ finite\ on\ C}$ provided $C \subset \cup\mathcal{H}$ and for every $x \in C$ the collection $(\mathcal{H})_x$ is finite (there is an open neighborhood $V$ of $x$ such that $\{H \in \mathcal{H} : V \cap H \neq \phi\}$ is finite).

Suppose $C$ is a subset of the space $X$. The following definitions of the most natural properties of relative paracompactness type are due to Gordienko,[8]. The subspace $C$ is said to be $\underline{1-paracompact\ in\ X}$ provided every open cover of $X$ has an open refinement locally finite on $C$. The subspace $C$ is $\underline{2-paracompact\ in\ X}$ provided every open cover of $X$ has an open partial refinement covering $C$ and locally finite on $C$. The subspace $C$ is $\underline{3-paracompact\ in\ X}$ provided every open cover of $X$ has a partial refinement consisting of sets open in $C$ locally finite on $C$.

By replacing "locally finite" with "point finite" in Gordienko's definitions we obtain relative metacompact analogs. We say that a subspace $C$ of a space $X$ is $\underline{strongly\ metacompact\ in\ X}$ provided every open cover of $X$ has an open refinement point finite on $C$. For a subspace $C$ of a space $X$ we say that $C$ is $\underline{metacompact\ in\ X}$

provided every open cover of $X$ has an open partial refinement point finite on $C$. Clearly for a space $X$ strongly metacompactness in $X$ is a natural relatively metacompact analog of $1-$ paracompactness in $X$ and metacompactness in $X$ is the corresponding relative metacompact analog of both $2-$ and $3-$ paracompactness in $X$.

## 2. Characterizations of 1-, 2-, 3-paracompactness and relative (strong) metacompactness

There are several characterizations of paracompactness and metacompactness. In [16] 1-, 2-, 3-paracompactness were characterized by means of Michael's type. In this section we characterize 1-, 2-, 3-paracompactness and relative (strong) metacompactness type of characterizations of original versions; monotone property type.

**Theorem 2.1** (Original version [1], [13], [17]). *A space $X$ is compact (paracompact, metacompact) if and only if every monotone open cover of $X$ has a finite subcover (locally finite open refinement, point finite open refinement, respectively).*

A collection $\mathcal{S}$ of sets is said to be *monotone* provided that for all $S, S' \in \mathcal{S}$ either $S \subset S'$ or $S' \subset S$.

More generally, collection $\mathcal{S}$ of sets is said to be *directed* provided that for all $S, S' \in \mathcal{S}$ there is a $T \in \mathcal{S}$ such that $S \cup S' \subset T$.

Clearly that a monotone collection is directed, but the converse is not true. For 1-paracompactness and strong metacompactness we have similar results for directed open covers as follows:

**Theorem 2.2.** *Suppose that $C$ is a subset of the space $X$. $C$ is 1-paracompact (strongly metacompact) in $X$ if and only if every directed open cover has an open refinement that is locally finite (point finite) on $C$.*

In Theorem 2.2 "directed" cannot be replaced with "monotone". In the following example we show that there are regular space $X$ and a dense subspace $A$ of $X$ such that every monotone open cover of $X$ has an open refinement locally finite on $A$ but $A$ is not 1-paracompact (or even strongly metacompact) in $X$.

**Example 2.3.** Let $Y = \Pi_{i=1}^{\infty}(\omega_i+1)$. For each natural number $k$ let $X_k = (\Pi_{i=1}^{k}(\omega_i+1)) \times (\Pi_{i=k+1}^{\infty}(\omega_i))$ and let $X = \cup_{k=1}^{\infty} X_k$ with the usual subspace topology [15]. For each natural number $k$ let $A_k = (\Pi_{i=1}^{k}(\omega_i+1)) \times (\Pi_{i=k+1}^{\infty}\{0\})$ and note that $A_k$ is compact. Then

$$A = \cup_{k=1}^{\infty} A_k$$

is a dense subset of $X$. Then every monotone open cover of $X$ has an open refinement locally finite on $A$ but $A$ is not 1-paracompact (or even strongly metacompact) in $X$.

Clearly, metacompactness in $X$, and $2-(3-)$ paracompactness in $X$ can also be characterized in terms of directed open covers as in Theorem 2.2. But we do not know if being metacompact or 2-,3- paracompact in a space can be characterized in terms of monotone open covers.

However if we make obvious modifications to Junnila's proof that paracompact (metacompact) spaces can be characterized in terms of monotone open covers [11], we get the following lemma:

**Lemma 2.4.** *For a space $X$ and $C \subset X$, if every monotone open cover of $X$ has an open refinement locally finite (point finite) on $C$, then $C$ is 2-paracompact (metacompact) in $X$.*

For a closed subset of a space $X$, monotone open covers can be used to characterize strong metacompactness and 3- paracompactness in $X$.

**Theorem 2.5.** *For a space $X$ and a closed subset $C$ of $X$, the subspace $C$ is strongly metacompact (3- paracompact) in $X$ if and only if every monotone open cover of $X$ has an open refinement that is point finite on $C$ (a partial refinement consisting of sets open in $C$ locally finite on $C$).*

**Theorem 2.6.** *Suppose that the space $X$ has Lindelöf degree $L(X) \leq \omega_1$ and $C$ is an $F_\sigma$ subset of $X$. Then $C$ is strongly metacompact in $X$ if and only if every monotone open cover of $X$ has an open refinement that is point finite on $C$. If $X$ is normal, then $C$ is 1- paracompact in $X$ if and only if every monotone open cover of $X$ has an open refinement locally finite on $C$.*

## 3. Relative paracompactness and metacompactness from inside and outside

In [2] Arkhangel'skii introduced the following types of relative topological properties. Let $Y$ be a subspace of a space $X$ and let $\mathcal{P}$ be a topological property. We say that $Y$ *has property $\mathcal{P}$ in $X$ from inside*, if every subspace of $Y$ closed in $X$ has property $\mathcal{P}$.

If there is a subspace $Z$ of $X$ having property $\mathcal{P}$ containing $Y$ then we say that $Y$ *has property $\mathcal{P}$ in $X$ from outside.* In [3] the following general question is posed:

**Question** Suppose $C$ is relatively $\mathcal{P}$ in $X$. Does $C$ have property $\mathcal{P}$ from outside?

For paracompactness and metacompactness the next result is clear:

$*$ $C$ is $F_\sigma$ subset in $X$
$C$ is metacompact in $X \Rightarrow C$ is metacompact.
$C$ is metacompact in $X \Rightarrow C$ is matacompact in $X$ from outside.

But "$C$ being metacompact in a space $X$ from outside" does not imply that $C$ is strongly metacompact in $X$. For example, take $X = \omega_1$ with the order topology and let $C$ be the set of isolated points of $X$. Since $C$ is metacompact, $C$ is metacompact in $X$ from outside but $C$ is not strongly metacompact in $X$.

**Theorem 3.1.** *Suppose that $C$ is a subset of the space $X$ and $C$ is paracompact (metacompact) in $X$ from outside. Then $C$ is 3-paracompact (metacompact) in $X$.*

**Remark 3.2.** The converse of Theorem 3.1 is not true. In [3] the authors give an example of a Tychonoff space $X$ and a subspace C of $X$ which is 1- paracompact in $X$ but not paracompact in $X$ from outside. In our next example we construct a Tychonoff space $Z$ and a subspace $C$ of $Z$ which is 2- paracompact (i.e. metacompact) in $Z$ but which is not metacompact in $Z$ from outside.

**Example 3.3.** Let $X$ be an non-metacompact 0-dimensional $T_1$ space. Let $Y$ be a compact 0-dimensional $T_1$ space such that $Y - \{p\}$ is not metacompact for some $p \in Y$. (For example $X = \beta\mathbb{N} - \{p\}$ and $Y = \beta\mathbb{N}$ where $p \in \beta\mathbb{N} - \mathbb{N}$.) Let $Z = X \times Y$ and define a topology on $Z$ as follows:

1. For $x \in X$ and $y \in Y - \{p\}$ basic open neighborhoods of $(x, y)$ are of the form $\{x\} \times V$ where $V$ is open neighborhood of $y$ in $Y$.
2. For $x \in X$ basic open neighborhoods of $(x, p)$ are of the form $\cup\{\{x\} \times V_x : x \in U\}$ where $U$ is an open neighborhood of $x$ in $X$ and $\{V_x : x \in U\}$ is a collection of open neighborhoods of p in $Y$.

Then $Z$ is a 0-dimensional $T_1$ space (and thus a Tychonoff space). Let

$$C = \{(x, y) \in X \times Y : y \neq p\}.$$

Then $C$ is 2- paracompact (i.e. metacompact) in $Z$ but which is not metacompact in $Z$ from outside.

**Theorem 3.4.** *Suppose that $C$ is a subset of the space $X$ and $C$ is 3-paracompact (metacompact) in $X$. Then $C$ is paracompact (metacompact) in $X$ from the inside.*

"A subspace $C$ being metacompact (paracompact) in a space $X$ from inside" does not imply that $C$ is metacompact in $X$, there is a counterexample.

## 4. Relative cp-paracompactness and cp-metacompactness

**Theorem 4.1** (Michael 1957 [14], Original version). *A regular space is paracompact if and only if every open cover has a closed closure preserving refinement.*

**Theorem 4.2** (Junnila 1979 [11], Original version). *A space is metacompact if and only if every directed open cover has a closed closure preserving refinement.*

We shall define relative versions. For a space $X$ and a subset $C$ of $X$, a collection $\mathcal{F}$ of closed subsets of $X$ is said to be *closure preserving with respect to $C$* provided that for all $\mathcal{F}' \subset (\mathcal{F})_C$ either $C \subset \cup\mathcal{F}'$ or $\cup\mathcal{F}'$ is closed in $X$.

The collection $\mathcal{F}$ is said to be *weakly closure preserving with respect to $C$* provided that for all $\mathcal{F}' \subset (\mathcal{F})_C$, one has $(\cup\mathcal{F}') \cap C = (\overline{\cup\mathcal{F}'}) \cap C$. We say that a subspace $C$ of a space $X$ is [*weakly*] *cp− paracompact* (*metacompact*) in $X$ provided every (directed) open cover of $X$ has a closed partial refinement covering $C$ which is [weakly] closure preserving with respect to $C$.

The following characterization of "cp- paracompact in a regular space $X$" demonstrates that it is a natural property of relative paracompactness type.

**Theorem 4.3.** *Suppose $X$ is a regular space and $C \subset X$. The subspace $C$ is cp-paracompact in $X$ if and only if $\overline{C}$ is a paracompact subspace of $X$.*

**Corollary 4.4.** *Suppose $X$ is a regular space and $C \subset X$. If the subspace $C$ is cp-paracompact in $X$ then $C$ is paracompact in $X$ from outside.*

**Remark 4.5.** There is a space $X$ and its subspace $C$ such that $C$ is cp- paracompact in $X$ but $C$ is not 2-paracompact in $X$.

**Theorem 4.6.** *For a regular space $X$ and a closed $C \subset X$ the following conditions are equivalent:*
(1) *$C$ is paracompact,*
(2) *$C$ is 3-paracompact in $X$,*
(3) *$C$ is weakly cp- paracompact in $X$,*
(4) *$C$ is cp- paracompact in $X$.*

The metacompact version of this result is as following.

**Theorem 4.7.** *For a space $X$ and a closed $C \subset X$ the following conditions are equivalent:*
(1) *$C$ is metacompact,*
(2) *$C$ is metacompact in $X$,*
(3) *$C$ is weakly cp- metacompact in $X$,*
(4) *$C$ is cp- metacompact in $X$.*

**Corollary 4.8.** *For a (regular) space $X$ and a closed $C \subset X$, if $C$ is weakly cp-metacompact (weakly cp- paracompact) in $X$, then $C$ is metacompact (paracompact) in $X$ from inside.*

There is a classical result about covers by compact subsets:

**Theorem 4.9** ([12] and [18], Original version). *Every space $X$ with a closure preserving closed cover by compact sets is metacompact.*

Since any cover of a space $X$ consisting of compact sets will refine every directed open cover, we have the following relative version of this theorem.

**Theorem 4.10.** *Suppose $C \subset X$ and that there is a collection of compact subsets of $X$ covering $C$ which is (weakly) closure preserving with respect to $C$ then $C$ is (weakly) cp-metacompact in $X$. In particular, if $C$ is a countable subset of $X$ then $C$ is cp-metacompact in $X$.*

**Theorem 4.11.** *Suppose that $X$ is a regular space and $C \subset X$ is metacompact in $X$. Then $C$ is weakly cp$-$ metacompact in $X$.*

We do not know if regularity is needed in Theorem 4.11. For the strongly metacompact analog it is not needed:

**Theorem 4.12.** *Suppose that $X$ is space and $C \subset X$ is strongly metacompact in $X$. Then $C$ is weakly cp– metacompact in $X$.*

## 5. Diagrams

The following diagrams shows the relationships among the properties of paracompactness and metacompactness types studied here.

**[diagram 1] (paracompact type)**

$$
\begin{array}{ccccc}
 & \text{1-paracompact in } X & & & \\
 & \Downarrow & & & \\
 & \text{2-paracompact in } X & & & \text{cp-paracompact in } X \\
 & \Downarrow & & & \Downarrow T_3 \\
 & \text{3-paracompact in } X & & \Longleftarrow & \text{paracompact in } X \text{ from outside} \\
\Downarrow & & \Downarrow T_3 & & \\
\text{paracompact in } X \text{ from inside} & & \text{weakly cp-paracompact in } X & &
\end{array}
$$

**[diagram 2] (metacompact type)**

$$
\begin{array}{ccccc}
\text{cp-metacompact in } X & & \text{strongly metacompact in } X & & \text{metacompact in } X \text{ from outside} \\
\Downarrow & \Downarrow & & \Downarrow & \Downarrow \\
\text{weakly cp-metacompact in } X & & \overset{T_3}{\Longleftarrow} & & \text{metacompact in } X \\
 & \Downarrow & & \Downarrow & \\
 & & \text{metacompact in } X \text{ from inside} & &
\end{array}
$$

## 6. Characterizations of original properties by relative properties

That is can topological properties of a space be characterized using relative topological properties. In 1996, Arhangel'skiĭ and Tartir showed that a $T_2$ space $Y$ is regular in every larger $T_2$ space if and only if Y is compact [6]. It is natural to consider the following question.

> Let $\mathcal{P}$ be a topological property. If C is $\mathcal{P}$ in every larger space, what can be said about topological properties of $C$ itself?

For relative paracompact and metacompact properties we have the following answer:

**Theorem 6.1** ([16]). *The following conditions are equivalent for a space $C$.*

(1) *$C$ is 3-paracompact (metacompact) in every larger space $X$.*
(2) *$C$ is 3-paracompact (metacompact) in every larger space $X$ in which $C$ is closed.*

(3) *C is 3-paracompact (metacompact) in some larger space X in which C is closed.*
(4) *C is paracompact (metacompact).*

**Theorem 6.2** ([16]). *Let C be a regular space. The following conditions are equivalent:*

(1) *C is 2-paracompact in every larger regular space X.*
(2) *C is 2-paracompact in every larger regular space X in which C is closed.*
(3) *C is 2-paracompact in every larger normal space X in which C is closed.*
(4) *C is Lindelöf.*

**Theorem 6.3** ([16]). *Let C be a regular space. The following conditions are equivalent:*

(1) *C is 1-paracompact in every larger regular space X.*
(2) *C is 1-paracompact in every larger regular space X in which C is closed.*
(3) *C is 1-paracompact in every larger Tychonöff space X in which C is closed.*
(4) *C is compact.*

**Theorem 6.4.** *Let C be a normal space. The following conditions are equivalent:*

(1) *C is 1-paracompact in every larger normal space X in which C is closed.*
(2) *C is Lindelöf.*

**Theorem 6.5.** *The following conditions are equivalent for a space C :*

(1) *C is strongly metacompact in every larger space X in which C is closed.*
(2) *C is strongly metacompact in some larger space X in which C is closed.*
(3) *C is metacompact.*

**Theorem 6.6.** *Let C be a normal space. The following conditions are equivalent:*

(1) *C is strongly metacompact in every larger regular space X.*
(2) *C is compact.*

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