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On certain conditions for starlikeness

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Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be strongly starlike of order α.

1 Introduction.

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in A is said to be starlike in U if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in U).$$

We denote by S^* the subclass of A consisting of all starlike functions $f(z)$ in U. Further a function $f(z)$ belonging to A is said to be strongly starlike of order α in U if it satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in U)$$

for some α $(0 < \alpha \leq 1)$. We denote by $SS^*(\alpha)$ the subclass of A consisiting of all strongly starlike functions of order α in U.
From the definition for strongly starlike functions of order α, we note that $f(z) \in SS^*(\alpha)$ is univalent and starlike in U. Recently, Tuneski [2] obtained the following theorem.

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Theorem A. *Let a function $f(z) \in A$ satisfy*

$$\frac{f(z)f''(z)}{f'(z)^2} \prec 2 - \frac{2}{(1-z)^2} \quad (z \in U),$$

where the symbol " $\prec$ " means the subordination. Then $f(z) \in S^*$.

To derive our main theorem, we need the following lemma due to Nunokawa [1].

Lemma. *Let $p(z)$ be analytic in U with $p(0) = 1$ and $p(z) \neq 0$ $(z \in U)$. If there exists a point $z_0 \in U$ such that*

$$|\arg(p(z))| \leq \frac{\pi}{2}\alpha \quad for \quad |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\alpha \quad (\alpha > 0),$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where $k \geqq 1$ when $\arg(p(z_0)) = (\pi/2)\alpha$ and $k \leqq -1$ when $\arg(p(z_0)) = -(\pi/2)\alpha$.

2 Strongly starlikeness of order α

Now we derive

Theorem. *Let $f(z)$ in A satisfy the following inequalities*

$$\pi - \frac{\pi}{2}\alpha - \mathrm{Tan}^{-1}\alpha \; < \; \arg\left(\frac{f(z)f''(z)}{f'(z)^2} - 1\right) \; < \; \pi + \frac{\alpha}{2}\alpha + \mathrm{Tan}^{-1}\alpha \quad (z \in U)$$

for some $\alpha (0 < \alpha \leqq 1)$. Then $f(z)$ belongs to the class $SS^*(\alpha)$ in U.

Proof. From the assumption in the theorem, we see that $f'(z) \neq 0$ in U. Let us define the function $p(z)$ by $p(z) = zf'(z)/f(z)$. Then $p(z)$ satisfies

$$\frac{f(z)f''(z)}{f'(z)^2} = 1 + \frac{zp'(z)}{p(z)^2} - \frac{1}{p(z)}$$

and so

$$\frac{f(z)f''(z)}{f'(z)^2} - 1 = \frac{1}{p(z)}\left(-1 + \frac{zp'(z)}{p(z)}\right).$$

If there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha \quad for \ |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\alpha,$$

then Lemma gives us that
(i) for the case $\arg(p(z_0)) = (\pi/2)\alpha$,

$$\begin{aligned}\arg\left(\frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1\right) &= \arg\left\{\frac{1}{p(z_0)}\left(\frac{z_0p'(z_0)}{p(z_0)} - 1\right)\right\} \\ &= -\frac{\pi}{2}\alpha + \arg\left(-1 + \frac{z_0p'(z_0)}{p(z_0)}\right) \\ &= -\frac{\pi}{2}\alpha + \arg(-1 + ik\alpha) \\ &\leqq \pi - \frac{\pi}{2}\alpha - \mathrm{Tan}^{-1}\alpha.\end{aligned}$$

This contradicts our condition in the theorem.
(ii) for the case $\arg(p(z_0)) = -(\pi/2)\alpha$, the application of the same method as in (i) shows that

$$\arg\left(\frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1\right) \geqq \pi + \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1}\alpha.$$

This also contradicts the assumption of the theorem. Thus we complete the proof of our main theorem.

Putting $\alpha = 1$ in Theorem, we have the following corollary.

Corollary. *If $f(z) \in A$ satisfies*

$$\frac{\pi}{4} < \arg\left(\frac{f(z)f''(z)}{f'(z)^2} - 1\right) < \frac{7\pi}{4} \quad (z \in U),$$

then $f(z) \in S^*$.

References

[1] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. **69**(1993), 234 - 237.

[2] N. Tuneski, *On certain condotions for starlikeness*, Internat. J. Math. Math. Sci. **23**(2000), 521 - 527.

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