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Singular Values of Differences of Positive Semidefinite Matrices[1]

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1 Introduction

Let M_n be the space of $n\times n$ complex matrices. For simplicity we treat matrices here, but all our results hold for compact operators on a Hilbert space. Suppose $A, B\in M_n$ are positive semidefinite. We shall study the relations between the singular values of

$$A-B \quad \text{and} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

and those of

$$A-|z|B, \quad A+zB, \quad \text{and} \quad A+|z|B$$

where z is a complex number.

A norm $|||\cdot|||$ on M_n is called *unitarily invariant* if $|||UAV|||=|||A|||$ for all A and all unitary U, V. Every unitarily invariant norm is a symmetric gauge function of the singular values. See [3, 7]. We always denote the singular values of A by $s_1(A)\geq\cdots\geq s_n(A)$, and put $s(A)\equiv(s_1(A),\ldots,s_n(A))$. Familiar examples of unitarily invariant norms are the Ky Fan k-norms defined by $||A||_{(k)}=\sum_1^k s_j(A)$ and the Schatten p-norms: $||A||_p=(\sum_1^n s_j^p(A))^{1/p}$, $p\geq 1$. Note that $||\cdot||_\infty$ is just the operator (spectral) norm and $||\cdot||_2$ is the Frobenius norm.

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A unitarily invariant norm may be considered as defined on M_n for all orders n by the rule

$$|||A||| = \left|\left|\left| \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right|\right|\right|,$$

i.e., adding or deleting zero singular values does not affect the value of the corresponding symmetric gauge function.

Given a real vector $x = (x_i) \in \mathbb{R}^n$, rearrange its components as $x_{[1]} \geq \cdots \geq x_{[n]}$. For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^n$, if

$$\sum_1^k x_{[i]} \leq \sum_1^k y_{[i]}, \quad k = 1, 2, \ldots, n,$$

then we say x is *weakly majorized* by y, denoted $x \prec_w y$. If the components of x and y are nonnegative and

$$\prod_1^k x_{[i]} \leq \prod_1^k y_{[i]}, \quad k = 1, 2, \ldots, n$$

we say x is *weakly log-majorized* by y, denoted $x \prec_{wlog} y$. See [6] for a discussion of this topic.

Denote the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ by $A \oplus B$. Bhatia and Kittaneh [4, Remark 5] observed that if $A, B \in M_n$ are positive semidefinite then

$$|||A - B||| \leq |||A \oplus B||| \tag{1.1}$$

for every unitarily invariant norm. By the Fan dominance principle [3, 7], (1.1) is equivalent to $s(A - B) \prec_w s(A \oplus B)$. We shall show that in fact each singular value of $A - B$ is not greater than the corresponding singular value of $A \oplus B$.

In another paper, Bhatia and Kittaneh [5, Thm 1] proved that for positive semidefinite $A, B \in M_n$ and any complex number z

$$|||A - |z|B||| \leq |||A + zB||| \leq |||A + |z|B||| \tag{1.2}$$

for all unitarily invariant norms. Again (1.2) is equivalent to

$$s(A - |z|B) \prec_w s(A + zB) \prec_w s(A + |z|B).$$

We shall prove that the corresponding weak log-majorizations hold. Since weak log-majorization implies weak majorization [6,7], our result strengthens (1.2).

2 Main Results

Our first result sharpens (1.1).

Theorem 1 *Let $A, B \in M_n$ be positive semidefinite. Then*

$$s_j(A - B) \leq s_j(A \oplus B), \quad j = 1, 2, \ldots, n. \tag{2.1}$$

The following result sharpens (1.2).

Theorem 2 *Let $A, B \in M_n$ be positive semidefinite. Then for any complex number z*

$$s(A - |z|B) \prec_{wlog} s(A + zB) \prec_{wlog} s(A + |z|B). \tag{2.2}$$

The special case $z = i = \sqrt{-1}$ of Theorem 2 says

$$s(A - B) \prec_{wlog} s(A + iB) \prec_{wlog} s(A + B). \tag{2.3}$$

It has been proved in [2] that for positive A, B and $p > 1$

$$s(A^p + B^p) \prec_w s((A + B)^p). \tag{2.4}$$

When $p \geq 2$, the above relation is refined as follows:

$$s(A^p + B^p) \prec_w s((A^2 + B^2)^{p/2}) \prec_w s(|A + iB|^p) \prec_{wlog} s((A + B)^p). \quad (2.5)$$

The first relation in (2.5) follows from (2.4) and the third relation follows from (2.3). To see the second relation let $T = A + iB$. This is the *Cartesian decomposition.* From $A^2 + B^2 = (T^*T + TT^*)/2$ we get

$$s(A^2 + B^2) \prec_w s(|A + iB|^2).$$

Note that $f(t) = t^{p/2}$ is convex and increasing on $[0, \infty)$. By a majorization principle [3, 7], applying this f to the preceding weak majorization yields the second relation in (2.5).

From (2.3) and the results in [1] and [2] it follows that for $0 < p \leq 1$,

$$\begin{aligned} s(A^p - B^p) \prec_w s(|A - B|^p) \prec_{wlog} s(|A + iB|^p) & \prec_{wlog} s((A + B)^p) \\ & \prec_w s(A^p + B^p). \end{aligned}$$

One might wonder whether the weak majorization (2.4) can be replaced by the stronger log-majorization. The answer is no, even for $p = 2$. Consider the example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We have $\det(A^2 + B^2) = 2 > 1 = \det[(A + B)^2]$.

Recently we have generalized Theorem 1 and the second majorization result in Theorem 2 to the case of τ-measurable operators affiliated with a semifinite von Neumann algebra.

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