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INEQUALITIES BETWEEN OPERATOR MEANS BASED ON THE MOND-PEČARIĆ METHOD AND ITS APPLICATIONS

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1. Introduction. This report is based on [17].

J.I.Fujii and E.Kamei [8] introduced the relative operator entropy $S(A|B)$ for positive operators A and B on a Hilbert space H as a relative version of the Nakamura-Umegaki operator entropy [15]:

$$S(A|B) = A^{\frac{1}{2}} \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

On the other hand, it is also expressed by

$$S(A|B) = \mathrm{s}-\lim_{\alpha\to 0}\frac{A \sharp_\alpha B - A \nabla_\alpha B}{\alpha} + B - A,$$

where $\sharp_\alpha$ is the weighted geometric mean and ∇_α is the weighted arithmetic mean. From point of veiw, they defined the following operator version of α-divergence in the differential geometry (cf. [6]): For positive operators A and B on H,

$$D_\alpha(A, B) \equiv \frac{1}{\alpha(1-\alpha)}(A \nabla_\alpha B - A \sharp_\alpha B) \quad (0 < \alpha < 1).$$

In particular,

$$D_1(A, B) \equiv \mathrm{s}-\lim_{\alpha\uparrow 1} D_\alpha(A, B) = A - B - S(B|A)$$

$$D_0(A, B) \equiv \mathrm{s}-\lim_{\alpha\downarrow 0} D_\alpha(A, B) = B - A - S(A|B).$$

For the case of $\alpha = 1/2$, it follows that α-operator divergence coinsides with by four times the difference of the geometric mean and the arithmetic mean. For the case of density operators, it coinsides with a relative entropy introduced by Beravkin and Staszewski [2] in C*-algebra setting.

In this paper, we shall consider the estimates of α-operator divergence by terms of the spectra of positive operators. For this purpose, we shall investigate the estimates of the difference of two operator means in general setting. We prove that for positive invertible operators A, B and a given $\alpha > 0$, there exists the most suitable real number β such that

$$\Phi(A \ \sigma_1 \ B) \geq \alpha\Phi(A) \ \sigma_2 \ \Phi(B) + \beta\Phi(A) \tag{1}$$

where Φ is a unital positive linear map and σ_1, σ_2 are operator means. In particular, if we put $\alpha = 1$ and Φ is the identity map in (1), then we have the lower bound of the difference of $A \ \sigma_1 \ B$ and $A \ \sigma_2 \ B$:

$$A \ \sigma_1 \ B - A \ \sigma_2 \ B \geq \beta A.$$

Consequently we obtain the estimates of α-operator divergence by terms of the spectra of positive operators.

2. A general theorem. Let $\Phi(\cdot)$ be a unital positive linear map from the space of $B(H)$ to $B(K)$, where $B(H)$ is the C*-algebra of all bounded linear operators on a Hilbert space H. Jensen's inequality asserts that if $f(t)$ is an operator concave function on an interval I, then

$$f(\Phi(A)) \geq \Phi(f(A))$$

for every selfadjoint operator A on a Hilbert space H whose spectrum is contained in I (cf. [3, 5]).

Mond and Pečarić [13, 14] established that the problem of determining the upper estimates of the difference and the ratio in Jensen's inequality is reduced to solving a single variable maximaization or minimization problem by using the concavity of $f(t)$, also see [12]. By using the Mond-Pečarić method, we show the following complimentary inequalities to Jensen's one.

Theorem 1. *Let A be a positive operator on H satisfying $M \geq A \geq m > 0$. Let $\Phi(\cdot)$ be a unital positive linear map from the space of $B(H)$ to $B(K)$. Let $f(t), g(t)$ be real valued continuous functions on $[m, M]$. Moreover let $f(t)$ be a concave function. Then for a given $\alpha > 0$*

$$\Phi(f(A)) \geq \alpha g(\Phi(A)) + \beta I$$

holds for $\beta = \beta(m, M, f, g, \alpha) = \min_{m \leq t \leq M}\{a_f t + b_f - \alpha g(t)\}$, where

$$a_f = \frac{f(M) - f(m)}{M - m} \quad \textit{and} \quad b_f = \frac{Mf(m) - mf(M)}{M - m}.$$

Proof. Put $h(t) = a_f t + b_f - \alpha g(t)$ and $\beta = \min_{m \le t \le M} h(t)$. Then it follows that

$$a_f t + b_f \ge \alpha g(t) + \beta \qquad \text{for } t \in [m, M].$$

Applying this inequality to $\Phi(A)$ we have

$$a_f \Phi(A) + b_f I \ge \alpha g(\Phi(A)) + \beta I.$$

On the other hand, since $f(t)$ is concave, by definition

$$f(t) \ge a_f t + b_f \qquad \text{for } t \in [m, M],$$

so that the inequality applied to A and then to $\Phi(\cdot)$ implies that

$$\Phi(f(A)) \ge a_f \Phi(A) + b_f I.$$

Combining these two inequalities we obtain

$$\Phi(f(A)) \ge \alpha g(\Phi(A)) + \beta I.$$

□

Remark 2. *If $g(t)$ is a strictly concave differentiable function on $[m, M]$, then a value of β in Theorem 1 may be determined more precisely as follows:*

$$\beta = a_f t_o + b_f - \alpha g(t_o),$$

where $t_o \in [m, M]$ is defined as the unique solution of $g'(t) = a_f/\alpha$ when $g'(M) \le a_f/\alpha \le g'(m)$, otherwise t_o is defined as M or m according as $a_f/\alpha \le g'(M)$ or $g'(m) \le a_f/\alpha$.

As an application of Theorem 1, we have the following corollary:

Corollary 3. *Let A be a positive operator on a Hilbert space H satisfying $mI \le A \le MI$ where $0 < m < M$. Let $\Phi(\cdot)$ be a unital positive linear map from the space of $B(H)$ to $B(K)$. Let p, q any real number $0 < p, q < 1$. Then for a given $\alpha > 0$*

$$\Phi(A^p) \ge \alpha \Phi(A)^q + \beta I$$

holds for $\beta = \beta(m, M, t^p, t^q, \alpha) =$

$$\begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\frac{M^p - m^p}{M - m}\right)^{\frac{q}{q-1}} + b_{t^p} & \text{if} \quad qm^{q-1} \ge \frac{1}{\alpha}\frac{M^p - m^p}{M - m} \ge qM^{q-1} \\ \min\{M^p - \alpha M^q, m^p - \alpha m^q\} & \text{otherwise.} \end{cases}$$

3. Operator means inequality. In this section, we shall study the estimates of the difference of two operator means related to a positive linear map by virtue of Theorem 1. We recall the Kubo-Ando theory of operator means [10]: A map $(A, B) \to A \ \sigma \ B$ in the cone of positive invertible operators is called an operator mean if the following conditions are satisfied:

monotonity: $A \le C$ and $B \le D$ imply $A \ \sigma \ B \le C \ \sigma \ D$,
upper continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \ \sigma B_n \downarrow A \ \sigma \ B$,
transformer inequality: $T^*(A \ \sigma \ B)T \le (T^*AT) \ \sigma \ (T^*BT)$ for every operator T,
normalized condition: $A \ \sigma \ A = A$.

The normalized condition is rarely assumed here. A key for the theory is that there is a one-to-one correspondence between an operator mean σ and a nonnegative operator monotone function $f(t)$ on $[0, \infty)$ through the formula

$$f(t) = 1 \ \sigma \ t \qquad (t > 0),$$

or

$$A \ \sigma \ B = A^{\frac{1}{2}}(1 \ \sigma \ A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

for all $A, B \ge \varepsilon > 0$. We say that σ has the representing function f. In this case, notice that $f(t)$ is operator monotone if and only if it is operator concave.

Simple examples of operator means are the weighted arithmetic mean ∇_p and the weighted harmonic mean $!_p$ $(0 < p < 1)$ defined by

$$A \ \nabla_p \ B = (1-p)A + pB \quad \text{and} \quad A \ !_p \ B = \left((1-p)A^{-1} + pB^{-1}\right)^{-1}$$

respectively. Another one is the geometric mean $\sharp$ which is just corresponding to the operator monotonity of the square root. As a matter of fact, the p-power mean (the weighted geometric mean) $\sharp_p$, $0 \le p \le 1$, are determined by the operator monotone function t^p;

$$A \ \sharp_p \ B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^p A^{\frac{1}{2}}$$

and the geometric mean $\sharp$ is defined as $A \ \sharp \ B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$.

Now, let Φ be a positive linear map from $B(H)$ to $B(K)$. Ando [1] showed that for a given operator mean σ

$$\Phi(A\ \sigma\ B) \leq \Phi(A)\ \sigma\ \Phi(B)$$

holds for every positive operator A and B. Related to this, we have the following results. Let f_1 and f_2 be representing functions for operator means σ_1 and σ_2 respectively. Then the following statements are mutually equivalent:

(i) $\Phi(A\ \sigma_1\ B) \leq \Phi(A)\ \sigma_2\ \Phi(B)$ for every positive invertible operator A, B.

(ii) $\Phi(f_1(A)) \leq f_2(\Phi(A))$ for every positive invertible operator A.

(iii) $f_1 \leq f_2$

Thus, if f_1 and f_2 are independent, then $\Phi(A\ \sigma_1\ B)$ and $\Phi(A)\ \sigma_2\ \Phi(B)$ have no relation on the usual order. By applying Theorem 1, we obtain our main results as follows:

Theorem 4. *Let Φ be a unital positive linear map from $B(H)$ to $B(K)$. Suppose that two operator means σ_1 and σ_2 have representing functions f_1 and f_2 respectively, which are not affine. Let A and B be positive invertible operators satisfying $M_1 \geq A \geq m_1 > 0$ and $M_2 \geq B \geq m_2 > 0$. Put $m = m_2/M_1$ and $M = M_2/m_1$. Then for a given $\alpha > 0$*

$$\Phi(A\ \sigma_1\ B) \geq \alpha\Phi(A)\ \sigma_2\ \Phi(B) + \beta\Phi(A) \tag{2}$$

where β is determined as the minimum of the function $a_{f_1}t + b_{f_1} - \alpha f_2(t)$ on $[m, M]$ with

$$a_{f_1} = \frac{f_1(M) - f_1(m)}{M - m} \quad \text{and} \quad b_{f_1} = \frac{Mf_1(m) - mf_1(M)}{M - m}.$$

Proof. By the same technique in [1], we consider the unital positive linear map Ψ by

$$\Psi(X) = \Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}.$$

Since the representing functions f_1, f_2 are nonnegative operator concave functions, it follows from Theorem 1 that for a given $\alpha > 0$

$$\Psi(f_1(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})) \geq \alpha f_2(\Psi(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})) + \beta I$$

holds for $\beta = \beta(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f_1, f_2, \alpha)$ in Theorem 1. Therefore we have

$$\begin{aligned}\Phi(A\ \sigma_1\ B) &= \Phi(A)^{\frac{1}{2}}\Psi(f_1(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))\Phi(A)^{\frac{1}{2}} \\ &\geq \Phi(A)^{\frac{1}{2}}(\alpha f_2(\Psi(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})) + \beta I)\Phi(A)^{\frac{1}{2}} \\ &= \alpha\Phi(A)\ \sigma_2\ \Phi(B) + \beta\Phi(A).\end{aligned}$$

□

Remark 5. *The value $\beta = \beta(m, M, f_1, f_2, \alpha) = a_{f_1}t_0 + b_{f_1} - \alpha f_2(t_0)$ can be writen explicitly as*

$$t_0 = \begin{cases} \text{the unique solution of } f_2'(t) = \frac{a_{f_1}}{\alpha} & \text{if } f_2'(M) \leq \frac{a_{f_1}}{\alpha} \leq f_2'(m) \\ M & \text{if } \frac{a_{f_1}}{\alpha} \leq f_2'(M) \\ m & \text{if } f_2'(m) \leq \frac{a_{f_1}}{\alpha} \end{cases}$$

Remark 6. *If we put $\alpha = 1$ in (2) of Theorem 4, then we have the following:*

$$\Phi(A\ \sigma_1\ B) - \Phi(A)\ \sigma_2\ \Phi(B) \geq \beta\Phi(A)$$

holds for $\beta = a_{f_1}t_0 + b_{f_1} - f_2(t_0)$ and t_0 is defined as the unique solution of $f_2'(t) = a_{f_1}$ when $f_2'(M) \leq a_{f_1} \leq f_2'(m)$, otherwise t_0 is defined as M or m according as $a_{f_1} \leq f_2'(M)$ or $f_2'(m) \leq a_{f_1}$.

Further if we choose α such that $\beta = 0$ in (2) of Theorem 4, then we have the following corollary:

Corollary 7. *Assume that the conditions of Theorem 4 hold. Then*

$$\Phi(A\ \sigma_1\ B) \geq \min_{m \leq t \leq M}\{\frac{a_{f_1}t + b_{f_1}}{f_2(t)}\}\Phi(A)\ \sigma_2\ \Phi(B).$$

Corollary 8. *Let Φ be a unital positive linear map from $B(H)$ to $B(K)$. Let A and B be positive invertible operators satisfying $M_1 \geq A \geq m_1 > 0$ and $M_2 \geq B \geq m_2 > 0$. Put $m = m_2/M_1$ and $M = M_2/m_1$. Let $p, q \in (0,1)$ be given real numbers. Then for a given $\alpha > 0$*

$$\Phi(A\ \sharp_p\ B) - \alpha\Phi(A)\ \sharp_q\ \Phi(B) \geq \beta\Phi(A) \tag{3}$$

holds for $\beta = \beta(m, M, p, q, \alpha) =$

$$\begin{cases} \alpha(q-1)\left(\alpha q \dfrac{M^p - m^p}{M-m}\right)^{\frac{q}{1-q}} + \dfrac{Mm^p - mM^p}{M-m} & \text{if } \dfrac{m^{1-q}}{q} \leq \alpha \dfrac{M^p - m^p}{M-m} \leq \dfrac{M^{1-q}}{q} \\ \min\{M^p - \alpha M^q, m^p - \alpha m^q\} & \text{otherwise.} \end{cases}$$

Proof. This corollary follows from Theorem 4 since the representing function of the p-power mean $\sharp_p$ and the q-power mean $\sharp_q$ are $f_1(t) = t^p$ and $f_2(t) = t^q$ respectively. □

Following after [9], for a symmetric mean σ, a parametrized operator mean σ_t is called an interpolational path for σ if it satisfies

(1) $A\ \sigma_o\ B = A$, $A\ \sigma_{1/2}\ B = A\ \sigma\ B$ and $A\ \sigma_1\ B = B$

(2) $(A\ \sigma_p\ B)\sigma(A\ \sigma_q\ B) = A\ \sigma_{\frac{p+q}{2}}\ B$

(3) the map $t \to A\ \sigma_t\ B$ is norm continous for each A and B.

For example, it is easy to see that the p-power mean $\sharp_p$ is an interpolational path for a geometric mean $\sharp$, so $A\sharp_p B$ is called the geometric interpolation. Corach, Porta and Recht [4] pointed out that the geodesic from A to B is the path $A\ \sharp_p\ B$ for the Finsler metric with the distance $d(A, B) = \|\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|$ and the relative operator entropy $S(A|B)$ is the velocity vector of $A\ \sharp_p\ B$ at $p = 0$. Moreover, J.I.Fujii [7] showed that the path $A\ \nabla_p\ B$ is the geodesic from A to B for the distance $d(A, B) = \|A - B\|$. It easily follows that $A\sharp_p B$ and $A\sharp_q B$ have no order relation for $p \neq q$. By virtue of Corollary 8, we obtain the estimation of the difference of the geometric interpolation $A\ \sharp_p\ B$:

Corollary 9. *Let A and B be positive invertible operators satisfying $M_1 \geq A \geq m_1 > 0$ and $M_2 \geq B \geq m_2 > 0$. Put $m = m_2/M_1$ and $M = M_2/m_1$. Let $p, q \in (0,1)$ be given real numbers. Then*

$$-\beta' A \geq A\ \sharp_p\ B - A\ \sharp_q\ B \geq \beta A \tag{4}$$

hold for $\beta = \beta(m_2/M_1, M_2/m_1, p, q, \alpha = 1)$ and $\beta' = \beta(m_1/M_2, M_1/m_2, q, p, \alpha = 1)$, which are defined in Corollary 8.

Proof. If we put $\alpha = 1$ and Φ is the identity map in (3) of Corollary 8, then we have the right-hand sides of (4). Moreover, when the substitutions $p \to q$ and $q \to p$ are made in (3) of Corollary 8, we have the left-hand sides of (4). □

The following corollary obtain the estimate of the difference of two paths $A \nabla_p B$ and $A \sharp_p B$:

Corollary 10. *Let A and B be positive invertible operators satisfying $M_1 \geq A \geq m_1 > 0$ and $M_2 \geq B \geq m_2 > 0$. Put $m = m_2/M_1$ and $M = M_2/m_1$. Let $p \in (0,1)$ be a given real number. Then*

$$\max\{1-p+pm-m^p, 1-p+pM-M^p\}A \geq A \nabla_p B - A \sharp_p B \geq 0.$$

Proof. It follows that

$$x^p - (1-p+px) \geq \begin{cases} m^p - (1-p+pm) & \text{if } p \leq \frac{M^p-m^p}{M-m}, \\ M^p - (1-p+pM) & \text{if } p \geq \frac{M^p-m^p}{M-m}. \end{cases}$$

Put $\beta = \max\{1-p+pm-m^p, 1-p+pM-M^p\}$. Then we have

$$\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p - \left((1-p) + pA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \geq -\beta.$$

Therefore, it follows that $A \sharp_p B - A \nabla_p B \geq -\beta A$. $\square$

4. α-operator divergence. As applications, we obtain the estimates of α-operator divergence. Since $A \nabla_\alpha B \geq A \sharp_\alpha B$ $(0 \leq \alpha \leq 1)$, it follows that α-operator divergence is positive, that is, $D_\alpha(A,B) \geq 0$. By corollary 10, we obtain the upper bound of α-operator divergence.

Theorem 11. *Let A and B be positive invertible operators satisfying $0 < m_1 I \leq A \leq M_1 I$ and $0 < m_2 I \leq B \leq M_2 I$. Put $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$. Then*

$$\max\{\frac{1-\alpha+\alpha m - m^\alpha}{\alpha(1-\alpha)}, \frac{1-\alpha+\alpha M - M^\alpha}{\alpha(1-\alpha)}\}A \geq D_\alpha(A,B) \geq 0.$$

Corollary 12. *Let A and B be positive invertible operators satisfying $0 < m_1 I \leq A \leq M_1 I$ and $0 < m_2 I \leq B \leq M_2 I$. Put $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$. Then*

$$\max\{m - 1 - \log m, M - 1 - \log M\}A \geq D_0(A,B) = B - A - S(A|B),$$

$$\max\{1 - m + m\log m, 1 - M + M\log M\}A \geq D_1(A,B) = A - B - S(B|A).$$

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