数理解析研究所講究録
1189 巻 2001 年 138-143

行列の HLAWKA 不等式

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1. INTRODUCTION

For arbitrary complex numbers x, y, z, the inequality

$$|x+y|+|y+z|+|z+x| \leq |x|+|y|+|z|+|x+y+z|$$

is well known as Hlawka's inequality. Djoković [2] proved the following inequalities which contain the above one as a special case:

$$\sum_{1\leq i_1<i_2<\cdots<i_k\leq n} |x_{i_1}+\cdots+x_{i_k}| \leq \binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n}|x_i|+\left|\sum_{j=1}^{n}x_j\right|\right) \tag{1}$$

for complex numbers $x_1, \ldots, x_n$ and $2 \leq k \leq n$.

In this paper, we pay attention to the special case $k=n-1$ of (1), namely

$$\|\mathbf{a}-\operatorname{Tr}\mathbf{a}\|_1 \leq \|\mathbf{a}\|_1+(n-2)|\operatorname{Tr}\mathbf{a}| \tag{2}$$

for a vector $\mathbf{a}=(a_1, \ldots, a_n)$ in $\mathbb{C}^n$ $(n \geq 2)$, where $\|\cdot\|_1$ is ℓ_1 norm on $\mathbb{C}^n$ and $\operatorname{Tr}\mathbf{a}=\sum_{i=1}^{n} a_i$. A weighted extension of the inequality (2) is known and is stated as follows:

Proposition 1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $x_1, x_2, \ldots, x_n \in \mathbb{C}$ $(n \geq 2)$. Then*

$$\sum_{i=1}^{n}\alpha_i|x_i-\sum_{j=1}^{n}\alpha_j x_j| \leq \beta\sum_{i=1}^{n}\alpha_i|x_i|+(\sum_{i=1}^{n}\alpha_i-2\alpha)|\sum_{j=1}^{n}\alpha_j x_j|, \tag{3}$$

where $\alpha=\min\{\alpha_i : \alpha_i>0\}$ and $\beta=\max\{2\alpha-1, 1\}$.

In view of the inequality (2), it might be natural to write the matrix version of Hlawka's inequality as follows:

$$\|A-\operatorname{Tr}A\|_1 \leq \|A\|_1+(n-2)|\operatorname{Tr}A| \tag{4}$$

for a complex $n\times n$ matrix A, where $\operatorname{Tr}A$ is the trace of A and $\|A\|_1$ is the trace norm of A, i.e. $\|A\|_1=\operatorname{Tr}|A|$ with $|A|=(A^*A)^{1/2}$. In this paper we prove the inequality (4) in a more general form stated in (5) of the following theorem. Indeed, the inequality (5) is not only a matrix extension of (3) but also a weighted extension of (4).

Theorem 2. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let A, B be complex $n\times n$ matrices. If $AB=BA$ and $B \geq 0$, then*

$$\operatorname{Tr}(B|A-\operatorname{Tr}BA|) \leq \max\{2\gamma(B)-1, 1\}\operatorname{Tr}B|A|+(\operatorname{Tr}B-2\gamma(B))|\operatorname{Tr}BA|, \tag{5}$$

where $\gamma(B)$ denotes the minimum positive eigenvalue of B.

PROOF OF THEOREM 2

Proof. Since $B \geq 0$ and $AB = BA$, we can write $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with invertible B_1 after some unitary conjugation. So it is enough to assume that B is invertible.

By approximation, we may assume that $\operatorname{Tr} B \neq 1$. If we put $A_\varepsilon = A + \varepsilon I$, then $A_\varepsilon - \operatorname{Tr} BA_\varepsilon = A - \operatorname{Tr} BA + \varepsilon(1 - \operatorname{Tr} B)$ is invertible for small $\varepsilon > 0$. So it suffices to prove the case where $A - \operatorname{Tr} BA$ is invertible. Since $AB = BA$, there is a unique unitary matrix U such that $U(A - \operatorname{Tr} BA) = |A - \operatorname{Tr} BA|$ and $UB = BU$. Hence there exists a unitary matrix V and diagonal matrices D_B and D_U such that

$$B = V^* D_B V, \qquad U = V^* D_U V.$$

So we have

$$\begin{aligned} \operatorname{Tr}(B|A - \operatorname{Tr} BA|) &= \operatorname{Tr}(BU(A - \operatorname{Tr} BA)) \\ &= \operatorname{Tr}(V^* D_B D_U V(A - \operatorname{Tr} V^* D_B V A)) \\ &= \operatorname{Tr}(D_B D_U (VAV^* - \operatorname{Tr} D_B VAV^*)). \end{aligned}$$

In this way, we can suppose that $B = \operatorname{diag}(b_1, b_2, \dots, b_n)$ with $b_i > 0$ and $U = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$. Then, by using Proposition 1, we have

$$\begin{aligned} \operatorname{Tr}(B|A - \operatorname{Tr} BA|) &= |\sum_{i=1}^n e^{i\theta_i} b_i (A - \operatorname{Tr} BA)_{ii}| \\ &\leq \sum_{i=1}^n b_i |a_{ii} - \sum_{j=1}^n b_j a_{jj}| \\ &\leq \max\{2 \min b_i - 1, 1\} \sum_{i=1}^n b_i |a_{ii}| + (\sum_{i=1}^n b_i - 2 \min b_i) |\sum_{j=1}^n b_j a_{jj}| \\ &\leq \max\{2\gamma(B) - 1, 1\} \sum_{i=1}^n b_i |a_{ii}| + (\operatorname{Tr} B - 2\gamma(B)) |\operatorname{Tr} BA|. \end{aligned}$$

Moreover, note that B commutes with A and A^* so that $|BA| = B|A|$. Since

$$\sum_{i=1}^n |\langle Xe_i, e_i \rangle| \leq \operatorname{Tr} |X|$$

for a general matrix X with the canonical basis $e_1, \dots, e_n$ of $\mathbb{C}^n$, it follows that

$$\sum_{i=1}^n b_i |a_{ii}| = \sum_{i=1}^n |\langle BAe_i, e_i \rangle| \leq \operatorname{Tr} |BA| = \operatorname{Tr} B|A|.$$

Therefore, the desired inequality (5) is obtained. □

Remark. The inequality (5) fails to hold for some non-commuting pairs of matrices. For example, choose

$$A = \begin{pmatrix} 1 & 0.5 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Corollary 3. *For every $n \in \mathbb{N}$ with $n \geq 2$ and every complex $n \times n$ matrix A,*

$$\|A - \operatorname{Tr} A\|_1 \leq \|A\|_1 + (n-2)|\operatorname{Tr} A| \tag{6}$$

and

$$\|A\|_1 \leq \|A - \frac{1}{n-1}\operatorname{Tr} A\|_1 + \frac{n-2}{n-1}|\operatorname{Tr} A|. \tag{7}$$

Proof. The inequality (6) is a specialization of (5) to the case $B = I$. Replace A by $A - \frac{1}{n-1}\operatorname{Tr} A$ in (6) to obtain the inequality (7). $\square$

Let A be a complex $n \times n$ matrix with $n \geq 2$, and consider the function $f(t) = \|A - t\operatorname{Tr} A\|_1$ for $t \geq 0$. When $0 < t \leq 1$, it follows from the convexity of f and (6) that

$$\frac{f(t) - f(0)}{t} \leq \frac{f(1) - f(0)}{1} \leq (n-2)|\operatorname{Tr} A|.$$

Therefore,

$$\|A - t\operatorname{Tr} A\|_1 \leq \|A\|_1 + t(n-2)|\operatorname{Tr} A| \qquad (0 \leq t \leq 1).$$

This inequality is also obtained by putting $B = tI$ in (5). Similarly, from the convexity of f and (7), we can show that

$$\|A\|_1 \leq \|A - t\operatorname{Tr} A\|_1 + t(n-2)|\operatorname{Tr} A| \qquad (t \geq \tfrac{1}{n-1}).$$

3. Related inequalities

Let $M_n(\mathbb{C})$ be the space of complex $n \times n$ matrices. For $1 \leq p < \infty$ let $\|A\|_p$ denote the Schatten p-norm of $A \in M_n(\mathbb{C})$, i.e. $\|A\|_p = (\operatorname{Tr}|A|^p)^{1/p}$. Also, the operator norm of A is denoted by $\|A\|_\infty$.

In this section, we discuss some inequalities comparing $\|A - \operatorname{Tr} A\|_p$ with $\|A\|_p$. To get such inequalities, we introduce some norms $\|\cdot\|_{(1,1)}$ and $|||\cdot|||$ on $M_n(\mathbb{C})$ and determine their dual norms.

Define the norm $\|\cdot\|_{(1,1)}$ on $M_n(\mathbb{C})$ by

$$\|A\|_{(1,1)} = \|A - \operatorname{Tr} A\|_1 \qquad (A \in M_n(\mathbb{C})).$$

Note that the linear mapping $\Phi(A) = A - \operatorname{Tr} A$ on $M_n(\mathbb{C})$ has the inverse $\Phi^{-1}(A) = A - \frac{1}{n-1}\operatorname{Tr} A$, so $\|\cdot\|_{(1,1)}$ is actually a norm on $M_n(\mathbb{C})$. Consider the canonical duality $\langle A, B\rangle = \operatorname{Tr} AB$ for $A, B \in M_n(\mathbb{C})$. Then we have

$$\begin{aligned}\max\{|\langle A, B\rangle| : \|A\|_{(1,1)} \leq 1\} &= \max\{|\langle A, B\rangle| : \|A - \operatorname{Tr} A\|_1 \leq 1\} \\ &= \max\{|\langle A - \frac{1}{n-1}\operatorname{Tr} A, B\rangle| : \|A\|_1 \leq 1\} \\ &= \max\{|\langle A, B - \frac{1}{n-1}\operatorname{Tr} B\rangle| : \|A\|_1 \leq 1\} \\ &= \|B - \frac{1}{n-1}\operatorname{Tr} B\|_\infty.\end{aligned}$$

This says that the dual norm of $\|\cdot\|_{(1,1)}$ on $M_n(\mathbb{C})$ is equal to

$$\|A\|_{(1,1)}^* = \|A - \frac{1}{n-1}\operatorname{Tr} A\|_\infty. \tag{8}$$

We next define the norm $|||\cdot|||$ on $M_n(\mathbb{C})$ by

$$|||A||| = \|A\|_1 + (n-2)|\operatorname{Tr} A| \qquad (A \in M_n(\mathbb{C})).$$

Then the inequality (4) is rewritten as $\|A\|_{(1,1)} \le |||A|||$, so the dual form of (4) is given as

$$|||A|||^* \le \|A\|_{(1,1)}^*, \tag{9}$$

where $|||\cdot|||^*$ is the dual norm of $|||\cdot|||$ with respect to the canonical duality. Since $\|A\|_{(1,1)}^* = \|A - \frac{1}{n-1}\operatorname{Tr} A\|_\infty$ by (8), the above (9) is equivalent to

$$|||A - \operatorname{Tr} A|||^* \le \|A\|_\infty. \tag{10}$$

Now let us determine the dual norm $|||\cdot|||_1^*$. To do so, define a semi-norm on the direct sum $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ by

$$|||X \oplus Y|||_1 = \|X\|_1 + (n-2)|\operatorname{Tr} Y| \qquad (X, Y \in M_n(\mathbb{C})).$$

Clearly, the mapping $X \mapsto X \oplus X$ is isometric from $(M_n(\mathbb{C}), |||\cdot|||)$ into $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), |||\cdot|||_1)$. Let $A \in M_n(\mathbb{C})$ satisfy $|||A|||^* \le 1$. Then the norm of the functional $X \oplus X \mapsto \langle X, A\rangle$ on the subspace $\{X \oplus X : X \in M_n(\mathbb{C})\}$ of $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), |||\cdot|||_1)$ is equal to $|||A|||^*$. By the Hahn-Banach extension theorem, this can be extended to a linear functional φ on $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), |||\cdot|||_1)$ which has the norm ≤ 1, namely

$$|\varphi(X \oplus Y)| \le |||X \oplus Y|||_1 = \|X\|_1 + (n-2)|\operatorname{Tr} Y|. \tag{11}$$

On the other hand, it is obvious that we can choose matrices $B, C \in M_n(\mathbb{C})$ such that $\varphi(X \oplus Y) = \langle X, B\rangle + \langle Y, C\rangle$. Then (11) implies that B and C must satisfy $\|B\|_1^* = \|B\|_\infty \le 1$ and $|\langle Y, C\rangle| \le (n-2)|\operatorname{Tr} Y|$. The latter implies $C = \zeta I$ for some $|\zeta| \le n-2$, so we have $\langle X, A\rangle = \langle X, B\rangle + \langle X, \zeta I\rangle$ for every $X \in M_n(\mathbb{C})$. This means that $A = B + \zeta I$ for some $|\zeta| \le n-2$. Conversely, it is immediate to see that if $A = B + \zeta I$ with $\|B\|_\infty \le 1$ and $|\zeta| \le n-2$, then $|||A|||^* \le 1$. In this way, we conclude the following proposition.

Proposition 4.

$$\{A \in M_n(\mathbb{C}) \ : \ |||A|||^* \le 1\} = \{X + \zeta I : |\zeta| \le n-2,\ X \in M_n(\mathbb{C}),\ \|X\|_\infty \le 1\}.$$

Put $t := |||A - \operatorname{Tr} A|||^*$. From the above proposition, we have

$$\|\frac{1}{t}(A - \operatorname{Tr} A) - \zeta\|_\infty \le 1,$$

for some ζ with $|\zeta| \le n-2$. So we have

$$\min_{|\zeta|\le t} \|A - \operatorname{Tr} A - (n-2)\zeta\|_\infty \le |||A - \operatorname{Tr} A|||^*$$

Since the inequality (4) can be rewritten as $t \le \|A\|_\infty$, we have

$$\min_{|\zeta|\le \|A\|_\infty} \|A - \operatorname{Tr} A - (n-2)\zeta\|_\infty \le |||A - \operatorname{Tr} A|||^*.$$

This and (6) imply that

$$\|A - \operatorname{Tr} A\|_p \le (n-1)\|A\|_p \qquad (p = 1, \infty)$$

for all $A \in M_n(\mathbb{C})$. So the following is a consequence of the complex interpolation method (cf. [5, Appendix to IX.4]).

Proposition 5. *For every complex $n \times n$ matrix A with $n \ge 2$,*

$$\|A - \operatorname{Tr} A\|_p \le (n-1)\|A\|_p \qquad (1 \le p \le \infty). \tag{12}$$

As in the last part of Sect. 1 we can get

$$\|A - t\operatorname{Tr} A\|_p \le (tn - 2t + 1)\|A\|_p \qquad (1 \le p \le \infty,\ 0 \le t \le 1) \tag{13}$$

from (12) and the convexity of $t \mapsto \|A - t\operatorname{Tr} A\|_p$. According to (8) the dual form of (13) is given as

$$\|A - \frac{t}{tn-1}\operatorname{Tr} A\|_q \ge \frac{1}{tn-2t+1}\|A\|_q$$

for $1 \le q \le \infty$ and $0 \le t \le 1$ with $t \ne 1/n$. Rewriting this we obtain

$$\|A - t\operatorname{Tr} A\|_p \ge \frac{tn-1}{2tn-2t-1}\|A\|_p \qquad (1 \le p \le \infty,\ t \ge \tfrac{1}{n-1}).$$

Acknowledgment

The author would like to express his thanks to T. Ando and F. Hiai for their very helpful suggestions.

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