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SATO-WELTER GAME AND KAC-MOODY LIE ALGEBRAS

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ABSTRACT. This is a revised version of [11]. The purpose of this paper is to give a large class of impartial 2-player games which are "completely solvable" in the sense that they have good formulas or good algorithms for the Sprague-Grundy numbers. Nim and Sato-Welter game are included here as very special cases. The notion of minuscule elements of Weyl groups, due to D. Peterson, and the classification of them, due to P. A. Proctor, are essential in our construction.

1. BASICS ON GAMES

The reference for this section is Conway [2].

1.1. Game graphs
We consider complete information games played by two players. More precisely, we consider only those games which can be represented by a graph g as follows. A finite directed graph g with the following properties is called a *game graph*:

1. The graph g has no cycle, i.e. there is no sequence $v_1, v_2, ..., v_n$ of vertices such that $v_i \to v_{i+1}$ and that $v_n = v_1$ unless $n = 1$.
2. There exista a (necessarily unique) vertex v_g of g such that, for any vertex v of g, there exists a sequence $v_1, v_2, ..., v_n$ of vertices such that $v_i \to v_{i+1}$ and that $v_1 = v_g, v_n = v$.

Given such a graph g, two players can play a game as follows. Place a "stone" at the beginning position v_g. The first player moves the stone to any vertex v connected to v_g by an edge directed toward v. Similarly, the second moves the stone to any vertex w connected to v by an edge directed toward w, and so on. The player first unable to move is the loser. The game considered in the present paper is isomorphic to this game corresponding to a game graph g. We shall identify the game itself and the game graph representing it. It is not difficult to see that one of the players, the first one or the second one, has a winning strategy.

Remark. We can define more general games by considering *colors.* We assume each edge of the game graph g is colored by one of the two colors, red or blue, say. Before beginning the game two players choose their colors; if one of them chooses red, say,then the other must take the rest, blue. Red (resp. blue) player can play only red (resp. blue) moves. In [2] games with colors are called *partizan* and games without color *impartial.* Since all the games considered in this paper are impartial, we donot discuss about partizan games any more and refer the interested readers to [2].

1.2. Sums of games
Let a and b be game graphs. Then the *sum* $a+b$ of a and b is a new game graph defined as follows. The set of vertices of $a+b$ is the set of pairs (x,y) of vertices x of a and vertices y of b. Two vertices (x,y) and (x',y') of $a+b$ are connected by an edge directed toward (x',y') if and only if either $x \to x'$ and $y=y'$, or $x=x'$ and $y \to y'$. As a game $a+b$ can also be defined as follows. The first player of the game $a+b$ chooses either a or b and plays a first move in the chosen game. The second player also chooses a or b and plays a second or first move in the chosen game according as the second chooses the same game as the first or not.... At the end, one of the player will be unable to play anymore neither in a nor in b, which means that the player is the loser.

1.3. Energy of games
Let g be the game graph of a game. Let $\mathbf{N}_0$ be the set of non-negative integers. For each vertex $v \in g$ we attach its *energy* $E(v) \in \mathbf{N}_0$ as follows. Let $S_v = \{w_1.w_2, ..., w_k\}$ be the set of successors of v in g. We define:

$$E(v) = \ [\mathbf{N}_0 - \{E(w_i); w_i \in S_v\}] \ .$$

In particular, $E(v)=0$ if v is an ending position of g, i.e. if no edge goes outside from v. The energy $E(g)$ of a game g is defined by:

$$E(g) = E(v_g).$$

In general, for $v \in g$, the energy $E(v)$ of v is the energy of the game which is "generated" by v, i.e. the "sub"game of g whose beginning position is v. The importance of the notion of energy in impartial (i.e. uncolored) games will become apparent by the following:

Theorem 1. (R. P. Sprague, P. M. Grundy, see [2])

(i) *The second player has a winning strategy in the game g, if and only if*

$$E(g) = 0.$$

(ii) *Let a and b be game graphs. Then*

$$E(a+b) = E(a) \oplus E(b) ,$$

where, for $m, n \in \mathbf{N}_0$, $m \oplus n \in \mathbf{N}_0$ (called "nim sum" of m and n) is defined by the following rules:

$$\begin{aligned} m \oplus m &= 0 , \\ 2^k \oplus m &= 2^k + m, && \text{if } m < 2^k , \\ m \oplus 0 &= m , \\ m \oplus n &= n \oplus m && (\textit{commutativity}) , \\ (l \oplus m) \oplus n &= l \oplus (m \oplus n) && (\textit{associativity}) . \end{aligned}$$

If one wants to calculate the energy $E(g)$ of a game g directly from the definition of $E(g)$, then one needs the whole graph g. Hence, in most cases, the actual calculation is hopeless. Thus examples of games which have good formulas for their energy are very precious. In literatures [2], energy of a game g is called the Sprague-Grundy number or Grundy numbers of g. We have used the word "energy" deliberately to stress its importance.

2. Some examples of games

Here we collect some examples of games.

2.1. Nim ([2])

This is a very well-known game. For a non-negative integer k, let L_k be a totally ordered set with k elements. Then

$$L = \cup_i L_{k_i} \quad \text{(finite disjoint union)}$$

is a partially ordered set. A component L_{k_j} of L is called a *string*. The first player choose a string L_{k_j} and reduce it to L_{h_j} $(0 \leq h_j < k_j)$. The second player also choose a (non-empty) string (L_{k_i} $(i \neq j)$ or L_{h_j}) and reduce it to a strictly shorter string, and so on. In other words, if L_k denotes the (trivial) game with just one string L_k, then this game is the sum

$$L_{k_1} + L_{k_2} + \ldots$$

of games L_{k_i} in the sense explained in 1.2. It is easy to see that

$$E(L_k) = k.$$

Hence, by Theorem 1, the energy of the game L is given by

$$E(L) = k_1 \oplus k_2 \oplus \ldots.$$

2.2. Sato-Welter game ([2][8][10])

In 1950's, M. Sato [8] and C.P. Welter [10] independently built a beautiful theory for the following game.

We have finitely many particles. Each particle is placed in one of the energy levels labelled by $\{0, 1, 2, 3, \ldots\}$. We assume a 'physical law' asserting that different particles can never be in the same energy lebel. Each player, in its turn, chooses a particle and reduces its energy to a strictly smaller level. Of course, if an energy level is already occupied by a particle, then no other particle can move into that energy level. If a player finds, in its turn, that it can move no particle any more (this is equivalent to say that the energy levels $0, 1, 2, \ldots, n-1$, with $n =$the number of particles, are already occupied), then that player is the loser.

For our purpose, it is essential to notice that Sato-Welter game can also be played on Young diagrams.

Let Y be a Young diagram. This means Y is a finite subset of $\mathbf{N} \times \mathbf{N}$ such that

$$Y = \cup_{i=1}^{l} \{(i,j);\ 1 \leq j \leq n_i\} ,$$

with

$$n_1 \geq n_2 \geq \ldots \geq n_l \geq 0 .$$

(If $n_i = 0$, then the i-th row of Y is empty.) The diagram Y can be viewed as a partially ordered set by

$$(i,j) \geq (i',j') \quad \text{if } i \leq i' \text{ and } j \leq j' .$$

In particular, $(1,1)$ is the unique maximum element of Y if Y is nonempty. For any $(i,j) \in Y$, the subset

$$H(i,j) = \{(k,l) \in Y;\ k = i \text{ or } l = j, (k,l) \leq (i,j)\}$$

of Y is called the *hook* of Y at (i,j), and the number $|H(i,j)|$ of elements in $H(i,j)$ the *hook-length* at (i,j). We now define the '*removal of hooks and pushing up*' procedure well-known in the representation theory of symmetric groups. This gives a procedure to obtain, for a fixed element (i,j) of a given Young diagram Y, a smaller Young diagram Y'. If the set theoretical difference $Y - H(i,j)$ is again a Young diagram, then we simply put $Y' = Y - H(i,j)$. In general, we put

$$Y' = \{(k,l) \in Y;\ k < i \text{ or } l < j\} \cup \{(k-1,l-1);\ (k,l) \in Y,\ (k,l) \leq (i+1,j+1)\} .$$

The difference $H(i,j)_* = Y - Y'$ is called the *rim-hook* of Y at (i,j). Note that a rim-hook $I = H(i,j)_*$ is an *order ideal* of Y, i.e. $a \in I$ and $b < a$ implies $b \in I$. We now give the Young diagramatic formulation of Sato-Welter game :

Let Y be a given Young diagram. Each player, in its turn, chooses an element (i,j) of Y and removes the corresponding hook $H(i,j)$ and pushes up (or, equivalently, remove the corresponding rim-hook $H(i,j)_*$). At the end, the empty Young diagram will be left, and the last player is the winner.

Remark. It seems Sato's first formulation of the game was in terms of Young diagrams. (One of his motivation was Nakayama conjecture in the representation theory of symmetric groups.) This formulation does not appear in Welter [10] nor in Conway[2].

Theorem 2. ([8][10], see also [2])
Let Y be a Young diagram. Then the energy $E(g_Y)$ of the Sato-Welter game g_Y played on Y is given by:

$$E(g_Y) = \sum_{(i,j)\in Y} \oplus\, N(|H(i,j)|) \ , \tag{2.1}$$

where $N(\ .\ )$ is the Sato-Welter norm defined by

$$N(k) = k \oplus (k-1) \ , \qquad k \in \mathbf{N} \ .$$

Remark. As Sato [9] remarked, the structure of the above formula for $E(g_Y)$ has a striking resemblanse to that of the "hook formula" for the dimensions of irreducible representations of symmetric groups. This observation has been crucial for the present work. See Theorem 6 below.

Remark. One can of course play Sato-Welter game on a disjoint union of a finite number of Young diagrams. (Nim is a very special case of this.) The energy of such multi-component Sato-Welter game can be calculated using Theorems 1 and 2.

2.3. Playing games on partially ordered sets
Let $\mathcal{P}$ be a collection of (isomorphism classes of) finite partially ordered sets. Suppose, for any $P \in \mathcal{P}$, and any $p \in P$, an order ideal $H(p)_*$ ($\neq \emptyset$) of P, called the *rim-hook* of P at p, is assigned in such a way that $P - H(p) \in \mathcal{P}$. Then just as in the case of Sato-Welter game (Young diagramatic formulation), we can consider a game g_P played on P; two players alternatively remove rim-hooks.

3. The parity condition

3.1. Nim sum for integers
Let $m \in \mathbf{N}$. We formally put

$$-m = (-1) \oplus (m-1) \ ,$$

and assume

$$(-1) \oplus (-1) = 0 .$$

This enables us to extend the nim sum $\oplus$ to the whole set $\mathbf{Z}$ of integers. For $m, n \in \mathbf{Z}$, we put

$$(m|n) = m \oplus n \oplus (m \oplus n - 1) = N(m \oplus n) = N(m - n).$$

Theorem 3. *Let $a_1, a_2, ..., a_n$ be distinct elements of $\mathbf{Z}$. We define a $\mathbf{Z}$-valued function f on $\mathbf{Z}$ by*

$$f(t) = \sum_{i=1}^{n} \oplus\ (t|a_i), \quad t \in \mathbf{Z} .$$

Then, f cannot be constant.

3.2. Games satisfying the parity condition

Let g be a game graph. Let $v_1, v_2, ..., v_k$ be vertices of g directly concted to the beginning position v_g by edges in g. For $n \in \mathbf{N}_0$, we put

$$a_n = |\{1 \leq i \leq k \ ; \ E(v_i) = n\}| .$$

Then the definition of the energy $E(g) = E(v_g)$ implies:

$$a_n \neq 0 \text{ if } n < E(g), \text{ and } a_n = 0 \text{ if } n = E(g).$$

We say that the game g satisfies the *parity condition* if

$$a_n \text{ is odd if } n < E(g), \text{ and } a_n \text{ is even if } n \geq E(g).$$

By Theorem 3, this condition is equivalent to say that g satisfies the following *parity equality*:

$$\sum_{i=1}^{k} \oplus\ (t|E(v_i)) = t \oplus (t - E(g)) . \tag{3.1}$$

If we put $t = E(g)$ in (3.1), then we have

$$E(g) = \sum_{i=1}^{k} \oplus\ (E(g)|E(v_i)) = \sum_{i=1}^{k} \oplus\ N(E(g) - E(v_i)) . \tag{3.2}$$

Compare (3.2) with (2.1).

4. P-GAMES

4.1. Miniscule elements of Weyl groups

Let W be the Weyl group of a Kac-Moody Lie algebra [4] with simply-laced Dynkin diagram. Let λ be a dominant integral weight. Following D. Peterson (unpublished; but see [1][6][7]), we say that an element w

of W is λ-*minuscule* if there exists a reduced expression $w = s_{i_l}...s_{i_2}s_{i_1}$ (s_i =the reflection associated to simple root α_i) of w such that

$$s_{i_j}(s_{i_{j-1}}...s_{i_1}\lambda) = (s_{i_{j-1}}...s_{i_1}\lambda) - \alpha_{i_j} .$$

We also say λ is *minuscule* if it is λ-minuscule for some λ. In his study of Schubert calculus in the Kac-Moody setting, Peterson proved the following "hook formula":

Theorem 4. (D. Peterson; see [1]) *Let $w \in W$ be minuscule. Then the number of reduced expressions of w is equal to*

$$l(w)! \prod_{\substack{\alpha>0 \\ w^{-1}(\alpha)<0}} h(\alpha)^{-1} ,$$

where $l(w)$ is the length of w, and $h(\alpha)$ is the height of the root α.

If W is of type A, then this gives an unusual way of stating the famous hook formula for the number of standard tableaux for a given Young diagram. See the last paragraph of 3.2 below. It is announced [6][7] that a q-analogue of the above theorem will appear in a forthcoming paper of Peterson and R. A. Proctor. See also [3].

4.2. Classification of minuscule elements of Weyl groups
The results in this subsection is due to R. A. Proctor [6][7].

Theorem 5. (R. A. Proctor) *The classification of minuscule elements in Weyl groups are equivalent to the classification of "d-complete posets". A d-complete poset can be explicitly described as a disjoint union of "slant sums" of "irreducible d-complete posets".*

See [6] and [7] for unexplained terminologies. In short, a *d*-complete poset is a finite partially ordered set P which satisfy, among other technical conditions, the following:

1. If $x, y, w \in P$ satisfy both $x \longleftarrow w$ (i.e. $x > w$ and no element of P lies between x and w) and $y \longleftarrow w$, then there exists a unique $v \in P$ satisfying both $v \longleftarrow x$ and $v \longleftarrow y$.
2. Assume, for $v, w \in P$ with $v > w$, the interval $[w, v] = \{a \in P;\ w \leq a \leq v\}$ looks like:

$$\begin{array}{ccccccccc} v & \longleftarrow & x & & & & & & \\ \uparrow & & \uparrow & & & & & & \\ y & \longleftarrow & w_k & \longleftarrow & \dots & \longleftarrow & w_2 & \longleftarrow & w_1 = w \end{array}$$

for some $k \geq 1$. Then there exists a unique sequemce $v_1 > v_2 > \ldots > v_k = v$ of elements of P such that the interval $[w_1, v_1]$ looks like:

$$\begin{array}{ccccccccccccc}
v_1 & \longleftarrow & v_2 & \longleftarrow & \ldots & \longleftarrow & v_k & \longleftarrow & x & & & & \\
 & & & & & & \uparrow & & \uparrow & & & & \\
 & & & & & & y & \longleftarrow & w_k & \longleftarrow \ldots \longleftarrow & w_2 & \longleftarrow & w_1 = w\ .
\end{array} \tag{4.1}$$

The partially ordered set (4.1) is called a double-tailed diamond. Rooted trees, Young diagrams, shifted Young diagrams (see [5]) and double-tailed diamonds are classical examples of d-complete posets. A non-classical example (which includes diamonds as special cases) is given in 3.4 below. See [7] for a lot of more exotic examples.
In the classification of d-complete posets P, we can assume P is connected with respect to the relation $\longleftarrow$. Then there exists a unique subset T of P called the "top tree" of P (see [7] for the definition), which should be considered as a simply-laced Dynkin diagram endowed with an extra structure of rooted tree. In the case P is a rooted tree, its top tree coincides with P itself. In the case of a double-tailed diamond (4.1), the top tree is:

$$\begin{array}{ccccccccc}
v_1 & \longleftarrow & v_2 & \longleftarrow & \ldots & \longleftarrow & v_k & \longleftarrow & x\ , \\
 & & & & & & \uparrow & & \\
 & & & & & & y & &
\end{array}$$

which is a Dynkin diagram of type D. In the case of a Young diagram:

$$\begin{array}{cccccccc}
v_m & \longleftarrow & v_{m+1} & \longleftarrow & \ldots & & \ldots \longleftarrow & v_n \\
\uparrow & & \uparrow & & \uparrow & & \vdots & \\
v_{m-1} & \longleftarrow & a & \longleftarrow & b & & \ldots. & \\
\uparrow & & \uparrow & & \vdots & & & \\
\vdots & .. & & .. & & .. & & \\
\uparrow & & & & & & & \\
v_2 & & & & & & & \\
\uparrow & & & & & & & \\
v_1 & & & & & & &
\end{array} \tag{4.2}$$

the top tree is:

$$v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_{m-1} \longrightarrow v_m \longleftarrow v_{m+1} \longleftarrow \ldots \longleftarrow v_n\ ,$$

which is a Dynkin diagram of type A. Let $W = W(T)$ be the Weyl group of the Kac-Moody Lie algebra with Dynkin diagram T, and

S the set of simple reflections of W. Then elements of T are in 1-1 correspondence with elements of S. This extends uniquely to a correspondence

$$\Psi : P \longrightarrow S$$

with the following property : If $v_1 > w_1$ ($\in P$) are such that the interval $[w_1, v_1]$ looks like (4.1) for some $k \geq 1$, then, for any i, $\Psi(v_i) = \Psi(w_i)$. (So, e.g. in (4.2), we have $\Psi(a) = \Psi(v_m), \Psi(b) = \Psi(v_{m+1})$.) Note that, for any $s \in S$, $\Psi^{-1}(s)$ is a totally ordered subset of P.

We now fix a total order on P compatible with the original partial order. In the case of a double-tailed diamond (4.1), the number of ways fixing such a total order is 2. In the case of a Young diagram, the number of ways is equal to the number of corresponding standard tableaux, or to the dimension of the corresponding irreducible representation of symmetric groups. We now pick elements $s_{i_1}, s_{i_2}, \ldots$ of S corresponding to elements $p_1, p_2, \ldots$ of P in the order fixed above, and take the product

$$w = s_{i_l} \ldots s_{i_2} s_{i_1}, \tag{4.3}$$

where $l = |P|$. Then w is a minuscule element of W independent of the choice of the total order on P, and (4.3) is a reduced expression of w. See Theorem 4 and Theorem 5.

4.3. Definitions of hooks and P-games

Let P be a d-complete poset. We are going to introduce a game g_P (which we call the P-*game*) played on P. For that purpose, it is enough to define, for each $p \in P$, the corresponding 'rim-hook' $H(p)_* \subset P$ in such a way that $H(p)_*$ is an order ideal of P. (Then the difference $P - H(p)_*$ is again a d-complete poset.) See 2.3.

We can assume P is connected. Let T be the top tree, and $\Sigma = \Sigma(T)$ and $W = W(T)$ the corresponding root system and Weyl group. Let

$$w = s_{i_l} \ldots s_{i_2} s_{i_1},$$

be the element of W corresponding to P as in (4.3). Let $p \in P$. If p is the j $(= j(p))$-th element of P with respect to the fixed total order of P, then we put

$$\alpha^{(p)} = -s_{i_l} s_{i_{l-1}} \ldots s_{i_j}(\alpha_{i_j}) \ (\in \Sigma) \ .$$

Since this is a positive root, we can write

$$\alpha^{(p)} = \sum_{\alpha_i \in S} c_i^{(p)} \alpha_i \ , \quad c_i^{(p)} \in \mathbf{N}_0 \ .$$

Let

$$M^{(p)} = \{j(p) \le m \le l;\ s_{i_m}s_{i_{m+1}}...s_{i_l}(\alpha^{(p)}) < s_{i_{m+1}}...s_{i_l}(\alpha^{(p)})\}\ .$$

We put

$$H(p) = \{p_m;\ m \in M^{(p)}\}$$

and call it the *hook* of P at p. Let $\Psi : P \longrightarrow S$ be as in 3.2. For each p and i such that $c_i^{(p)} \neq 0$, let q_{i_1} be the minimal element of the totally ordered set $\Psi^{-1}(\alpha_i)$, and $I(p;i)_* = [q_{i_1}, q_{i_c}] = \{q_{i_1}, q_{i_2}, ..., q_{i_c}\}$ $(c = c_i^{(p)})$ be the lowest interval of $c_i^{(p)}$ elements in $\Psi^{-1}(\alpha_i)$. (Let $I(p;i) = \emptyset$ if $c_i^{(p)} = 0$.) We put

$$H(p)_* = \cup_{\alpha_i \in S}\ I(p;i)_*$$

and call it the *rim-hook* of P at p. Similarly, we define $I(p;i)^*$ to be the highest interval of $c_i^{(p)}$ elements in $\Psi^{-1}(\alpha_i)$. We put

$$H(p)^* = \cup_{\alpha_i \in S}\ I(p;i)^*\ .$$

Then $H(p)^*$ is a d-complete poset.

Remark. If P is a Young diagram or a shifted Young diagram, then the above definition of hooks coincide with the known one. See [5], 5.1.4, Ex.21 for the graphical definition of hooks of a shifted Young diagram. (This exercise gave the motivation for the present work.)

4.4. Main Theorem

Let P be a (connected or disconnected) d-complete poset, and consider a P-game g_P. We have the following generalization of Theorem 2:

Theorem 6. *The game g_P satisfies the parity condition. Moreover, its energy $E(g_P)$ satisfies:*

$$E(g_P) = \sum_{p \in P} \oplus N(E(g_{H(p)^*}))\ .$$

The proof will appear elsewhere. If P is a Young diagram, then we always have $N(E(g_{H(p)^*})) = N(|H(p)|)$ (although, in general, $E(g_{H(p)^*}) \neq |H(p)|$). Hence Theorem 2 is a special case of Theorem 6.

Example. Let P be a d-complete poset of the following form (called "Inset" in [7]).

$$
\begin{array}{cccccccccccccccc}
v_1 & \longleftarrow & v_2 & \longleftarrow & \dots & \longleftarrow & v_k & \longleftarrow & x & \longleftarrow & \bullet & \longleftarrow & \dots\bullet & \longleftarrow & \bullet \longleftarrow & \bullet \\
 & & & & & & \uparrow & & \uparrow & & \uparrow & & & & & \\
 & & & & & & y & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \dots & & & \\
 & & & & & & & & \uparrow & & \uparrow & & & & & \\
 & & & & & & & & \bullet & \longleftarrow & \bullet & \longleftarrow & \dots & & & \\
 & & & & & & & & \vdots & & \vdots & & & & & \\
 & & & & & & & & \bullet & \longleftarrow & \dots & & & & & \\
 & & & & & & & & \uparrow & & & & & & & \\
 & & & & & & & & \bullet & & & & & & &
\end{array}
$$

More precisely, P is the union of a Dynkin diagram of type D

$$
\begin{array}{ccccccccc}
v_1 & \longleftarrow & v_2 & \longleftarrow & \dots & \longleftarrow & v_k & \longleftarrow & x\ , \\
 & & & & & & \uparrow & & \\
 & & & & & & y & &
\end{array}
$$

and a Young diagram

$$Y = \cup_{i=1}^{k+1}\{(i,j);\ 1 \leq j \leq n_i\}\ , \quad n_1\ \geq \dots \geq n_{k+1} \geq 0$$

with an identification $x = (1,1)$ if $n_1 \geq 1$, and an extra relation $y \longleftarrow (2,1)$ if $n_2 \geq 1$. The top tree of P is :

$$
\begin{array}{cccccccccccccc}
v_1 & \longleftarrow & v_2 & \longleftarrow & \dots & \longleftarrow & v_k & \longleftarrow & x & \longleftarrow & \bullet & \longleftarrow & \dots\bullet & \longleftarrow \bullet \longleftarrow \bullet \\
 & & & & & & \uparrow & & & & & & & \\
 & & & & & & y & & & & & & &
\end{array}
$$

which is, in general, not a Dynkin diagram of finite type.

By Theorem 6, we get

$$E(g_P) = E(g_Y) \oplus \sum_{1 \leq m \leq k+1} \oplus N(1 + E(g_{Y_m}))\ ,$$

where, for $1 \leq m \leq k+1$, Y_m is the Young diagram given by :

$$Y_m = \cup_{i=1}^{k}\{(i,j);\ 1 \leq j \leq n_i^{(m)}\}\ ,$$

where

$$n_1^{(m)} = n_1 + 1, n_2^{(m)} = n_2 + 1, \dots, n_{m-1}^{(m)} = n_{m-1} + 1,$$
$$n_m^{(m)} = n_{m+1}, n_{m+1}^{(m)} = n_{m+2}, \dots, n_k^{(m)} = n_{k+1}\ .$$

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