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ON THE ORDER OF UNIFORMLY CONVEX FUNCTIONS

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Abstract. We let $UC(\alpha)$ denote a subclass of uniformly convex functions in the open unit disk U. The object of the present paper is to derive the order of starlikeness, convexity, strongly starlikeness and strongly convexity of $UC(\alpha)$.

1. INTRODUCTION.

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in A is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$. We denote by $S^\star(\alpha)$ the subclass of A consisiting of all starlike functions of order α in U. A function $f(z)$ in A is said to be convex of order α if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U) \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$. We denote by $C(\alpha)$ the subclass of A consisiting of all convex functions of order α in U. A function $f(z)$ in A is said to be strongly starlike of order α if it satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in U) \tag{1.4}$$

for some α $(0 < \alpha \leq 1)$. We denote by $SS^\star(\alpha)$ the subclass of A consisiting of all strongly starlike functions of order α in U. A function $f(z)$ in A is said to be strongly convex of order α if it satisfies

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad (z \in U) \tag{1.5}$$

for some α $(0 < \alpha \leq 1)$. We denote by $SC(\alpha)$ the subclass of A consisiting of all strongly convex functions of order α in U. In particular,

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we denote by $S^\star(0) = SS^\star(1) = S^\star$ and $C(0) = SC(1) = C$. It is well-known that $C \subset S^\star$.

Goodman [3] introduced the concept of uniformly convex and uniformly starlike functions as the following

Definition 1. A function $f(z)$ is said to be uniformly convex (starlike) in U if $f(z)$ is in C $(S^\star)$ and has the property that for every circular arc γ contained in U, with center ζ, also in U, the arc $f(\gamma)$ is convex (starlike with respect to $f(\zeta)$).

We denote by UC and $US^\star$ the subclass of A consisiting of functions which are uniformly convex and uniformly starlike functions in U respectively.

Applying the above definition, Goodman obtained the following theorem.

Theorem A.

(a) $f(z) \in UC$ if and only if

$$\operatorname{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geq 0 \qquad (z, \zeta) \in U \times U. \tag{1.6}$$

(b) $f(z) \in US^\star$ if and only if

$$\operatorname{Re}\frac{f(z) - f(\zeta)}{(z-\zeta)f'(z)} \geq 0 \qquad (z, \zeta) \in U \times U. \tag{1.7}$$

As Marx [5] and Strohhäcker [11] showed that $C \subset S^\star(1/2)$, it is also not trivial to research for the order of starlikeness, convexity, strongly starlikeness and strongly convexity of subclasses of C. In the present paper, we derive the order of starlikeness, convexity, strongly starlikeness and strongly convexity of UC.

We need the following lemmas. These are due to MacGregor [4], Nunokawa [6] and Rønning [10] respectively.

Lemma 1([4]). *If $f(z) \in C(\alpha)$, then $f(z) \in S^\star(\beta)$, where*

$$\beta = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & \text{if } \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\log 2} & \text{if } \alpha = \dfrac{1}{2}. \end{cases} \tag{1.8}$$

Lemma 2([6]). *If $f(z) \in SC(\alpha)$, then $f(z) \in SS^*(\beta)$, where*

$$\alpha = \beta + \frac{2}{\pi}\arctan\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta)+\beta q(\beta)\cos\frac{\pi}{2}(1-\beta)}, \tag{1.9}$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}} \quad \textit{and} \quad q(\beta) = (1-\beta)^{\frac{\beta-1}{2}}.$$

Lemma 3([10]). *$f(z) \in UC$ if and only if*

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in U). \tag{1.10}$$

Furthermore, we take the following definition due to Owa [7].

Definition 2([7]). $f(z) \in A$ is said to be a member of the class $UC(\alpha)$ if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} - \alpha > \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in U) \tag{1.11}$$

for some α $(-1 < \alpha < 1)$.

2. Research for the order of $UC(\alpha)$.

We begin with the statement and the proof of the following result.

Theorem. *If $f(z) \in UC(\alpha)$, $(-1 < \alpha < 1)$, then*

$$f(z) \in C\left(\frac{1+\alpha}{2}\right), \tag{2.1}$$

$$f(z) \in SC\left(\frac{2}{\pi}\arctan\sqrt{\frac{1-\alpha}{1+\alpha}}\right), \tag{2.2}$$

$$f(z) \in \begin{cases} S^*\left(\dfrac{-\alpha}{2^{1-\alpha}(1-2^{\alpha})}\right) & \textit{if } \alpha \neq 0, \\ S^*\left(\dfrac{1}{2\log 2}\right) & \textit{if } \alpha = 0 \end{cases} \tag{2.3}$$

and

$$f(z) \in SS^*(\beta), \tag{2.4}$$

where

$$\frac{2}{\pi}\arctan\sqrt{\frac{1-\alpha}{1+\alpha}} = \beta + \frac{2}{\pi}\arctan\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta)+\beta q(\beta)\cos\frac{\pi}{2}(1-\beta)},$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}} \quad \textit{and} \quad q(\beta) = (1-\beta)^{\frac{\beta-1}{2}}.$$

Proof. Let $w = u + iv = 1 + \frac{zf''(z)}{f'(z)}$, then from (1.11), we have

$$u - \alpha > \sqrt{(u-1)^2 + v^2}. \tag{2.5}$$

and it follows that

$$\begin{aligned} (u-\alpha)^2 &> (u-1)^2 + v^2 \\ u^2 - 2\alpha u + \alpha^2 &> u^2 - 2u + 1 + v^2 \\ 2(1-\alpha)u &> v^2 + (1-\alpha^2). \end{aligned} \tag{2.6}$$

From (2.6), we have that the domain of values of w is the parabolic region $2(1-\alpha)\mathrm{Re}\, w > (\mathrm{Im}\, w)^2 + (1-\alpha^2)$. Therefore, by a brief calculation, we obtain

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1+\alpha}{2} \tag{2.7}$$

and

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\left(\frac{2}{\pi}\arctan\sqrt{\frac{1-\alpha}{1+\alpha}}\right). \tag{2.8}$$

These show (2.1) and (2.2).

Applying Lemma 1 and Lemma 2 to (2.7) and (2.8), we obtain (2.3) and (2.4). □

Remark 1. Note that $UC(\alpha) \subset C$ for $-1 < \alpha < 1$ and $UC(\alpha) \subset UC$ for $0 \le \alpha < 1$. Also we denote by $UC(0) = UC$.

Remark 2. (2.3) is the improvement of Owa's result ([7], Theorem 4).

Putting $\alpha = 0$, then we have

Corollary. *If $f(z) \in UC$, then*

$$f(z) \in S^*(0.721348\cdots),$$

$$f(z) \in C(1/2),$$

$$f(z) \in SS^{*}(0.350459\cdots)$$

and

$$f(z) \in SC(1/2).$$

Remark 3. Since the order of starlikeness, convexity, strongly starlikeness and strongly convexity of UC are obtained, we can have some distortion theorems, rotation theorems and some properties of UC by applying the results due to Robertson [9], Brannan and Kirwan [1], Brannan, Clunie and Kirwan [2] and Pinchuk [8], but these properties of UC are not necessarily sharp.

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