

ノルムに到達するティープリッツ及びハンケル作用素の構造

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Let μ be the normalized Lebesgue measure on the Borel sets of the unit circle in the complex plane \mathbb{C} . If $e_n(z) = z^n$ for $|z| = 1$ and $n = 0, \pm 1, \pm 2, \dots$, then the bounded measurable functions e_n constitute an orthonormal basis for $L^2 = L^2(\mu)$. And the functions e_n , $n = 0, 1, 2, \dots$ constitute an orthonormal basis for H^2 . Let $H_0^2 = \{f \in H^2 : f(0) = 0\}$. L^∞ denotes the set of all essentially bounded measurable functions on the unit circle and $H^\infty = H^2 \cap L^\infty$. For a $\varphi \in L^\infty$ the Laurent operator L_φ is given by $L_\varphi f = \varphi f$ for $f \in L^2$ as the multiplication operator on L^2 . And the Laurent operator induces, in a natural way, twin operators on H^2 called the Toeplitz operator T_φ given by $T_\varphi f = PL_\varphi f$ for $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2 and the Hankel operator H_φ given by $H_\varphi f = J(I - P)L_\varphi f$ for $f \in H^2$ where J is the unitary operator on L^2 defined by $J(z^{-n}) = z^{n-1}$, $n = 0, \pm 1, \pm 2, \dots$.

次の2つの結果は昨年の数理解析の研究集会で示した (講究録 1137, 108-111) .

Theorem 1. 次は同値である.

- (1) $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2\} \neq \{0\}$ (i.e., T_φ is norm-achieved).
- (2) $\frac{\varphi}{\|T_\varphi\|} = g$ for some $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(H_g)$.

In this case, $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2\} = \mathcal{N}_{H_g}$ and it is invariant under T_z .

- (3) $\frac{\varphi}{\|T_\varphi\|} = \bar{q}h$ for some inner functions q and h such that q and h have no common non-constant inner factor.

Theorem 2. 次は同値である.

- (1) $\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\| \|f\|_2\} \neq \{0\}$ (i.e., H_φ is norm-achieved).
- (2) $\frac{\varphi}{\|H_\varphi\|} = g + \psi$ for some $\psi \in H^\infty$ and $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(T_g)$.

In this case, $\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\| \|f\|_2\} = \mathcal{N}_{T_g}$.

The following results are known.

Proposition 1. ([1]) Let \mathcal{M} be an invariant subspace of L_z . Then, in the case where $L_z \mathcal{M} = \mathcal{M}$, there exists a characteristic function χ_E of some subset E of the unit circle such that $\mathcal{M} = L_{\chi_E} L^2$ and, in the case where $L_z \mathcal{M} \subset \mathcal{M}$, there exists a unitary Laurent operator L_g uniquely, except a constant multiple of absolute value one, such that $\mathcal{M} = L_g H^2$.

Lemma 1. ([2]) If φ is non-analytic (i.e., $\varphi \notin H^\infty$), then the only invariant subspace of L_φ which contained in H^2 is $\{0\}$ itself.

Lemma 2. If φ is non-co-analytic (i.e., $\bar{\varphi} \notin H^\infty$), then the only invariant subspace of L_φ which contained in $L^2 \ominus H_0^2$ is $\{0\}$ itself.

Proof. Let

$$\mathcal{M} = \vee \{L_\varphi^{*n} f : f \in H_0^2, n = 0, 1, 2, \dots\}.$$

Then it is the smallest invariant subspace of L_φ^* which includes H_0^2 . Hence we have only to prove $\mathcal{M} = L^2$. Since L_z commutes with L_φ^* and since H_0^2 is invariant under L_z , \mathcal{M} is invariant under L_z .

If \mathcal{M} reduces L_z , then $\bar{z}^{n-1} = L_z^{*n} z \in \mathcal{M}$ ($n = 1, 2, \dots$) because $z \in H_0^2 \subseteq \mathcal{M}$ and hence $\mathcal{M} = L^2$.

If \mathcal{M} is a non-reducing invariant subspace of L_z , then $L_z \mathcal{M} \subset \mathcal{M}$ because L_z is unitary and, by Proposition 1, $\mathcal{M} = L_g H^2$ for some unitary Laurent operator L_g and $L_g L_\varphi^* H^2 = L_\varphi^* \mathcal{M} \subseteq \mathcal{M} = L_g H^2$ and hence $L_\varphi^* H^2 \subseteq H^2$. Since $1 \in H^2$, $\bar{\varphi} \in H^2$ and $\bar{\varphi} \in H^2 \cap L^\infty = H^\infty$. This contradicts the hypothesis that φ is non-co-analytic. \square

Theorem 3. ([2]) For a T_φ such as $\|T_\varphi\| = 1$, if

$$\{f \in H^2 : \|T_\varphi^n f\|_2 = \|f\|_2, n = 0, 1, 2, \dots\} \neq \{0\},$$

then T_φ is an isometry.

Proof. For a non-zero $f \in \{f \in H^2 : \|T_\varphi^n f\|_2 = \|f\|_2, n = 0, 1, 2, \dots\}$, we have $\|f\|_2 = \|PL_\varphi f\|_2 \leq \|L_\varphi f\|_2 \leq \|f\|_2$ because $\|L_\varphi\| = \|T_\varphi\| = 1$. This implies that $T_\varphi f = PL_\varphi f = L_\varphi f$ and

$$\|f\|_2 = \|T_\varphi^2 f\|_2 = \|T_\varphi L_\varphi f\|_2 = \|PL_\varphi^2 f\|_2 \leq \|L_\varphi^2 f\|_2 \leq \|f\|_2$$

and hence $T_\varphi^2 f = PL_\varphi^2 f = L_\varphi^2 f$. Similarly, we have $T_\varphi^n f = PL_\varphi^n f = L_\varphi^n f$ for all $n \geq 0$.

Let $\mathcal{N} = \vee\{L_\varphi^n f : n = 0, 1, 2, \dots\}$. Then $\mathcal{N} \neq \{o\}$ is an invariant subspace of L_φ contained in H^2 and, by Lemma 1, φ is analytic, i.e., $\varphi \in H^\infty$. Since, by Theorem 1, $\varphi = \bar{q}h$ for some inner functions q and h such that q and h have no common non-constant inner factor, $h = q\varphi$ and $q = e^{i\theta_0}1$ for some $\theta_0 \in [0, 2\pi)$ and hence $\varphi = e^{-i\theta_0}h$ is inner. \square

Corollary 1. ([2]) For a non-constant function φ in L^∞ , if T_φ is a contraction (i.e., $\|T_\varphi\| \leq 1$), then it is completely non-unitary.

Proof. It is known that the unitary part of T_φ is the restriction of T_φ on

$$\mathcal{H}_{T_\varphi}^{(u)} = \{f \in H^2 : \|T_\varphi^n f\|_2 = \|T_\varphi^{*n} f\|_2 = \|f\|_2, n \geq 0\}.$$

Hence we have only to prove $\mathcal{H}_{T_\varphi}^{(u)} = \{o\}$. If $\mathcal{H}_{T_\varphi}^{(u)} \neq \{o\}$, then $\varphi \in H^\infty \cap \overline{H^\infty} = \{\mathbb{C}1\}$ by Theorem 3. \square

The matrix representation $(a_{i,j})_{i,j=0}^\infty$ of H_φ where $a_{i,j} = \langle H_\varphi z^j, z^i \rangle$ by the basis $\{z^n : n \geq 0\}$ is determined by $\{a_{i,0} : i \geq 0\}$ and we remark here $\sum_{i=0}^\infty a_{i,0} z^i = H_\varphi 1$. And hence we have the following.

Theorem 4. $H_\varphi = O$ if and only if $H_\varphi 1 = o$.

It is known that there is no invertible Hankel operator. Moreover, we have the following.

Corollary 2. There is no isometric Hankel operator.

Proof. If $H_\varphi^* H_\varphi = I$, then

$$H_\varphi^* H_\varphi = I = T_z^* T_z = T_z^* H_\varphi^* H_\varphi T_z = H_\varphi^* T_z T_z^* H_\varphi$$

and $H_\varphi^*(I - T_z T_z^*)H_\varphi = O$ and hence $(I - T_z T_z^*)H_\varphi = O$ because $I - T_z T_z^*$ is a projection. And then $H_\varphi H^2 \subseteq T_z H^2$ and $\langle H_\varphi^* 1, H^2 \rangle = \langle 1, H_\varphi H^2 \rangle = 0$. Therefore $H_\varphi^* 1 = H_\varphi^* 1 = o$ and, by Theorem 4, $H_\varphi^* = H_{\varphi^*} = O$ which contradicts $H_\varphi^* H_\varphi = I$. \square

Theorem 5. For a H_φ such as $\|H_\varphi\| = 1$, if

$$\{f \in H^2 : \|H_\varphi^n f\|_2 = \|f\|_2, n = 0, 1, 2, \dots\} \neq \{o\},$$

then H_φ is normal.

Proof. Since, by Theorem 2, $\varphi = g + \psi$ for some $\psi \in H^\infty$ and $g \in L^\infty$ such as $|g| = 1$ a.e. and $0 \in \sigma_p(T_g)$. Hence $H_\varphi = H_g$.

For a non-zero $f \in \{f \in H^2 : \|H_\varphi^n f\|_2 = \|f\|_2, n \geq 0\}$,

$$\|f\|_2 = \|J(I - P)L_g f\|_2 \leq \|f\|_2$$

because $\|L_g\| = \|g\|_\infty$ and $(I - P)L_g f = L_g f$ and hence $H_g f = JL_g f$. Since

$$\|f\|_2 = \|H_g^2 f\|_2 = \|(I - P)L_g J L_g f\|_2 \leq \|f\|_2,$$

$(I - P)L_g J L_g f = L_g J L_g f = J L_{\bar{g}^*} L_g f$ and $H_g^2 f = L_{\bar{g}^*} L_g f$. And

$$\|f\|_2 = \|H_g^3 f\|_2 = \|(I - P)L_g L_{\bar{g}^*} L_g f\|_2 \leq \|f\|_2.$$

Hence $(I - P)L_g L_{\bar{g}^*} L_g f = L_g L_{\bar{g}^*} L_g f$ and $H_g^3 f = J L_g L_{\bar{g}^*} L_g f$. Similarly, for all $n \geq 0$,

$$(I - P)(L_g L_{\bar{g}^*})^n L_g f = (L_g L_{\bar{g}^*})^n L_g f$$

$$\text{and } (I - P)J(L_{\bar{g}^*} L_g)^n f = J(L_{\bar{g}^*} L_g)^n f.$$

Let $\mathcal{N} = \vee\{L_{g\bar{g}^*}^n L_g f : n \geq 0\}$. Then $\mathcal{N} \neq \{o\}$ is an invariant subspace of $L_{g\bar{g}^*}$ contained in $L^2 \ominus H_0^2$ and, by Lemma 2, $g\bar{g}^*$ is co-analytic and hence $u = \bar{g}g^*$ is inner because $|\bar{g}g^*| = |g| |g^*| = 1$ a.e. Since $u^* u = \overline{g\bar{g}^*} \bar{g}g^* = 1$, $\bar{u} = u^* \in H^\infty$ and hence u is a constant of absolute value one because $u \in H^\infty \cap \overline{H^\infty} = \mathbb{C}1$. Therefore $g^* = e^{i\theta_0} g$ for some $\theta_0 \in [0, 2\pi)$ and we have the conclusion. \square

Corollary 3. If H_φ is a non-normal contraction, then it is completely non-unitary.

Proof. We have only to prove that $\mathcal{H}_{H_\varphi}^{(u)} = \{o\}$. If $\mathcal{H}_{H_\varphi}^{(u)} \neq \{o\}$, then H_φ is normal by Theorem 5. \square

We say that a bounded linear operator A on a Hilbert space \mathcal{H} is paranormal if $\|Ax\|^2 \leq \|A^2x\| \|x\|$ for all $x \in \mathcal{H}$.

Theorem 6. If T_φ is norm-achieved paranormal, then T_φ is a scalar multiple of an isometry.

Proof. We may assume that $\|T_\varphi\| = 1$. Let

$$\mathcal{M} = \{f \in H^2 : \|T_\varphi f\|_2 = \|f\|_2\}.$$

Then, by the hypothesis, $\mathcal{M} \neq \{o\}$ and $T_\varphi \mathcal{M} \subseteq \mathcal{M}$ by the paranormality of T_φ . In fact, if $f \in \mathcal{M}$, then we have

$$\|f\|_2^2 \geq \|f\|_2 \|T_\varphi^2 f\|_2 \geq \|T_\varphi f\|_2^2 = \|f\|_2 \|T_\varphi f\|_2 = \|f\|_2^2$$

and this implies that $\|T_\varphi^2 f\|_2 = \|T_\varphi f\|_2$ and hence $T_\varphi f \in \mathcal{M}$. Therefore

$$\{f \in H^2 : \|T_\varphi^n f\|_2 = \|f\|_2, n = 0, 1, 2, \dots\} \neq \{o\}$$

and T_φ is an isometry by Theorem 3. \square

Theorem 7. If H_φ is norm-achieved paranormal, then H_φ is normal.

Proof. We may assume that $\|H_\varphi\| = 1$. Let

$$\mathcal{M} = \{f \in H^2 : \|H_\varphi f\|_2 = \|f\|_2\}.$$

Then, by the hypothesis, $\mathcal{M} \neq \{o\}$ and, by the paranormality of H_φ , $H_\varphi \mathcal{M} \subseteq \mathcal{M}$. In fact, if $f \in \mathcal{M}$, then we have

$$\|f\|_2^2 \geq \|f\|_2 \|H_\varphi^2 f\|_2 \geq \|H_\varphi f\|_2^2 = \|f\|_2 \|H_\varphi f\|_2 = \|f\|_2^2$$

and $\|H_\varphi^2 f\|_2 = \|H_\varphi f\|_2$ and hence $H_\varphi f \in \mathcal{M}$. Therefore

$$\{f \in H^2 : \|H_\varphi^n f\|_2 = \|f\|_2, n = 0, 1, 2, \dots\} \neq \{o\}$$

and H_φ is normal by Theorem 5. □

References

- [1] Beurling, A., *On two problems concerning linear transformations in Hilbert space*, Acta Math., 81(1949), 239–255.
- [2] Yoshino, T., *Note on Toeplitz operators*, Tohoku Math. Journ., 26(1974), 535–540.