数理解析研究所講究録
1188 巻 2001 年 127-137

GO-spaces and orderability of compactifications

東京学芸大学　田中祥雄 (Yoshio Tanaka)

In this paper, we give some characterizations for certain compactifications of GO-spaces to be orderable by means of " cuts " in GO-spaces.

Let $(X, \leq)$ be a linearly ordered set. Then, a *linearly ordered topological space* (abbreviated LOTS) is a triple $(X, \tau(\leq), \leq)$, where $\tau(\leq)$ is the usual *order topology* (i.e., open-interval topology) by the order $\leq$. Also, $(X, \tau, \leq), \tau$ is a topology on X, is a *generalized ordered space* (abbreviated GO-space) if (i) $\lambda(\leq) \subset \tau$; and (ii) every point of X has a local τ-base consisting of (possibly degenerate) intervals of X. For a space (X, τ), there exists a linear order $\leq$ of X such that $(X, \tau, \leq)$ is a GO-space iff it is a (closed) subspace of a LOTS; see [L].

A space X is *orderable* (resp. *suborderable*) if X is homeomorphic to a LOTS (resp. GO-space) [N]. Thus, a space X is orderable iff the topology of X coincides with the order topology by some linear order of X [VRS].

For a space X, a compactification $c(X)$ of X is a compact space such that X is homeomorhic to a dense subset of $c(X)$. We call a compactification $c(X)$ of X *orderable* if the topology of $c(X)$ is the order topology by some order of $c(X)$.

If no confusion, for a GO-space (or LOTS) $(X, \tau, \leq)$, we shall omit " τ " or " $\leq$ ". Also, we shall sometimes use " LOTS " (resp. " GO-spaces ") instead of " orderable spaces " (resp. " suborderable spaces ").

Let $(X, \tau, \leq)$ be a GO-space. Let us consider a space Y containing a subspace (X, τ) such that the order $\leq$ on X can be so extended to some linear order $\preceq$ on Y as to yield the given topology of Y as the order topology $\tau(\preceq)$ by $\preceq$. Then, we say that Y is a *linearly ordered extension* of X. Also, let us call a compactification $c(X)$ of X a *linearly ordered compactification* of X if $c(X)$ is a linearly ordered extension of X. (When a GO-space $(X, \tau, \leq)$ is homeomorphic to a dense subspace D of $c(X)$ under a map f, we shall consider a GO-space $(D, f(\tau), \leq_f)$ instead of $(X, \tau, \leq)$, here $f(\tau) = \{f(G) : G \in \tau\}$, and $d <_f d'$ if $x < x'$ for $d = f(x),\ d' = f(x')$).

Let $(X, \tau, \leq)$ be a GO-space, and $\lambda = \tau(\leq)$ be the order topology on X defined by $\leq$. Let $R = \{x \in X : [x, +\infty) \in \tau - \lambda\}$, and $L = \{x \in X :$

$(-\infty, x] \in \tau - \lambda\}$, and Z be the set of all integers.

Define subsets X^* and $\widetilde{X}$ of $X \times Z$ as follows. Let X^*; $\widetilde{X}$ be a LOTS having the order topology by the lexicographic order on X^*; $\widetilde{X}$ respectively.

$$X^* = (X \times \{0\}) \cup \{\langle x, n\rangle : x \in R,\ n < 0\} \cup \{\langle x, m\rangle : x \in L,\ m > 0\}.$$
$$\widetilde{X} = (X \times \{0\}) \cup (R \times \{-1\}) \cup (L \times \{1\}).$$

If X is not a LOTS, then $\widetilde{X}$ is not a subspace of X^*, and X^* is not a linearly ordered extension of $\widetilde{X}$ under the natural correspondence. For X^*; $\widetilde{X}$, see [L] (or [N]); [MK] respectively.

Remark 1. (1) For a GO-space X, X^* (resp. $\widetilde{X}$) is a minimal (in the sense of inclusion) linearly ordered extension of X containing X as a closed (resp. dense) subset [L] (resp. [MK]).

(2) For the Sorgenfrey line S, S is separable, and perfect (i.e., every closed subset is a G_δ-set), ans so is $\widetilde{S}$. But, S^* is neither separable nor perfect ([L]).

We note that there exists a perfect, GO-space X, but X has no perfect, linearly ordered extensions containing X as a closed or dense subset ([MK]).

(3) For a GO-space X, if X is metrizable; first countable; locally compact; paracompact, then so is X^* respectively ([L]). But, $\widetilde{X}$ need not be metrizable even if X is a discrete, GO-space ([MK]).

Let $(X, \leq)$ be a linearly ordered set. A pair $(A|B)$ of subsets of X is caled a *cut* of X, if $X = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, and if $x \in A$ and $y \in B$, then $x < y$.

For every cut $(A|B)$ of X, exactly one of the following four cases arises. A cut $(A|B)$ is a *jump* if it satisfies (C1), and a *gap* if it satisfies (C4); see ([E]). Note that, for cuts $(A|B)$ and $(C|D)$ of X, $A \subset C$ or $C \subset A$.

(C1) There exist $Max\ A$ and $min\ B$.
(C2) There exists $Max\ A$, but no $min\ B$.
(C3) There exists $min\ B$, but no $Max\ A$.
(C4) There exists neither $Max\ A$ nor $min\ B$.

Let $(X, \leq)$ be a GO-space. A cut $(A|B)$ of X is called a *pseudo-gap* if A and B are disjoint *open sets* satisfying (C2) or (C3); see [N]. We note that a GO-space $(X, \leq)$ is a LOTS iff $(X, \leq)$ has no pseudo-gaps.

Let $(X, \leq)$ be a GO-space. Define a subset $X^\sim$ of $X \times \{0, \pm 1\}$ by

$X^\sim = (X \times \{0\}) \cup \{\langle MaxA, 1\rangle : (A|B)$ is a *pseudo-gap* of X having $MaxA\} \cup \{\langle minB, -1\rangle : (A|B)$ is a *pseudo-gap* of X having $minB\}$.

Let $X^\sim$ be a LOTS having the order topology defined by the lexicographic order on $X^\sim$. Then $\widetilde{X} = X^\sim$.

For a LOTS $(X, \leq)$, define

$X^+ = X \cup \{c = (A|B) : c \text{ is a gap of } X\} \cup \{\pm\infty\}$.

Let X^+ be a LOTS having the order topology by a linear order $\preceq$ on X^+ as follows: (i) For a gap $c = (A|B)$, $a \prec c$ for all $a \in A$, and $c \prec b$ for all $b \in B$; and (ii) For gaps $c = (A|B)$ and $c' = (A'|B')$, $c \prec c'$ if $A \subset A'$ and $A \neq A'$. Also, let $-\infty \prec x$ and $x \prec +\infty$ for all $x \in X$, but put $-\infty = minX$ if $minX$ exists, and put $+\infty = MaxX$ if $MaxX$ exists.

For a GO-space $(X, \leq)$, X^+ is defined by the closure of X in $(X^*)^+$. Then, $X^+ = (X^+, \tau(\preceq), \preceq)$ is a linearly ordered compactification of X. See [N; Example VIII.3]. X^+ is called *Dedekind compactification* of X.

Let $(X^\sim)^+ = X^\sim \cup \{\langle \alpha, 0\rangle : \alpha = (A|B) \text{ is a gap of } X\} \cup \{\langle \pm\infty, 0\rangle\}$ be a subset of $X^+ \times \{0, \pm 1\}$. Let $(X^\sim)^+$ be a LOTS having the order topology by the lexcographic order on $(X^\sim)^+$. Then, $X^+ = (X^\sim)^+ = (\widetilde{X})^+$, so X^+ is a linearly ordered compactification of $\widetilde{X}$.

Remark 2. For a GO-space $(X, \leq)$, a compact LOTS ℓX was defined in [K1] as the *minimal* linearly ordered compactification of X in the following sense: For each linearly ordered compactification L of X, there exists a continuous map $f : L \to \ell X$ such that $f|X$ is the identity map on X. (In [K1], ℓX is used in the study on normality of products of GO-spaces and cardinals). We can assume that $X^+ = \ell X$ (in view of [K1]).

For a space X, let us consider the following compactifications of X.
$\alpha(X)$: *Alexandroff's one-point compactification.*
$\beta(X)$: *Stone-Čech compactification.*
X^+: *Dedekind compactification*, but X is a GO-space.

The following facts are well-known. See [E] or [N], for example.

Fundamental Facts: (1) Every GO-space is hereditarily (collectionwise) normal, and hereditarily countably paracompact.

(2) For a LOTS $(X, \leq)$, X is compact $\Leftrightarrow$ X has no gaps, and there exist $minX$ and $MaxX$ $\Leftrightarrow$ For every $A \subset X$, there exists $supA$, here $sup\ \emptyset = minX$, and $supX = MaxX$.

(3) For a LOTS $(X, \leq)$, X is connected $\Leftrightarrow$ X has no jumps and no gaps.

(4) For a GO-space $(X, \tau, \leq)$, $\tau = \tau(\leq)$ if X is compact or connected.

Example 1. (1) (i) Let $X = (0,1) \cup \{2\}$ be a space with the usual topology. Then, X is a GO-space which is the topological sum of LOTS (0, 1) and $\{2\}$.

But, X is not orderable.

(ii) None of the following subspaces of the Euclidean plane is suborderable: The circle S^1; The square $[0,1] \times [0,1]$; The space obtained from the topological sum of n (≥ 3) many intervals [0, 1] by identifying all zero-points.

(2) The Sorgenfrey line and the Michael line are GO-spaces, but none of them is orderable (in view of [L]).

(3) (i) Let $X = \{0\} \cup (1,2]$ be a space with the usual topology. Hence X is a GO-space, but not a LOTS by the usual order. While, X is orderable by the usual order $\leq$, but let $x < 0$ for all $x \in (1,2]$.

(ii) Let $Y = ([0,\omega_1], \leq)$, where $\leq$ is the usual order. Let τ be the topology on Y obtained from the order topology by isolating every countable limit ordinal. Then, $(Y, \tau, \leq)$ is a GO-space, but not a LOTS. While, (Y, τ) is orderable by the lexicographic order on $([0, \omega_1) \times Z) \cup \{\langle \omega_1, 0 \rangle\}$ ([L]).

(4) (i) Let X be the unit square $[0,1] \times [0,1]$, and define the order topology on X by the the lexicographic order. Then, as is well-known, X is a first countable, compact, connected LOTS, but X is not separable, hence not metrizable.

(ii) Let Y be $[0,1] \times \{0,1\}$, and define the order topology on Y by the lexicographic order. Then, as is well-known, Y is a first countable, compact, separable LOTS, but Y is not metrizable.

Remark 3. (1) Related to (1) of Example 1, the following modifications hold: (i) Let Y be a topological sum of a connected LOTS $(X, \leq)$ and a point p. Then Y is suborderable, and Y is orderable iff $Max\ X$ or $min\ X$ exists. (ii) Any connected space X with $|X| \geq 2$ is not orderable if $X - \{p\}$ is connected for any point $p \in X$, or $X - \{q\}$ has at least three components for some point $q \in X$.

(2) Let X be suborderable. Then X is orderable if X is a topological group ([LiSaT]), or X is a metrizable space which is totally disconnected (i.e., any connected subset of X is a singleton).

(3) ([VRS]) If $X \times Y$ is suborderable, then X is totally disconnected, or Y is discrete. Conversely, for any orderable (resp. suborderable) space X, $X \times Y$ is so respectively if Y is discrete. While, even if $X \times Y$ is orderable with Y discrete, X need not be orderable. (In fact, let X be the space $(0,1) \cup \{2\}$ in Example 1(1), and let Y be a countably infinite discrete space).

Proposition 1. Let X be a GO-space. If X is separable metrizable, then X^*, and X^+ are separable metrizable, hence so is $\widetilde{X}$.

Corollary 2. Let $(X, \leq)$ be a GO-space. If X is separable metrizable, then X has at most countably many jumps and pseudo-gaps.

Remark 4. (1) Let X be a separable metrizable space. Then, as is well-known, $\alpha(X)$ is metrizable if X is locally compact, but, $\beta(X)$ is not even first countable if X is not compact.

(2) For a compactification Y of a space X, if Y is first countable, then $|Y| \leq c = 2^{\omega}$ (thus, $|X| \leq c$).

Proposition 3. For a LOTS $(X, \leq)$, the following are equivalent.
(a) $\alpha(X)$ is a linearly ordered compactification of $(X, \leq)$.
(b) One of the following (i), (ii), and (iii) holds.
(i) X has no gaps, and there exists $minX$, but no $MaxX$.
(ii) X has no gaps, and there exists $MaxX$, but no $minX$.
(iii) X has only one gap, and there exist $minX$ and $MaxX$.
(c) $\alpha(X) = X^{+}$.

Remark 5. The linearly ordered extension for $\alpha(X)$ in Proposition 3 is essential (by Example 2 below).

Example 2. Let $N = \{1, 2, \ldots\}$. Let $\mathbf{N}$ be a LOTS $(N, \leq)$ with the usual order $\leq$. Let $X = (N, \preceq)$ be a LOTS, but the order $\preceq$ is defined as follows: $\ldots \prec 4 \prec 2 \prec 1 \prec 3 \prec 5 \prec \ldots$. Then, $\alpha(\mathbf{N}) = \mathbf{N}^{+}$, but a linearly ordered compactification $\alpha(X)$ of $X = (N, \preceq)$ doesn't exist (by Proposition 3). While, $\mathbf{N} \cong X$, so $\alpha(\mathbf{N}) \cong \alpha(X)$, but $\mathbf{N}^{+} \ncong X^{+}$. Hence, $\alpha(X)$ is orderable, but $\alpha(X) \ncong X^{+}$.

Proposition 4. ([VRS]) Let Y be a space having a dense subset X. If Y is suborderable, then the following hold.
(1) If $|X| \geq \omega$, then the character $\chi(Y) \leq |X|$, and $|Y| \leq 2^{|X|}$.
(2) If X is connected, then Y is connected and $|Y - X| \leq 2$.

The following lemma is shown by refering to [E; 6.3.2].

Lemma 5. (1) Let X be a separable connected, compact space. If X is orderable, then X is homeomorphic to the closed unit interval $[a, b]$ in the Euclidean line $\mathbf{R}$.

(2) Let X be a separable connected space. If X is orderable, then X is homeomorphic to an interval of $\mathbf{R}$.

(3) Let X be a separable metrizable space. If X is suborderable, then X is homeomorphic to a subspace of $\mathbf{R}$.

Remark 6. (1) Not every separable compact LOTS is metrizable, also, not every compact connected LOTS is metrizable (by Example 1(4)).

(2) As is well-known, every separable suborderable space X is first countable, hereditarily separable, hereditarily Lindelöf, and $|X| \leq 2^{\omega}$.

Remark 7. (1) For a separable connected LOTS $(X, \leq)$, $X \cong \mathbf{R} \Leftrightarrow X$ has no Maxmal point and no minimal point $\Leftrightarrow X$ is a topological group.

(2) Let $(K, +, \times)$ be a *field*, here $(K, +)$ is an additive Abelian group, and $(K, \times)$ is a multiplicative Abelian group with respect to $K - \{0\}$. Then, K with a linearly order $\leq$ on K is called an *ordered field* if it is a LOTS $(K, \tau(\leq), \leq)$ satisfying: For any $a, b, c \in K$, $a < b \Rightarrow a + c < b + c$; and $a < b$ and $c > 0 \Rightarrow a \times c < b \times c$. An order field $(K, \leq)$ is *Archimedian* if, for each $a, b\ (> 0) \in K$, there exists $n \in N$ with $a < n \times b$. Every Archimedian order field is a separable metrizable LOTS, thus it is homeomorphic to a subspace of $\mathbf{R}$ (by Lemma 5(3)).

Let $(K, \tau(\leq), \leq)$ be an ordered field. For $x \in K$, define the *absolute value* $|x|$ by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$. Then, $\{V_\varepsilon(a) : a, \varepsilon \in K$ with $\varepsilon > 0\}$ is a base for the order topology $\tau(\leq)$, here $V_\varepsilon(a) = \{x \in K : |x - a| < \varepsilon\}$. For a function $f : K$ (or $[a, b] \subset K) \to K$, using absolute values, the following can be defined by the same way as in $\mathbf{R}$: f is *bounded, continuous, differentiable*, or *integrable*.

Let $K = (K, \tau(\leq), \leq)$ be an ordered field. Let us say that K is a *real number field* if it has no gaps (i.e., K is connected). As is well-known, every real number field is isomorhic, hence, homeomorphic to $\mathbf{R}$ (by (1)). We know many equivalent conditions for K to be $\mathbf{R}$ (for example, every upper bounded subset A of K has *sup* A). Besides, we have the following equivalences by means of cuts of K. Here, a *map* means a *continuous* function defined on a closed interval $[a, b]$ in K.

(Theorem): ([T2]) For an ordered field K, K is $\mathbf{R} \Leftrightarrow$ Any *map* to K is bounded and K is Archimedian $\Leftrightarrow$ Any *map* to $\mathbf{R}$ is bounded $\Leftrightarrow$ For any *map* f to K (or $\mathbf{R}$), $f([a, b])$ has the Maxmal (minimal) value $\Leftrightarrow$ For any *map* f to K (or $\mathbf{R}$), $f([a, b]) = [f(a), f(b)]$ if $f(a) \leq f(b) \Leftrightarrow$ Any differentiable *map* to K satisfies the Rolle's theorem $\Leftrightarrow$ Any bounded *map* to K is integrable.

Proposition 6. For a space X, the following are equivalent.
(a) X is a locally separable, metrizable, suborderable space.
(b) X is the topological sum of subspaces of $\mathbf{R}$.

Proposition 7. Let X be a separable connected space, and let $c(X)$ be a compactification of X. Then, (a) $\Leftrightarrow$ (b), and (b) $\Rightarrow$ (c) hold.
(a) $c(X)$ is orderable.
(b) $c(X) \cong [0, 1]$.
(c) X is homeomorphic to an interval of $\mathbf{R}$, and $|c(X) - X| \leq 2$

Remark 8. The implication (c) $\Rightarrow$ (a) (or (b)) in Proposition 7 doesn't hold. (In fact, put $c(\mathbf{R}) = \alpha(\mathbf{R})$, then $|c(\mathbf{R}) - \mathbf{R}| = 1$, but $c(\mathbf{R}) \cong S^1$ is not orderable (by Example 1(1))).

Lemma 8. ([Sh]) Let $(X,\leq)$ and $(Y,\preceq)$ be connected LOTS. For a homeomorphism $f : X \cong Y$, (a) or (b) below holds.

(a) For all $x, y \in X, x < y$ iff $f(x) \prec f(y)$.

(b) For all $x, y \in X, x < y$ iff $f(y) \prec f(x)$.

Theorem 9. ([Sh]) Let X be a connected LOTS, and let $c(X)$ be a compactification of X. Then the following are equivalent.

(a) $c(X)$ is orderable.

(b) $c(X) \cong X^+(= X \cup \{\pm\infty\})$

Remark 9. (1) The connectedness of X in Theorem 9 is essential. (In fact, for a case $c(X) = \alpha(X)$ (resp. $c(X) = \beta(X)$), see Example 2 (resp. Example 3(2))).

(2) Let $c(X)$ be a linearly ordered compactification of a connected LOTS X such that $|c(X) - X| \leq 2$. If $c(X) = \beta(X)$, then $c(X)$ is orderable (by Corollary 19), however, if $c(X) = \alpha(X)$, $c(X)$ need not be orderable (by Remark 8).

For the following lemma, refer to [GJ], [E], or [T1]. Recall that a space X *has countable tightness* (abbreviated $t(X) \leq \omega$) if, whenever $x \in clA$, there exists a countable subset C of A with $x \in clC$.

Lemma 10. (1) Let X be a normal space. If X is not countably compact, then $\beta(X) - X$ contains a copy of $\beta(\mathbf{N})$ as well as $\beta(\mathbf{N}) - \mathbf{N}$.

(2) $\beta(\mathbf{N})$ is neither hereditarily normal nor hereditarily countably paracompact, in particular, $\beta(\mathbf{N})$ is not orderable. Also, $|\beta(\mathbf{N})| = 2^c$ $(c = 2^\omega)$, and $t(\beta(\mathbf{N})) > \omega$.

Lemma 11. For a suborderable space X, as is known, the following hold.

(1) If X is countably compact, then X is sequentially compact.

(2) If $t(X) \leq \omega$, then X is first countable, thus, every countably compact subset is closed.

Proposition 12. ([VRS]) Let $\beta(X)$ be orderable. Then X is countably compact, hence sequentially compact.

Corollary 13. Let $\beta(X)$ be orderable. Then X is compact if (a) or (b) below holds. (For F-spaces and P-spaces, see [GJ]).

(a) $\beta(X)$ has countable tightness.

(b) X satisfies one of the following properties: Paracompact space; Realcompact space; Separable space; F-space; P-space.

For a GO-space $(X, \leq)$, define a subset $X^\sharp$ of $X^+ \times \{0, \pm 1\}$ by the following. Let $X^\sharp$ be a LOTS having the order topology defined by the lexicographic order on $X^\sharp$.

$X^\sharp = (X \times \{0\}) \cup \{\langle MaxA, 1\rangle : (A|B)$ is a *pseudo-gap* of X having $MaxA\} \cup \{\langle minB, -1\rangle : (A|B)$ is a *pseudo-gap* of X having $minB\} \cup \{\langle c, 1\rangle, \langle c, -1\rangle : c = (A|B)$ is a *gap* of $X\} \cup \{\langle \pm\infty, 0\rangle\}$.

Namely, $X^\sharp = \widetilde{X} \cup \{\langle c, 1\rangle, \langle c, -1\rangle : c = (A|B)$ is a *gap* of $X\} \cup \{\langle \pm\infty, 0\rangle\}$. Obviously, if X has no gaps, $X^\sharp = X^+$. If X has a gap, then $X^\sharp$ is not *minimal* (in the sense of Remark 2).

Proposition 14. Let $(X, \leq)$ be a GO-space. Then the following hold.

(1) $X^\sharp$ and X^+ are linearly ordered compactfications of X, as well as $\widetilde{X}$.

(2) $X^\sharp$ is connected $\Leftrightarrow \beta(X)$ is connected $\Leftrightarrow X$ is connected. While, X^+ is connected $\Leftrightarrow X$ has no jumps and no pseudo-gaps.

(3) $X^\sharp$ is metrizable $\Leftrightarrow X$ is a separable metrizable space having at most countably many gaps $\Leftrightarrow X$ is a separable metrizable space with $|X^\sharp - X| \leq \omega$.

Lemma 15. For a countably compact GO-space $(X, \leq)$, the following (1) and (2) hold.

(1) For every continuous real-valued function f on X, there exist $a, b \in X$ with $a \leq b$ such that f is constant on $R_b = \{x \in X : x \geq b\}$, and on $L_a = \{x \in X : x \leq a\}$.

(2) Every continuous real-valued function f on X can be continuously extendable over $X^\sharp$ (hence, $\beta(X) \cong X^\sharp$).

(In fact, for (1), assuming X has no Maximal point, we show that each real valued function f on X is constant on some R_b as in the poof of the Vickery's result on the ordinal space $[0, \omega_1)$ (see [D; p.81], etc.). For (2), note that for a cut $c = (A|B)$ of X, A and B are clopen in X (so, they are countably compact GO-spaces) if c is a gap, a pseudo-gap, or a jump. Then, using (1), we can define a continuous extension F of f over $X^\sharp$ naturally).

Theorem 16[1]. Let $(X, \leq)$ be a GO-space. Then following are equivalent.

(a) $\beta(X)$ is orderable.

(b) X is countably compact (equivalently, sequentially compact).

(c) $\beta(X) \cong X^\sharp$.

(d) $\beta(X)$ is a linear ordered compactification of X.

(e) $\beta(\widetilde{X})$ is orderable with $\beta(X) \cong \beta(\widetilde{X})$.

(f) $\beta(X) \cong \beta(\widetilde{X}) \cong X^\sharp$.

[1]S. Purisch [P] (resp. R. Kaufman [Ka]) has already proved that the equivalence (a) $\Leftrightarrow$ (b) for a GO-space (resp. LOTS) holds by a different proof.

Corollary 17. For a GO-space X, let $R(X) = \beta(X) - X$ be the remainder of $\beta(X)$. Then the following are equivalent.

(a) $\beta(X)$ is orderable.

(b) $R(X)$ is suborderable.

(c) $R(X)$ is hereditarily normal.

(d) $R(X)$ is hereditarily countably paracompact.

(e) $R(X)$ contains no copy of $\beta(\mathbf{N})$.

Corollary 18. For a GO-space X, $\beta(X)$ is orderable if $R(X)$ satisfies one of the following properties: $|R(X)| < 2^c$; $t(R(X)) \leq \omega$; Each point of $R(X)$ is a G_δ-set in $R(X)$.

Corollary 19. For a conected LOTS X, the following are equivalent.

(a) $\beta(X)$ is orderable.

(b) $|R(X)| \leq 2$.

(c) $\beta(X) \cong X^+ (= X \cup \{\pm\infty\})$.

Corollary 20. For a GO-space X, $\beta(X) \cong X^+ \Leftrightarrow X$ is a countably compact space with $X^\sharp \cong X^+$.

Corollary 21. For GO-spaces $(X, \leq)$ and $(Y, \preceq)$ with $X \cong Y$, if X is countably compact, then $X^\sharp \cong Y^\sharp$.

Remark 10. (1) In Theorem 16, even if $\beta(\widetilde{X})$ is orderable with $\beta(\widetilde{X}) \cong X^\sharp$, $\beta(X)$ need not be orderable (by Example 3(1)). Note that, for a GO-space Y, if $\beta(Y)$ is orderable, then so is $\beta(\widetilde{Y})$, but the converse doesn't hold.

(2) In Corollary 17, we can replace " $R(X)$ " by " $\beta(X)$ ". We can't omit " hereditarily " in (c) and (d). (In fact, let X be a GO-space which is locally compact, in particular, connected, but X is not countably compact).

(3) (i) Related to Corollary 18, as a special case, the following holds [2].

For $|R(X)| = 1$, X is *orderable* iff $\beta(X)$ is orderable. But, for $|R(X)| = 2$, the " if " part need not hold (by Example 3(2)).

(ii) For Corollary 18, even if $|R(X)| = 2^c$, $\beta(X)$ need not be orderable. (In fact, let X be a GO-space which is separable, but not compact).

The author has the following question[3]: Is there a GO-space X such that $|R(X)| = 2^c$, but $\beta(X)$ is orderable (equivalently, X is countably compact) ?

(4) In Corollary 19, for the implications (a) $\Rightarrow$ (b) or (c), and (b) $\Rightarrow$ (c), the connectedness of X is essential (by Example 3(2)). Also, in Corollary 21, the countable compactness of X is essential (by Example 2).

---

[2]K. Miyazaki announced this fact (with a different proof).

[3]N. Kemoto gave an affirmative answer to this question (in general, for any cardinal κ with $cf\kappa \geq \omega_2$, there exists a countably compact GO-space X with $|R(X)| = \kappa$) in [K2].

Example 3. (1) Let $X = [0,1] \cup (2,3]$. Then X is a GO-space by the usual topology (also, X is orderable). Then, $\widetilde{X} = X^\sharp = X^+ = X \cup \{\langle 1,1\rangle\}$ is a compact LOTS. Thus, $\beta(\widetilde{X}) = X^\sharp$ is orderable. But, $\beta(X)$ is not orderable (by Proposition 11).

(2) Let Ω be the Long line; that is, Ω is a space $(\Omega, \tau(\leq), \leq)$ obtained by replacing all jumps in the ordinal space $[0, \omega_1)$ by the closed intervals, where $\leq$ is the obvious order. Then Ω is a connected and countably compact LOTS, but Ω is neither separable nor compact. Also, $\alpha(\Omega) \cong \beta(\Omega) \cong \Omega^+ = \Omega \cup \{+\infty\}$ (by means of Lemma 15).

(i) Define $(-\Omega)$ by a LOTS $(\Omega, \tau(\leq'), \leq')$, but $\leq'$ is defined as follows: $x' <' x$ if $x < x'$. Let $\Sigma = (-\Omega) \cup \Omega$ be a LOTS defined by an order $\preceq$: $x \prec x'$ if $x <' x'$ in $(-\Omega)$, $y \prec y'$ if $y < y'$ in Ω, and $x \prec y$ if $x \in (-\Omega)$ and $y \in \Omega$. Then, Σ is a countably compact, connected space having no Maxmal point and no minimal point. Let T be the topological sum of $(\Sigma, \preceq)$ and a point $+\infty$. Let τ be the topology of the space T, and define the obvious order $\preceq'$ of T with the Maximal point $+\infty$. Then $(T, \tau, \preceq')$ is a countably compact GO-space which is not orderable, and $\beta(T) \cong T^+ = T \cup \{\pm\infty\} \cup \{\langle +\infty, -1\rangle\}$ (hence, $|R(T)| = 2$).

(ii) For $n \in N$ $(n \neq 1)$, let X be the topological sum of n many LOTS $(\Sigma, \preceq)$. Then X is a countably compact disconnected LOTS having gaps but no jumps. Then, X^+ is a connected space with $|X^+ - X| = n + 1$. While, $\beta(X)$ is a disconnected space with $|\beta(X) - X| = 2n$. Thus, $\beta(\Gamma)$ is orderable such that $|\beta(X) - X| = 2n$ $(|X^+ - X| = n + 1)$, but $\beta(X) \not\cong X^+$.

(iii) Let $\Gamma = \Omega \cup (-\Omega)$ be a LOTS defined by a similar way as Σ. Then Γ is a countably compact space having only one gap $\omega_1 = (\Omega | (-\Omega))$ and no jumps. Then, $\Gamma^+ = \Gamma \cup \{\omega_1\}$ is connected. While, $\beta(\Gamma) \cong \Gamma \cup \{\langle \Omega, \pm 1\rangle\}$ is disconnected. Hence, $\beta(\Gamma)$ is orderable, but $\beta(\Gamma) \not\cong \Gamma^+$.

Acknowlegement. The author would like to thank Professors N. Kemoto and T. Miwa for their valuable suggestions.

REFERENCES

[D] J. Dugundji, Topology, Allyn and Bacon, Inc., Boston, 1967.

[E] R. Engelking, General Topology, PWN-Polish Scientific Pub. Warszawa, 1977.

[GJ] L. Gillman and M. Jerison, Rings of continuous functions, van Nostrand, Princeton, 1960.

[Ka] R. Kaufman, Ordered sets and compact spaces, Colloquim Math., 17(1967), 35-39.

[K1] N. Kemoto, Normality of products of GO-spaces and cardinals, Topology Proc., 18(1993), 133-142.

[K2] N. Kemoto, personal communication.

[LiSaT], C. Liu, M. Sakai, and Y. Tanaka, Orderability of topological groups and biradial spaces, to appear in Questions and Answers in General Topology, 19(2001).

[L] D. J. Lutzer, On generalized ordered spaces, Dissertationes Math., Warszawa, 1971, 1-36.

[MK] T. Miwa and N. Kemoto, Linearly ordered extensions of GO-spaces, Topology and Appl., 54(1993), 133-140.

[N] J. Nagata, Moderan General Topology, North-Holland, Amsterdam, Newyork, Oxford, 1983.

[P] S. Purisch, On the orderability of Stone-Čech compactifications, Proc. Amer. Math. Soc., 1973, 55-56,

[Sh] T. Shinoda, Linearly ordered topological spaces and their generalization, Master-thesis (Tokyo Gakugei univ.) 2000, 1-55.

[T1] Y. Tanaka, On closedness of C- and C^*-embeddings, Pacific J. Math., 68(1977), 283-292.

[T2] Y. Tanaka, Ordered fields and the axiom of continuity, Bull. Tokyo Gakugei Univ., Sect. 4, 46(1994), 1-6.

[VRS] M. Venkataraman, M. Rajagopalan and T. Soundararajan, Orderable spaces, Gen. Topology and Appl., 2(1972), 1-10.

Department of Mathematics, Tokyo Gakugei University, Koganei, Tokyo, 184-8501, JAPAN

e-mail address: ytanaka@u-gakugei.ac.jp