数理解析研究所講究録
1193 巻 2001 年 1-19

TWO TOPICS ON FLEMING-VIOT PROCESSES

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1. INTRODUCTION

For a Fleming-Viot process Y_t (which is a probability measure-valued process) on a compact metric space S, it is well-known that if its mutation operator A is bounded, then Y_t is pure atomic for every $t>0$ (Ethier and Kurtz in [2], [4]). We shall extend this result to some jump-type measure-valued processes which are called "jump-type Fleming-Viot processes" introduced by the author in [6].

It is also well-known that the normalized binary branching process is a time inhomogeneous Fleming-Viot process. We shall introduce another new class of probability measure-valued diffusion, which are called "space-time inhomogeneous Fleming-Viot processes" and show that the normalized space inhomogeneous binary branching process is a space-time inhomogeneous Fleming-Viot process.

Let S be a compact metric space, fix $r\geq 0$ and set $D_r=\mathbf{D}([r,\infty)\to S)$ be a path space of right continuous functions with left-hand limit. Let $(w(t),P_x)_{t\geq r,x\in S}$ be a S-valued Markov process starting from x at $t=r$ with sample paths in D_r. We denote the transition semi-group by (P_t) and the generator by $(A,D(A))$, where $D(A)$ is a domain of A. We suppose that (P_t) is a Feller semi-group on $(C(S),\|\cdot\|)$, where $C(S)$ is a family of continuous functions on S and $\|\cdot\|=\|\cdot\|_\infty$ denotes the supremum norm.

Let $\mathcal{M}_F=\mathcal{M}_F(S)$ be a family of finite Radon measures on S with the weak topology, that is, $\mu_n\to\mu$ in $\mathcal{M}_F \iff \langle\mu_n,f\rangle\to\langle\mu,f\rangle$ for every $f\in C(S)$, where $\langle\mu,f\rangle=\int f d\mu$. Then, $\mathcal{M}_F$ is a Polish space, i.e., complete separable metrizable space. The family of probability measures on S, $\mathcal{M}_1=\mathcal{M}_1(S)\subset\mathcal{M}_F$, is a compact metric space (cf. Chap. 3 of [3]). For $\mu\in\mathcal{M}_F\setminus\{0\}$, we always denote the normalized measure as $\overline{\mu}=\mu/\langle\mu,1\rangle$.

Let $(Y_t,\mathbf{P}_\mu^{FV})$ be a Fleming-Viot process on S, with a mutation operator A, that is, $(Y_t,\mathbf{P}_\mu^{FV})$ is an $\mathcal{M}_1$-valued process on S such that $\mathbf{P}_\mu^{FV}(Y_0=\mu)=1$ and

$$\langle Y_t,f\rangle=\langle Y_0,f\rangle+\int_0^t\langle Y_s,Af\rangle+M_t(f),$$

where $\{M_t(f)\}$ is a continuous martingale with quadratic variation

$$\langle\langle M(f)\rangle\rangle_t=c\int_0^t\left(\langle Y_s,f^2\rangle-\langle Y_s,f\rangle^2\right)ds\quad(c>0),$$

A is a generator (with a domain $D(A)\subset(C(S),\|\cdot\|)$) of a conservative Feller process $(w(t),P_x)_{t\geq 0,x\in S}$; a S-valued Markov process starting from x with $w(\cdot)\in\mathbf{D}=\mathbf{D}([0,\infty)\to S)$ and the transition semi-group (P_t).

The generator $\mathcal{L}$ of this process is given as, for $\eta\in\mathcal{M}_1, f\in D(A)$,

$$\mathcal{L}e^{-\langle\cdot,f\rangle}(\eta)=-\langle\eta,Af\rangle e^{-\langle\eta,f\rangle}+\frac{c}{2}\left[\langle\eta,f^2\rangle-\langle\eta,f\rangle^2\right]e^{-\langle\eta,f\rangle}.$$

It is well-known that if its mutation operator A is bounded, then Y_t is pure atomic for every $t > 0$ (Ethier and Kurtz in [2], [4], see also Th. 8.2.1 in [1]). In particular, if $A = 0$ and if we denote $\eta = \sum_i m_i \delta_{x_i}$, then the generator $\mathcal{L}$ can be expressed as, for a function $\phi(\mathbf{m})$ of $\mathbf{m} = (m_1, m_2, \dots)$,

$$G\phi(\mathbf{m}) = \frac{c}{2}\sum_{i,j} m_i(\delta_{ij} - m_j)\partial_{ij}^2\phi(\mathbf{m}).$$

The corresponding weight process $\{m_i(t)\}$ is given as

$$dm_i(t) = \sum_j (\delta_{ij} - m_i(t))\sqrt{cm_j(t)}dB_j(t) \quad (i \in S),$$

where $\{B_j(t)\}$ is a family of independent one-dimensional Brownian motions.

There is another well-known measure-valued process which is a branching process $(Z_t, \mathbf{P}_\mu)$, that is, $\mathcal{M}_F$-valued process such that $\mathbf{P}_\mu(Z_0 = \mu) = 1$ and

$$\mathbf{P}_\mu\left[e^{-\langle Z_t, f\rangle}\right] = e^{-\langle \mu, V_t f\rangle},$$

where $V_t f$ is a unique solution to the following equation

$$V_t f(x) = P_t f(x) - \int_0^t ds P_s \Psi(V_{t-s}f)(x),$$

or

$$\partial_t V_t f(x) = AV_t f(x) - \Psi(V_t f)(x), \quad V_0 f(x) = f(x)$$

with branching mechanism $\Psi(v)(x)$;

$$\Psi(v)(x) = \frac{1}{2}c(x)v^2 + \int_0^\infty \left[e^{-vu} - 1 + vu\right]\nu(x, du) \ (\geq 0),$$

where $c(x) \geq 0$ is a bounded function and $\nu(x, du)$ is a kernel on $S \times (0, \infty)$ satisfying that

$$\sup_{x \in S}\int_0^\infty (u \wedge u^2)\nu(x, du) < \infty.$$

In particular, if $\Psi(v)(x) = cv^2/2$ $(c > 0)$, then $(Z_t, \mathbf{P}_\mu)$ is called a *binary branching process* or a *binary branching measure-valued process*, or a *binary branching superprocess*, or a *Dawson-Watanabe process*, etc.

Let $\tau_0 = \inf\{t > 0; \in Z_t, 1 = 0\}$. For $t < \tau_0$ and $f \in D(A)$, $\langle Z_t, f\rangle$ has the following semi-martingale representation:

$$\langle Z_t, f\rangle = \langle Z_0, f\rangle + \int_0^t \langle Z_s, Af\rangle ds + M_t^c(f) + M_t^d(f),$$

where $\{M_t^c(f)\}$ is a continuous L^2-martingale with quadratic variation $\langle\langle M^c(f)\rangle\rangle_t$ such that

$$\langle\langle M^c(f)\rangle\rangle_t = \int_0^t \langle Z_s, cf^2\rangle ds,$$

and

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_F} \langle \eta, f\rangle \widetilde{N}(ds, d\eta),$$

where $\widetilde{N}(ds, d\eta)$ is a martingale measure with compensator

$$\widehat{N}(ds, d\eta) = ds\int_S Z_s(dx)\int_0^\infty \nu(du)\delta_{u\delta_x}(d\eta).$$

The generator of this process $\mathcal{L}^Z$ is given as

$$\mathcal{L}^Z e^{-\langle\cdot,f\rangle}(\eta) = [-\langle\eta, Af\rangle + \langle\eta, \Psi(f)\rangle]\, e^{-\langle\eta,f\rangle}.$$

It is also well-known that the normalized binary branching process is a time inhomogeneous Fleming-Viot process. More exactly, in 1991, Perkins [8] established that the conditional law of the binary branching process given the total mass process is a time inhomogeneous Fleming-Viot process.

Fix $r \geq 0$ and let

$$C_{r,+} \equiv \Big\{ g : [r,\infty) \to [0,\infty);\ g \text{ is continuous, and}$$
$$\text{there is } \tau_g \in (r,\infty] \text{ such that } g > 0 \text{ on } [r,\tau_g),\ g = 0 \text{ on } [\tau_g,\infty) \Big\}.$$

Let $\mu \in \mathcal{M}_1$, $g \in C_{r,+}$ and $c > 0$.

The time inhomogeneous Fleming-Viot process $(Y_t, \mathbf{P}_\mu^{FV})$ satisfies the following:

(i) $Y_r = \mu$, $Y_t = Y_{\tau_g -}$ $(t \geq \tau_g)$, $\mathbf{P}_{r,\mu}^{FV}$-a.s.,

(ii) For $f \in D(A)$, $\langle Y_t, f\rangle$ has the following semi-martingale representation:

$$\langle Y_t, f\rangle = \langle Y_r, f\rangle + \int_r^t \langle Y_s, Af\rangle ds + M_{r,t}(f)$$

such that $\{M_{r,t}(f)\}$ is a continuous L^2-martingale with quadratic variation

$$\langle\!\langle M(f)\rangle\!\rangle_{r,t} = c\int_r^t g(s)^{-1}\left[\langle Y_s, f^2\rangle - \langle Y_s, f\rangle^2\right] I(s<\tau_g)ds.$$

We denote normalized measure $\overline{\mu} = \mu/\langle\mu,1\rangle$ for $\mu \in \mathcal{M}_F \setminus \{0\}$.

Theorem 1 (Perkins '91). *Let $\mu \in \mathcal{M}_F \setminus \{0\}$ and set $y = \langle\mu,1\rangle$. For a binary branching process $(Z_t, \mathbf{P}_\mu)$, set $x_t = \langle Z_t, 1\rangle$ and $Q_y = \mathbf{P}_\mu \circ x_\cdot^{-1}$. Then*

$$\mathbf{P}_\mu\left(\overline{Z_\cdot} \in B \mid \langle Z_\cdot, 1\rangle = g(\cdot)\right) = \mathbf{P}_{0,\overline{\mu}}^{FV}(Y_\cdot \in B), \qquad Q_y\text{-a.a. } g \in C_{0,+},$$

where $(Y_t, \mathbf{P}_{0,\overline{\mu}}^{FV})$ is a time inhomogeneous Fleming-Viot process associated with (A,g,c) starting from $Y_0 = \overline{\mu}$.

We would like to extend these results to some wide class which include Fleming-Viot processes. The first one to "jump-type Fleming-Viot processes" introduced by the author in [6], and second one to "space-dependent Fleming-Viot processes" intorduced by this paper.

2. Pure atomic jump-type Fleming-Viot processes

According to [6], we give characterizations of jump-type Fleming-Viot processes.

For each $x \in S$, we define an operator T_x from the space of Dirac measures $\delta_x(dy)$ on S to $\mathcal{M}_1$ by

$$\langle T_x\delta, f\rangle = \langle \delta_x, Tf\rangle = Tf(x) = \int_S f(y)T(x,dy),$$

where $T(x,dy)$ is a non-negative kernel on S such that $T(x,S) = 1$.

We fix $\gamma > 0$ and let $\nu(dv)$ be a measure on $(0,\infty)$ such that

$$\int_0^\infty (v \wedge v^2)\nu(dv) < \infty.$$

Let $(Y_t, \mathbf{P}_\mu)$ be a jump-type Fleming-Viot process associated with (A, γ, ν, T_x) starting from $\mu \in \mathcal{M}_1$. That is, $(Y_t, \mathbf{P}_\mu)$ is an $\mathcal{M}_1$-valued process such that $\langle Y_t, f\rangle$ $(f \in D(A))$ has the following semi-martingale representation:

$$\langle Y_t, f\rangle = \langle Y_0, f\rangle + \int_0^t \langle Y_s, Af\rangle + M_t^c(f) + M_t^d(f),$$

where $\{M_t^c(f)\}$ is a continuous martingale with quadratic variation

$$\langle\langle M^c(f)\rangle\rangle_t = \gamma \int_0^t \left(\langle Y_s, f^2\rangle - \langle Y_s, f\rangle^2\right) ds \quad (\gamma > 0),$$

and $\{M_t^d(f)\}$ is a pure discontinuous martingale such that

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_F} \frac{\langle \eta, 1\rangle}{1 + \langle \eta, 1\rangle} \langle \overline{\eta} - Y_{s-}, f\rangle \widetilde{N}(ds, d\eta),$$

where $\widetilde{N}(ds, d\eta)$ is the martingale measure with compensator

$$\widehat{N}(ds, d\eta) = ds \int_S Y_s(dx) \int_0^\infty \nu(dv) \delta_{vT_x\delta}(d\eta)$$

Remark 1. In [6] the process is defined only the case of $T_x = I$, i.e., $T_x\delta = \delta_x$. However the extension is possible and easy.

The generator $\mathcal{L}$ of this process for Laplace functionals $e^{-\langle \mu, f\rangle}$ $(\mu \in \mathcal{M}_1, f \in D(A))$ is given as

$$\begin{aligned}
\mathcal{L}e^{-\langle \cdot, f\rangle}(\mu) &= -\langle \mu, Af\rangle e^{-\langle \mu, f\rangle} + \frac{\gamma}{2}\left[\langle \mu, f^2\rangle - \langle \mu, f\rangle^2\right] e^{-\langle \mu, f\rangle} \\
&\quad + \int \mu(dx) \int_0^\infty \nu(dv) \\
&\quad \left\{\exp\left[-\frac{v}{1+v}\langle T_x\delta - \mu, f\rangle\right] - 1 + \frac{v}{1+v}\langle T_x\delta - \mu, f\rangle\right\} e^{-\langle \mu, f\rangle}.
\end{aligned}$$

For a functional $F(\mu)$, a derivative at $x \in S$ is defined by

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[F(\mu + \epsilon\delta_x) - F(\mu)\right] \qquad \text{(if exists)},$$

and higher order derivatives $\delta^2 F(\mu)/(\delta\mu(x)\delta\mu(y)), \ldots$ are defined similarly.

Note that the generator can be expressed as

$$\begin{aligned}
\text{(2.1)} \qquad \mathcal{L}F(\mu) &= \langle \mu, A\frac{\delta F(\mu)}{\delta \mu(\cdot)}\rangle + \frac{\gamma}{2} \iint \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} Q(\mu; dx, dy) \\
&\quad + \int_{\mathcal{M}_F} \left[F(\mu + g(\mu, \eta)) - F(\mu) - \langle g(\mu, \eta), \frac{\delta F(\mu)}{\delta \mu(\cdot)}\rangle\right] n(\mu; d\eta),
\end{aligned}$$

where

$$Q(\mu; dx, dy) = \mu(dx)\delta_x(dy) - \mu(dx)\mu(dy),$$

$$g(\mu, \eta) = \frac{\mu + \eta}{1 + \langle \eta, 1\rangle} - \mu = \frac{\langle \eta, 1\rangle}{1 + \langle \eta, 1\rangle}(\overline{\eta} - \mu) \quad \in \mathcal{M}_F$$

and

$$n(\mu; d\eta) = \int \mu(dx) \int_0^\infty \nu(dv) \delta_{vT_x\delta}(d\eta).$$

Here we give a formal calculation for general measrue-valued processes, i.e., Ito's formula for measure-valued processes.

For a set B, let $\mathcal{M}_F^{\pm}(B)$ be the class of finite singed measures on B, i.e., $\eta \in \mathcal{M}_F^{\pm}(B) \Longleftrightarrow \eta = \eta^+ - \eta^-;\ \eta^{\pm} \in \mathcal{M}_F(B)$. We denote $\|\eta\| = (\eta^+ + \eta^-)(B)$. For simplicity, if $B = S$, then $\mathcal{M}_F^{\pm} = \mathcal{M}_F^{\pm}(S)$. Let $Q(\mu; dx, dy) : \mathcal{M}_F \to \mathcal{M}_F^{\pm}(S \times S)$ be measurable such that $Q(\mu; dx, dx) \le C\mu(dx)$ for some $C > 0$. Let $g(\mu, \eta) : \mathcal{M}_F \times \mathcal{M}_F^{\pm} \to \mathcal{M}_F^{\pm}$ and $n(\mu; d\eta) : \mathcal{M}_F \to \mathcal{M}_F(\mathcal{M}_F^{\pm})$ be measurable such that

$$\sup_{\mu \in \mathcal{M}_F; \mu(S) \le K} \int_{\mathcal{M}_F^{\pm}} \|g(\mu, \eta)\| \wedge \|g(\mu, \eta)\|^2 n(\mu; d\eta) < \infty \quad \text{for every } K > 1.$$

In general, let X_t be an $\mathcal{M}_F$-valued Markov process such that

$$X_t = X_0 + \int_0^t A^* X_s ds + M_t^c + M_t^d,$$

where $A^* X_s$ is defined by $\langle A^* X_s, f \rangle = \langle X_s, Af \rangle$, M_t^c is a continuous martingale and M_t^d is a pure discontinuous martingale;

$$M_t^c(dx) = \int_0^t M(ds, dx)$$

$$M_t^d(dx) = \int_0^t \int_{\mathcal{M}_F^{\pm}} g(X_{s-}, \eta)(dx) \widetilde{N}(ds, d\eta)$$

with a continuous martingale measure $M(ds, dx)$ and a pure discontinuous martingale measure $\widetilde{N}(ds, d\eta)$. Suppose that the covariance of $M^c(dx) M^c(dy)$ is given as

$$\langle\langle M^c(dx), M^c(dy) \rangle\rangle_t = \gamma \int_0^t Q(X_s; dx, dy) ds$$

and the compensator of $\widetilde{N}$ is $\widehat{N}(ds, d\eta) = dsn(X_s, d\eta)$ (cf. for martingale measures, see Walsh [10]).

Let $F(\mu)$ be a suitable functional of $\mu \in \mathcal{M}_F$ and let $\mathcal{L}F(\mu)$ be given as in (2.1). Then, by a formal calcutation we have the following Ito's formula:

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t \mathcal{L}F(X_s) ds + M_t^c(F) \\ &\quad + \int_0^t \int_{\mathcal{M}_F^{\pm}} [F(X_{s-} + g(X_{s-}, \eta)) - F(X_{s-})] \widetilde{N}(ds, d\eta), \end{aligned}$$

where

$$M_t^c(F) = \int_0^t \int \frac{\delta F(X_s)}{\delta X_s(x)} M(ds, dx)$$

is a martingale with quadratic variation

$$\langle\langle M^c(F) \rangle\rangle_t = \gamma \int_0^t ds \iint \frac{\delta F(X_s)}{\delta X_s(x)} \frac{\delta F(X_s)}{\delta X_s(y)} Q(X_s; dx, dy)$$

(for the stochastic integrals corresponding to the martingale measures, see Walsh [10] and also Dawson [1]).

We denote the space of pure atomic probability measures on S by $\mathcal{M}_{1,a} = \mathcal{M}_{1,a}(S)$. We also denote the pure atomic part of a measure η by η_a, and for a process Y_t by $Y_{t,a}$ or $(Y_t)_a$.

We set

$$\mathbf{M} = \{\mathbf{m} = (m_1, m_2, \dots); m_1 \geq m_2 \geq \dots \geq 0, \textstyle\sum_i m_i = 1\}$$
$$\overline{\mathbf{M}} = \{\mathbf{m} = (m_1, m_2, \dots); m_1 \geq m_2 \geq \dots \geq 0, \textstyle\sum_i m_i \leq 1\}$$

and let $M : \mathcal{M}_1 \to \overline{\mathbf{M}}$; $M(\eta) = M(\eta_a) =$ (the vector of descending order statistics of the masses of the atoms of η_a).

Theorem 2. *Let $(Y_t, \mathbf{P}_\mu)$ be a jump-type Fleming-Viot process associated with (A, γ, ν, T_x) starting from $\mu \in \mathcal{M}_1$. Suppose that A is a bounded operator having the following form:*

$$Af(x) = a(x) \int_S [f(y) - f(x)]\, B(x, dy),$$

where $a(x)$ is a nonnegative bounded function on S and $B(x, dy)$ is a nonnegative kernel such that $B(x, S) = 1$ for all $x \in S$. Also assume that $\gamma > 0$ and $T_x\delta \in \mathcal{M}_{1,a}$ for every $x \in S$. Then Y_t is pure atomic for all $t > 0$ a.s., i.e.,

$$\mathbf{P}_\mu(Y_t \in \mathcal{M}_{1,a} \quad \textit{for all } t > 0) = 1.$$

Moreover if $a(\cdot) \equiv a (\geq 0)$, $T_x\delta = \delta_x$ for every $x \in S$ and $B(x, \cdot)$ has no atoms for every $x \in S$, then $\{\mathbf{m}_t = M(Y_t)\}$ is a solution of the $\mathbf{D}([0,\infty), \mathbf{M})$-martingale problem for the infinite dimensional operator $(G, \mathcal{D}(G))$, where

$$G\phi(\mathbf{m}) = \frac{\gamma}{2} \sum_{i,j} m_i(\delta_{ij} - m_j)\partial^2_{ij}\phi(\mathbf{m}) - a \sum_i m_i \partial_i \phi(\mathbf{m})$$
$$+ \sum_i m_i \int_0^\infty \left[\phi\left(\left(\frac{m_j + v\delta_{ij}}{1+v} \right)_j \right) - \phi(\mathbf{m}) - \sum_j \frac{v}{1+v}(\delta_{ij} - m_j)\partial_j \phi(\mathbf{m}) \right] \nu(dv)$$

with $\partial_i = \partial/\partial m_i$, $\partial^2_{ij} = \partial^2/(\partial m_i \partial m_j)$ and $\mathcal{D}(G)$ is the algebra generated by $\{1, \phi^2, \phi^3, \dots\}$ with $\phi^\beta(\mathbf{m}) = \sum_i m_i^\beta$ $(\beta > 1)$. That is, Y_t is the size ordered atom process for the infinitely many neutral alleles (jump-type) model.

Proof. It is enough to consider the case that $a(\cdot) \equiv a > 0$ and $B(x, \cdot)$ has no atoms. Because if we set $\overline{S} = S \times [0,1]$, $\overline{a} = \sup a(x) > 0$ (note that the case of $\overline{a} = 0$ is trivial),

$$\overline{B}(x, u, dydv) = \frac{a(x)}{\overline{a}} B(x, dy)dv + \frac{\overline{a} - a(x)}{\overline{a}} \delta_x(dy)dv$$

and

$$\overline{A}f(x, u) = \overline{a} \int_0^1 \int_S [f(y, v) - f(x, u)]\, \overline{B}(x, u, dydv),$$

then $Y_t(\cdot) := \overline{Y}_t(\cdot \times [0,1])$, with the solution $\overline{Y}_t$ of the martingale problem for $\overline{A}$, is the solution of the martingale problem for A.

For $\mu \in \mathcal{M}_1$, if $\mu_a = \sum_i m_i \delta_{x_i}$, then set $F_\beta(\mu) = \phi_\beta(M(\mu)) = \sum_i m_i^\beta$ $(\beta > 1)$, and

$$F_{1+}(\mu) = \lim_{\beta \downarrow 1} F_\beta(\mu) = \sum_j m_j (= \langle \mu_a, 1 \rangle).$$

If $\beta > 2$, then

$$\frac{\delta F_\beta(\mu)}{\delta \mu(x)} = \sum_i \beta m_i^{\beta-1} 1_{x_i}(x) \quad \text{and} \quad \frac{\delta^2 F_\beta(\mu)}{\delta \mu(x) \delta \mu(y)} = \sum_i \beta(\beta - 1) m_i^{\beta-2} 1_{x_i}(x) 1_{x_i}(y),$$

We first give some formal calculations. For each fixed x we denote all atoms of $\mu + T_x$ by $\{x_j\}$. Since $B(x,\cdot)$ has no atoms, by (2.1) we have

$$\begin{aligned}
\mathcal{L}F_\beta(\mu) &= \sum_i \left[-a\beta m_i^\beta + \frac{\gamma}{2}\beta(\beta-1)\left(m_i - m_i^2\right)m_i^{\beta-2}\right] \\
&\quad + \int_S \mu(dx)\int_0^\infty \Bigg[F_\beta\left(\frac{\mu + vT_x\delta}{1+v}\right) - F_\beta(\mu) \\
&\quad - \frac{v}{1+v}\sum_j (T_x\delta - \mu)(\{x_j\})\beta m_j^{\beta-1}\Bigg]\nu(dv) \\
&= -a\beta F_\beta(\mu) + \frac{\gamma}{2}\beta(\beta-1)\left(F_{\beta-1}(\mu) - F_\beta(\mu)\right) \\
&\quad + \int_S \mu(dx)\int_0^\infty \Bigg[F_\beta\left(\frac{\mu + vT_x\delta}{1+v}\right) - F_\beta(\mu) \\
&\quad - \frac{\beta v}{1+v}\sum_j \left(T_x\delta(\{x_j\}) - \mu(\{x_j\})\right)\mu(\{x_j\})^{\beta-1}\Bigg]\nu(dv)
\end{aligned}$$

Moreover if we set $F_1(\mu) \equiv 1$, then the above formula is still valid for $\beta = 2$. Hence by formal Ito's formula

$$M_t(\beta) := M_t(F_\beta(Y_t)) = F_\beta(Y_t) - F_\beta(Y_0) - \int_0^t \mathcal{L}F_\beta(Y_s)ds \tag{2.2}$$

is an L^2-martingale such that $M_t(\beta) = M_t^{c,\beta} + M_t^{d,\beta}$, where $M_t^{c,\beta}$ is a continuous martingale with quadratic variation

$$\langle\langle M^{c,\beta}\rangle\rangle_t = \beta^2\gamma\int_0^t \left[F_{2\beta-1}(Y_s) - F_\beta(Y_s)^2\right]ds$$

and

$$M_t^{d,\beta} = \int_0^t\int\left[F_\beta\left(\frac{Y_{s-}+\eta}{1+\langle\eta,1\rangle}\right) - F_\beta(Y_{s-})\right]\widetilde{N}(ds,d\eta)$$

is a pure discontinuous martingale with compensator

$$\widehat{N}(ds,d\eta) = ds\int_S Y_s(dx)\int_0^\infty \nu(dv)\delta_{vT_x\delta}(d\eta).$$

To verify the above result we use an approximation method. Let $\{S_j^n, x_j^n, \rho_n; n, j \in \mathbf{N}\}$ be a partition family for S, i.e.,

(a) $S_j^n \subset S, x_j^n \in S_j^n, S_j^n \cap S_k^n = \emptyset$ if $j \neq k$,
(b) $\bigcup_j S_j^n = S$ for all $n \in \mathbf{N}$,
(c) for each j, there is k such that $S_j^{n+1} \subset S_k^n$,
(d) $\rho_n := \sup_j \mathrm{diam}(\mathrm{S}_j^n) \to 0$ $(n \to \infty)$.

For $\mu \in \mathcal{M}_1$, set $\xi^n(\mu) = \sum_j \mu(S_j^n)\delta_{x_j^n}$ and $\xi_t^n = \xi^n(Y_t)$. For $\beta \geq 1$, let $F_{\beta,n}(\mu) = \sum_j \mu(S_j^n) = \phi^\beta(\xi^n(\mu))$. Note that for $\beta \geq 1$, $F_{\beta,n}(\mu) \to F_\beta(\mu)$ as $n \to \infty$ by the definition of $F_\beta(\mu)$.

For each j, n, if we take $f_k^{j,n} \in D(A)$; $0 \leq f_k^{j,n} \uparrow 1_{S_j^n}$ $(k \uparrow \infty)$, and use Ito's formula for $\langle Y_t, f_k^{j,n}\rangle$, then by letting $k \to \infty$ and summing up on j we can get the following: If

$\beta \geq 2$, then

$$M_{n,t}(\beta) := M_t(F_{\beta,n}(Y_t)) = F_{\beta,n}(Y_t) - F_{\beta,n}(Y_0) - \int_0^t \mathcal{L}F_{\beta,n}(Y_s)ds$$

is a bounded (uniformly in n) L^2-martingale, where

$$\begin{aligned}\mathcal{L}F_{\beta,n}(\mu) &= \sum_i \Big[\frac{\gamma}{2}\beta(\beta-1)\left(\mu(S_i^n) - \mu(S_i^n)^2\right)\mu(S_i^n)^{\beta-2} \\ &\qquad + a\beta\left[\langle \mu, B(\cdot, S_i^n)\rangle - \mu(S_i^n)\right]\mu(S_i^n)^{\beta-1}\Big] \\ &\quad + \int_S \mu(dx)\int_0^\infty \sum_j \Big[\left(\frac{(\mu + vT_x\delta)(S_j^n)}{1+v}\right)^\beta - \mu(S_j^n)^\beta \\ &\qquad - \frac{v}{1+v}(T_x\delta - \mu)(S_j^n)\beta\mu(S_j^n)^{\beta-1}\Big]\nu(dv).\end{aligned}$$

Note that if we set $p = 1/(1+v), q = v/(1+v)$ and

$$h(a,b) := (pa+qb)^\beta - a^\beta - \beta q(b-a)a^{\beta-1} \geq 0 \tag{2.3}$$

for $0 \leq a, b \leq 1$, then by $\beta \geq 2$

$$\begin{aligned}h(a,b) &= \int_0^1 ds \int_0^s \beta(\beta-1)(tq(b-a)+a)^{\beta-2}q^2(b-a)^2 dt \\ &\leq \beta(\beta-1)q^2(a^2+b^2).\end{aligned}$$

Since $B(x,\cdot)$ has no atoms, we can see that as $n \to \infty$ $\mathcal{L}F_{\beta,n}(\mu) \to \mathcal{L}F_\beta(\mu)$ (bounded-pointwisely), and hence $M_{n,t}(\beta) \to M_t(\beta)$ (a.s., in L^2). Moreover the limit process $\{M_t(\beta)\}$ is an L^2-martingale with $M_t(\beta) = M_t^{c,\beta} + M_t^{d,\beta}$ as mentioned above (note that this decomposition can be shown by using the uniqueness of special martingales; cf. Theorem 6.1.3 in [1]).

By mean zero in (2.2) and by taking a limit of $\mathcal{L}F_\beta(\mu)$ as $\beta \downarrow 2$ carefully, we have

$$0 = \lim_{\beta\downarrow 2} \mathbf{E}_\mu\left[M_t(\beta) - M_t(2)\right] = \mathbf{E}_\mu\left[\gamma\int_0^t (1 - F_{1+}(Y_s))ds\right] \ (\geq 0).$$

By $\gamma > 0$ this implies

$$\langle (Y_t)_a, 1\rangle = F_{1+}(Y_t) = 1, \quad \text{i.e., } Y_t \in \mathcal{M}_{1,a} \text{ for a.a.} t > 0, \text{ a.s.}$$

In order to show that $Y_t \in \mathcal{M}_{1,a}$ for all $t > 0$, a.s., we mention that $M_t^{c,\beta}$, $M_t^{d,\beta}$ are still L^2-martingales for $\beta \in (1,2)$. In fact, to verify this, let $F_{\beta,\epsilon}(\mu) = \phi_{\beta,\epsilon}(M(\mu))$, where

$$\phi_{\beta,\epsilon}(\mathbf{m}) = \sum_i \psi_\epsilon(m_i) \quad \text{with} \quad \psi_\epsilon(m) = (m+\epsilon)^\beta - \epsilon^\beta - \beta\epsilon^{\beta-1}m.$$

Apply Ito's formula to $F_{\beta,\epsilon}(Y_t)$ and take the limit $\epsilon \downarrow 0$, then the corresponding martingale parts $M_t^{c,\beta,\epsilon}$, $M_t^{d,\beta,\epsilon}$ converge to $M_t^{c,\beta}$, $M_t^{d,\beta}$ respectively in L^2. Moreover for $M_t(\beta) = M_t^{c,\beta} + M_t^{d,\beta}$, the formula (2.2) also holds. These can be checked as follows: First note that for $1 < \beta < 2$,

$$\psi_\epsilon(m) = \int_0^1 ds \int_0^s \beta(\beta-1)(tm+\epsilon)^{\beta-2}m^2 dt \ \uparrow m^\beta \ (\beta \downarrow 1),$$

$\psi_\epsilon(m)$ is convex in $0 \le m \le 1$ and

$$\frac{\delta F_{\beta,\epsilon}(\mu)}{\delta\mu(x)} = \sum_i \beta\left\{(m_i+\epsilon)^{\beta-1} - \epsilon^{\beta-1}\right) 1_{x_i}(x),$$

$$\frac{\delta^2 F_{\beta,\epsilon}(\mu)}{\delta\mu(x)\delta\mu(y)} = \sum_i \beta(\beta-1)(m_i+\epsilon)^{\beta-2} 1_{x_i}(x) 1_{x_i}(y).$$

Hence

$$\begin{aligned}\mathcal{L}F_{\beta,\epsilon}(\mu) &= \sum_i \left[-a\beta\left\{(m_i+\epsilon)^{\beta-1} - \epsilon^{\beta-1}\right\} m_i + \frac{\gamma}{2}\beta(\beta-1)\left(m_i - m_i^2\right)(m_i+\epsilon)^{\beta-2}\right] \\ &\quad + \int_S \mu(dx) \int_0^\infty \left[F_{\beta,\epsilon}\left(\frac{\mu + vT_x\delta}{1+v}\right) - F_{\beta,\epsilon}(\mu)\right. \\ &\quad \left. - \frac{\beta v}{1+v} \sum_j (T_x\delta - \mu)(\{x_j\}) \left\{(m_i+\epsilon)^{\beta-1} - \epsilon^{\beta-1}\right\}\right] \nu(dv)\end{aligned}$$

converges to $\mathcal{L}F_\beta(\mu)$ as $\epsilon \downarrow 0$ by using

$$(m+\epsilon)^{\beta-1} - \epsilon^{\beta-1} = (\beta-1)m\int_0^1 (tm+\epsilon)^{\beta-2} dt \uparrow m^{\beta-1} \quad (\epsilon \downarrow 1)$$

and monotone convergence theorem. By the same way we also have

$$\int_0^t \mathcal{L}F_{\beta,\epsilon}(Y_s)ds \to \int_0^t \mathcal{L}F_\beta(Y_s)ds.$$

If we set $(Y_t)_a = \sum_j m_j(t)\delta_{x(t)}$, then as $\epsilon \downarrow 0$,

$$\begin{aligned}\langle\!\langle M^{c,\beta,\epsilon}\rangle\!\rangle_t &= \gamma\beta^2 \int_0^t \left[\sum_j \left\{(m_j(s)+\epsilon)^{\beta-1} - \epsilon^{\beta-1}\right\}^2 m_j(s)\right. \\ &\quad \left. - \left\{\sum_j \left((m_j(s)+\epsilon)^{\beta-1} - \epsilon^{\beta-1}\right) m_j(s)\right\}^2\right] ds \\ &\to \gamma\beta^2 \int_0^t \sum_j \left\{m_j(s)^{2\beta-1} - \left(\sum_j m_j(s)^\beta\right)^2\right\} ds \\ &= \langle\!\langle M^{c,\beta}\rangle\!\rangle_t.\end{aligned}$$

Hence

$$M_t^{c,\beta,\epsilon} = B_{\langle\!\langle M^{c,\beta,\epsilon}\rangle\!\rangle_t} \to M_t^{c,\beta} = B_{\langle\!\langle M^{c,\beta}\rangle\!\rangle_t},$$

where $\{B_t\}$ is a one-dimensional Brownian motion. The L^2 convergence can be shown by the folowing:

$$\begin{aligned}\langle\!\langle M^{c,\beta,\epsilon} - M^{c,\beta}\rangle\!\rangle_t &= \gamma\beta^2 \int_0^t \left[\sum_j \left\{(m_j(s)+\epsilon)^{\beta-1} - \epsilon^{\beta-1} - m_j(s)^{\beta-1}\right\}^2 m_j(s)\right. \\ &\quad \left. - \left\{\sum_j \left((m_j(s)+\epsilon)^{\beta-1} - \epsilon^{\beta-1} - m_j(s)^{\beta-1}\right) m_j(s)\right\}^2\right] ds \\ &\to 0 \quad \text{in } L^1.\end{aligned}$$

For the discontinuous parts, we can see that

$$\int_0^t ds \int Y_s(dx) \int_0^\infty \nu(dv) \Bigg| F_{\beta,\epsilon}\left(\frac{Y_s + vT_x\delta}{1+v}\right) - F_{\beta,\epsilon}(Y_s)$$
$$- \left\{ F_\beta \left(\frac{Y_s + vT_x\delta}{1+v}\right) - F_\beta(Y_s) \right\} \Bigg|^2 \to 0 \quad \text{in } L^1$$

by Lebesgue's convergence theorem. Hence $M_t^{d,\beta,\epsilon} \to M_t^{d,\beta}$ in L^2. Moreover by taking a suitable subsequence we get the a.s. convergence. Therefore (2.2) is also valid.

Form the above results as $\beta \downarrow 1$,

$$\mathbf{E}_\mu\left[(M_t^{c,\beta})^2\right] = \mathbf{E}_\mu\left[\langle\!\langle M^{c,\beta}\rangle\!\rangle_t\right] \to \gamma \mathbf{E}_\mu\left[\int_0^t F_{1+}(Y_s)\left(1 - F_{1+}(Y_s)\right) ds\right] = 0,$$

$$\begin{aligned} \mathbf{E}_\mu\left[(M_t^{d,\beta})^2\right] &= \mathbf{E}_\mu\left[\int_0^t ds \int Y_s(dx) \int_0^\infty \nu(dv) \left| F_\beta\left(\frac{Y_s + vT_x\delta}{1+v}\right) - F_\beta(Y_s)\right|^2\right] \\ &\to \mathbf{E}_\mu\left[\int_0^t ds \int Y_s(dx) \int_0^\infty \nu(dv) \left(\frac{v}{1+v}\right)^2 \langle (T_x\delta - Y_s)_a, 1\rangle^2\right] = 0 \end{aligned}$$

by $T_x \in \mathcal{M}_{1,a}$ for all $x \in S$ and $Y_t \in \mathcal{M}_{1,a}$ for a.a. $t > 0$, a.s. Furthermore

$$\begin{aligned} \lim_{\beta\downarrow 1} \mathbf{E}_\mu\left[M_t(\beta)^2\right] &\leq 2\lim_{\beta\downarrow 1}\left(\mathbf{E}_\mu\left[(M_t^{c,\beta})^2\right] + \mathbf{E}_\mu\left[(M_t^{d,\beta})^2\right]\right) \\ &= 0. \end{aligned}$$

Hence by Doob's maximal inequality and by taking a sequence $\{\beta_n\}; \beta_n \downarrow 1$,

$$\sup_{t\leq T} |M_t(\beta_n)| \to 0 \quad \text{a.s. for each } T > 0.$$

Note that

$$\begin{aligned} H^\beta(\mu) &:= \int_S \mu(dx) \int_0^\infty \Bigg[F_\beta\left(\frac{\mu + vT_x\delta}{1+v}\right) - F_\beta(\mu) \\ &\quad - \frac{\beta v}{1+v}\left(\sum_j T_x\delta(\{x_j\})\mu(\{x_j\})^{\beta-1} - F_\beta(\mu)\right)\Bigg] \nu(dv) \geq 0 \end{aligned}$$

and that for $\mu \in \mathcal{M}_1$,

$$\limsup_{\beta\downarrow 1} \mathcal{L}F_\beta(\mu) = -aF_{1+}(\mu) + \limsup_{\beta\downarrow 1}\left[\frac{\gamma}{2}\beta(\beta-1)F_{\beta-1}(\mu) + H^\beta(\mu)\right].$$

Thus if we set

$$R_t = \limsup_{n\to\infty} \int_0^t \left[\frac{\gamma}{2}\beta_n(\beta_n - 1)F_{\beta_n - 1}(Y_s) + H^{\beta_n}(Y_s)\right] ds$$

(which is nondecreasing in t), then by (2.2) and the above result we have

$$F_{1+}(Y_t) - F_{1+}(Y_0) + a\int_0^t F_{1+}(Y_s)ds - R_t = 0 \quad \text{for all } 0 < t \leq T, \text{ a.s.}$$

By $F_{1+}(Y_t) = 1$ for a.a.$t > 0$, a.s.,

$$F_{1+}(Y_t) - F_{1+}(Y_0) = R_t - at \quad \text{for all } t \geq 0, \text{ a.s.}$$

and

$$0 = F_{1+}(Y_t) - F_{1+}(Y_s) = R_t - R_s - a(t-s) \quad \text{for a.a. } t > s > 0, \text{ a.s.},$$

that is,

$$R_t - R_s = a(t-s) \quad \text{for a.a. } t > s > 0, \text{ a.s.}$$

The left hand side is nondecreasing in $t > s$. Hence it is easy to see that

$$R_t = at \quad \text{for all } t \geq 0, \text{ a.s.}$$

This implies that $Y_t \in \mathcal{M}_{1,a}$ for all $t > 0$, a.s. Finally in case of $T_x\delta = \delta_x$ $(x \in S)$, it is easy to check that $\{\mathbf{m_t}\}$ is a solution of the martingale problem for $(G, \mathcal{D}(G))$. □

Remark 2 (Pure jump case). In case of $\gamma = 0$, even if A is bounded, it is not ensure that Y_t is pure atomic for all $t > 0$ $\mathbf{P}_\mu$-a.s. For instance, suppose that $T_x\delta \in \mathcal{M}_{1,a}$ for every $x \in S$ and $Y_0 = \mu \in \mathcal{M}_{1,a}$ $\mathbf{P}_\mu$-a.s. Also assume that $a(\cdot) \equiv a > 0$ and $B(x, \cdot)$ has no atoms for each $x \in S$. Let $H^\beta(\mu)$ be defined as in the previous proof. If $\int_0^\infty v\nu(dv) < \infty$, then it is easy to see that

$$\lim_{\beta \downarrow 1} H^\beta(\mu) = 0.$$

In fact, for $\beta > 1$, let $h(a,b) \geq 0$ be in (2.3) with $p = 1/(1+v), q = v/(1+v)$, then it holds that for $0 \leq a, b \leq 1$,

$$\begin{aligned} h(a,b) &\leq pa^\beta + qb^\beta - a^\beta - \beta q(b-a)a^{\beta-1} \\ &= q\left(-a^\beta + b^\beta - \beta(b-a)a^{\beta-1}\right) \\ &\quad \begin{cases} \leq & q(1+\beta)(a+b), \\ \to & 0 \quad (\beta \downarrow 1). \end{cases} \end{aligned}$$

Threfore we can apply Lebegue's convergence theorem for $H(\eta)$. Now by mean zero

$$\begin{aligned} 0 &= \lim_{\beta \downarrow 1} \mathbf{E}_\mu[M_t^{d,\beta_n}] \\ &= \mathbf{E}_\mu\left[F_{1+}(Y_t) - F_{1+}(Y_0) + a\int_0^t F_{1+}(Y_s)ds\right]. \end{aligned}$$

This implies that $\mathbf{E}_\mu[F_{1+}(Y_t)] = F_{1+}(\mu)e^{-at} = e^{-at} < 1$, i.e., Y_t is not pure atomic.

(Note that if $a = 0$, then $\mathbf{E}_\mu[F_{1+}(Y_t)] = 1$, i.e., $\mathbf{E}_\mu[\int_0^T F_{1+}(Y_t)dt] = T$ for al $T > 0$. thus, $Y_t \in \mathcal{M}_{1,a}$ for a.a. $t > 0$, a.s. Moreover by the same way as in the previous proof we have

$$F_{1+}(Y_t) - 1 = R_t - at = H_t - at = 0 \quad \text{for all } t > 0, \text{ a.s.}$$

That is,

$$\mathbf{P}_\mu(Y_t \in \mathcal{M}_{1,a} \quad \text{for all } 0 \leq t \leq T) = 1.)$$

3. Space-time inhomogeneous Fleming-Viot processes

Next we would like to extend the Perkins result to the space(-time) inhomogeneous case.

Let $c(x) \geq 0$ be a bounded function.

According to Dawson [1], we first give a characterization of the space inhomogeneous binary branching process $(Z_t, \mathbf{P}_\mu)_{t\geq 0}$ $(\mu \in \mathcal{M}_F)$.

$(Z_t, \mathbf{P}_\mu)_{t\geq 0}$ is an $\mathcal{M}_F$-valued process such that $\mathbf{P}_\mu(Z_0 = \mu) = 1$, $Z_t = Z_{t\wedge\tau_0}$ $(\tau_0 = \inf\{t \geq 0; \langle Z_t, 1\rangle = 0\})$ and

$$\mathbf{P}_\mu\left[e^{-\langle Z_t, f\rangle}\right] = e^{-\langle\mu, V_t f\rangle},$$

where $V_t f$ is a unique solution to the following equation

$$V_t f(x) = P_t f(x) - \frac{1}{2}\int_0^t ds P_s\left(c(\cdot)(V_{t-s}f)(\cdot)^2\right)(x),$$

or

$$\partial_t V_t f(x) = AV_t f(x) - \frac{1}{2}c(x)(V_t f)^2(x), \quad V_0 f(x) = f(x).$$

Moreover $\langle Z_t, f\rangle$ $(f \in D(A))$ has the following semi-martingale representation:

$$\langle Z_t, f\rangle = \langle Z_0, f\rangle + \int_0^t \langle Z_s, Af\rangle ds + M_t(f),$$

where $\{M_t(f)\}$ is a continuous L^2-martingale with quadratic variation $\langle\!\langle M(f)\rangle\!\rangle_t$ such that

$$\langle\!\langle M(f)\rangle\!\rangle_t = \int_0^t \langle Z_s, cf^2\rangle ds \quad (t < \tau_0).$$

The generator of this process $\mathcal{L}^Z$ is given as

$$\mathcal{L}^Z e^{-\langle\cdot,f\rangle}(\mu) = \left[-\langle\mu, Af\rangle + \frac{1}{2}\langle\mu, cf^2\rangle)\rangle\right] e^{-\langle\mu,f\rangle}.$$

For a functional $F(\eta)$ of $\eta \in \mathcal{M}_F$, a derivative at $x \in S$ is defined by

$$\frac{\delta F(\eta)}{\delta\eta(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[F(\eta + \epsilon\delta_x) - F(\eta)\right] \quad \text{(if exists)},$$

and higher order derivatives $\delta^2 F(\eta)/(\delta\eta(x)\delta\eta(y)), \ldots$ are defined similarly. Note that the generator can be also expressed as

$$\mathcal{L}^Z F(\eta) = \langle\eta, A\frac{\delta F(\eta)}{\delta\eta(\cdot)}\rangle + \frac{1}{2}\iint \frac{\delta^2 F(\eta)}{\delta\eta(x)\delta\eta(y)} Q(\eta; dx, dy) \tag{3.1}$$

where $Q(\eta; dx, dy) = c(x)\eta(dx)\delta_x(dy)$.

Next in oreder to introduce space-time inhomogeneous Fleming-Viot processes we define a family of operators $\mathcal{L}^g = (\mathcal{L}^g_t)_{r\leq t<\tau_g}$ for a fixed $g \in C_{r,+}$ as follows: For functionals $\exp[-\langle\eta, f\rangle]$ $(\eta \in \mathcal{M}_1, f \in D(A))$ and $r \leq t < \tau_g$,

$$\begin{aligned}\mathcal{L}^g_t e^{-\langle\cdot,f\rangle}(\eta) &= \left\{-\langle\eta, Af\rangle - g(t)^{-1}\left[\langle\eta, c\rangle\langle\eta, f\rangle - \langle\eta, cf\rangle\right]\right\} e^{-\langle\eta,f\rangle} \\ &\quad + \frac{1}{2g(t)}\left[\langle\eta, cf^2\rangle + \langle\eta, c\rangle\langle\eta, f\rangle^2 - 2\langle\eta, cf\rangle\langle\eta, f\rangle\right] e^{-\langle\eta,f\rangle}.\end{aligned}$$

with a domain

$$\mathcal{D}_0(\mathcal{L}^g) := \text{lin span}\left\{e^{-\langle\cdot,f\rangle}; f \in D(A), f \geq 0\right\}.$$

This operator can be also expressed as in (3.1) by

$$\mathcal{L}_t^g F(\eta) = \langle \eta, A\frac{\delta F(\eta)}{\delta \eta(\cdot)}\rangle + g(t)^{-1}\left[\langle \eta, c\rangle\langle \eta, \frac{\delta F(\eta)}{\delta \eta(\cdot)}\rangle - \langle \eta, c\frac{\delta F(\eta)}{\delta \eta(\cdot)}\rangle\right] + \frac{1}{2}\iint \frac{\delta^2 F(\eta)}{\delta\eta(x)\delta\eta(y)} Q_t(\eta; dx, dy),$$

where

$$Q_t(\eta; dx, dy) = \frac{1}{g(t)}\left[c(x)\eta(dx)\delta_x(dy) + (\langle \eta, c\rangle - c(x) - c(y))\eta(dx)\eta(dy)\right].$$

We need the following condition.

Condition 3.1. $c \in C(S)$ satisfies $0 \le \sup c(x) \le 2\inf c(x)$.

This condition is equivalent to $c(x)+c(y) \ge c(z) \ge 0$ for every $x, y, z \in S$.

The following result gives the definition of the space-time inhomogeneous Fleming-Viot process $(Y_t, \mathbf{P}_{r,\mu}^{FV})$ associated with (A, g, c) starting from $\mu \in \mathcal{M}_1$ at $t = r$.

Theorem 3. *Let $\mu \in \mathcal{M}_1$, $g \in C_{r,+}$ and $c \in C(S); c(x) \ge 0$. For $\omega \in \mathbf{C}_{r,\tau_g-} := \mathbf{C}([r, \tau_g) \to \mathcal{M}_1)$, set $Y_t(\omega) = \omega(t)$. Then, under Condition 3.1 on $c(x)$, there is a solution $\mathbf{P}_{r,\mu}^{FV}$ on $\mathbf{C}_{r,\tau_g-}$ to the martingale problem for $(\mathcal{L}_t^g, \mathcal{D}_0(\mathcal{L}^g))_{t\in[r,\tau_g)}$ satisfying the following:*

(i) *$Y_r = \mu$, $\mathbf{P}_{r,\mu}^{FV}$-a.s.,*

(ii) *For $f \in D(A)$ and $r \le t < \tau_g$, $\langle Y_t, f\rangle$ has the following semi-martingale representation:*

$$\langle Y_t, f\rangle = \langle Y_r, f\rangle + \int_r^t \left\{\langle Y_s, Af\rangle + g(s)^{-1}\left[\langle Y_s, c\rangle\langle Y_s, f\rangle - \langle Y_s, cf\rangle\right]\right\} ds + M_{r,t}(f)$$

such that $\{M_{r,t}(f)\}$ is a continuous L^2-martingale with quadratic variation

$$\langle\langle M(f)\rangle\rangle_{r,t} = \int_r^t g(s)^{-1}\left[\langle Y_s, cf^2\rangle + \langle Y_s, c\rangle\langle Y_s, f\rangle^2 - 2\langle Y_s, cf\rangle\langle Y_s, f\rangle\right] ds.$$

Moreover if $A = 0$, then the solution $(Y_t, \mathbf{P}_\mu^{FV})$ is unique.

By using the same argument as in Perkins [8] and the author [6] it is possible to obtain the following result. Recall $\overline{\mu} = \mu/\langle \mu, 1\rangle$.

Corollary 1. *Let $\mu \in \mathcal{M}_F \setminus \{0\}$ and set $y = \langle \mu, 1\rangle$. For a fixed $r \ge 0$, let $(Z_t, \mathbf{P}_\mu)$ be a space-inhomogeneous binary branching process with $A = 0$ on $\mathbf{C}([r,\infty), \mathcal{M}_F)$ such that $Z_r = \mu$. Set $x_t = \langle Z_t, 1\rangle$ and $\tau_0 = \inf\{t \ge r; x_t = 0\}$. If $Q_y = \mathbf{P}_\mu \circ (\{x_t\}_{t\in[r,\tau_0)})^{-1}$, then*

$$\mathbf{P}_\mu\left(\overline{Z}_\cdot|_{[r,\tau_0)} \in B \mid \langle Z_\cdot, 1\rangle = g(\cdot)|_{[r,\tau_g)}\right) = \mathbf{P}_{r,\overline{\mu}}^{FV}(Y_\cdot|_{[r,\tau_g)} \in B), \quad Q_y\text{-a.a. } g \in C_{r,+},$$

where $(Y_t, \mathbf{P}_{r,\overline{\mu}}^{FV})$ is a space-time inhomogeneous Fleming-Viot process associated with $(0, g, c)$ starting from $Y_r = \overline{\mu}$.

Note that $x_t = \langle Z_t, 1\rangle$ has a decomposition $x_t = \langle \mu, f\rangle + m_t$, where $\{m_t\}$ is a continuous martingale starting from 0 with quadratic variation,

$$\langle\langle m\rangle\rangle_t = \int_r^t \langle Z_s, c\rangle ds \quad (r \le t < \tau_0).$$

Proof. For simplicity of the notations, as in [6] we set $Z_t(f) = \langle Z_t, f\rangle$, $|Z_t| = \langle Z_t, 1\rangle = x_t$. Recall that

$$dZ_t(f) = Z_t(Af)dt + dM_t(f), \quad Z_0(f) = \langle \mu, f\rangle,$$

where $\{M_t(f)\}$ is a continuous L^2-martingale with quadratic variation $d\langle\langle M(f)\rangle\rangle_t = \langle Z_t, cf^2\rangle dt$ Thus by using Ito's formula we have

$$d(1/|Z_t|) = -d|Z_t|/|Z_t|^2 + d\langle\langle M(1)\rangle\rangle_t/|Z_t|^3$$

and, noting that

$$d\langle\langle Z(f), (1/|Z|)\rangle\rangle_t = -d\langle\langle M(f), M(1)\rangle\rangle_t/|Z_t|^2 = -[\langle Z_t, cf\rangle/|Z_t|^2]dt,$$

we also have

$$d\overline{Z}_t(f) = \overline{Z}_t(Af)dt + dU_t(f) + [\overline{Z}_t(c)\overline{Z}_t(f) - \overline{Z}_t(cf)]/|Z_t|dt,$$

where

$$dU_t(f) = dM_t(f)/|Z_t| - [\overline{Z}_t(f)/|Z_t|]dM_t(1)$$

is a continuous L^2-martingale with quadratic variation

$$\begin{aligned} d\langle\langle U(f)\rangle\rangle_t &= \frac{d\langle\langle M(f)\rangle\rangle_t}{|Z_t|^2} + \left[\frac{\overline{Z}_t(f)}{|Z_t|}\right]^2 d\langle\langle M(1)\rangle\rangle_t - 2\left[\frac{\overline{Z}_t(f)}{|Z_t|^2}\right] d\langle\langle M(f), M(1)\rangle\rangle_t \\ &= \left[\overline{Z}_t(cf^2) + \overline{Z}_t(f)^2\overline{Z}_t(c) - 2\overline{Z}_t(f)\overline{Z}_t(cf)\right] |Z_t|^{-1}dt. \end{aligned}$$

Hence by the same way as in [8] we can show that $\{\overline{Z}_t\}$ under the condition $|Z_\cdot| = g(\cdot)$ is the space inhomogeneous Fleming-Viot process. □

Proof of Theorem 3.
Let $c(x) \geq 0$ be in $C(S)$ and satisfy Condition 3.1. Fix $g \in C_{r,+}$ and set $c_t^g(x) = c^g(t;x) := c(x)/g(t)$ $(t \leq \tau_g)$,

$$c_t^g(x,y,z) = c^g(t;x,y,z) := \frac{1}{2}\left[c^g(t,x) + c^g(t,y) - c^g(t,z)\right] \ (\geq 0).$$

It is enough to consider the uniqueness and the existence in $\mathbf{C}_{r,T} := \mathbf{C}([r,T] \to \mathcal{M}_1)$ for each fixed $r < T < \tau_g$.

We first show the existence of the solution. Fix $n \geq 1$. Let $\mu^{(n)} = \sum_{k=1}^n \delta_{x_k}$. Let $(X_t^0, \mathbf{Q}^0_{r,\mu^{(n)}})$ be the independent particle system associated with the motion process $(w(t), P_x)$ starting from $X_r^0 = \mu^{(n)}$. Let $(X_t, \mathbf{Q}_{r,\mu^{(n)}})$ be the Markov particle system starting from $X_r = \mu^{(n)}$ such that the Laplace functional $L_{r,t}(\mu^{(n)}) = \mathbf{Q}_{r,\mu^{(n)}}[\exp -\langle X_t, f\rangle]$ is the unique solution to the following equation:

$$
\begin{aligned}
L_{r,t}(\mu^{(n)}) = \frac{1}{n}\sum_{k=1}^{n} \mathbf{Q}^0_{r,\mu^{(n)}} \Bigg[& \exp\left(-\sum_m c^g_m(r,t) - \sum_{i\neq j} c^g_{i,j,k}(r,t) - \langle X^0_t, f\rangle\right) \\
+ & \sum_m \int_r^t ds\; c^g_m(s) \exp\left(-c^g_m(r,s) - \sum_{m'\neq m} c^g_{m'}(r,t) - \sum_{i\neq j} c^g_{i,j,k}(r,t)\right) \\
& \quad L_{s,t}\left(X^0_s - \delta_{w_m(s)} + \delta_{w_k(s)}\right) \\
+ & \sum_{i\neq j} \int_r^t ds\; c^g_{i,j,k}(s) \exp\left(c^g_{i,j,k}(r,s) - \sum_m c^g_m(r,t) - \sum_{i'\neq j';(i',j')\neq(i,j)} c^g_{i',j',k}(r,t)\right) \\
& \quad L_{s,t}\left(X^0_s - \delta_{w_i(s)} + \delta_{w_j(s)}\right)\Bigg],
\end{aligned}
$$

where $c^g_m(t) := c^g(t; w_m(t))$, $c^g_m(r,t) := \int_r^t c^g_m(s)ds$ and $c^g_{i,j,k}(t) := c^g(t; w_i(t), w_j(t), w_k(t))$, $c^g_{i,j,k}(r,t) := \int_r^t c^g_{i,j,k}(s)ds$ $(i \neq j)$.

This particle system can be constructed directly as follows: first n-particles $\{w_i(t); i = 1,2,\dots,n\}$ $(\subset D_r)$ move independently and one particle (e.g. k_1-th particle w_{k_1}) is selected with probability $1/n$ at the starting time $t = r$. Let $\sigma^{(p)}_m = \sigma^{(p)}_m$, $\tau^{(p)}_{i,j,k}$ $(p = 1,2,\dots,$ and $m,i,j,k \in \{1,\dots,n\}$; $i \neq j)$ be independent random variables such that $P_{\mathbf{w}}(\sigma^{(p)}_m > t) = \exp[-c^g_m(r,t)]$ and $P_{\mathbf{w}}(\tau^{(p)}_{i,j,k} > t) = \exp[-c^g_{i,j,k}(r,t)]$, where $\mathbf{w} = \{w_i(t); i = 1,2,\dots,n\}$, The index m_1 and the pair (i_1, j_1) is uniquely defined by $\min_m \sigma^{(1)}_m = \sigma^{(1)}_{m_1}$ and $\min_{i\neq j}\tau^{(1)}_{i,j,k_1} = \tau^{(1)}_{i_1,j_1,k_1}$. If $\sigma^{(1)}_{m_1} < \tau^{(1)}_{i_1,j_1,k_1}$, then after the random time $\sigma^{(1)}_{m_1}$, m_1-th particle jumps to the location of the k_1-th particle and at the same time k_2-th particle is selected with probability $1/n$. If $\sigma^{(1)}_{m_1} > \tau^{(1)}_{i_1,j_1,k_1}$, then after the random time $\tau^{(1)}_{i_1,j_1,k_1}$, i_1-th particle jumps to the location of the another j_1-th particle and at the same time k_2-th particle is selected with probability $1/n$. Again these particles move independently according to the same law. For these particles, we use the same notations $\{w_i(t)\}$. Next the random times $\sigma^{(2)}_{m_2}, \tau^{(2)}_{i_2,j_2,k_2}$ are defined as above by using $\{\sigma^{(2)}_m\}, \{\tau^{(2)}_{i,j,k_2}\}$. Then according to $\sigma^{(2)}_{m_2} > \tau^{(2)}_{i_2,j_2,k_2}$ or $\sigma^{(2)}_{m_2} < \tau^{(2)}_{i_2,j_2,k_2}$ particles moves similarly to above. These operations are continued.

This particle system $(X_t, \mathbf{Q}_{r,\mu^{(n)}})$ is called the *space-time dependent Moran particle system starting from* $\mu^{(n)}$ *at* $t = r$ *associated with the motion process* $(w(t), P_{r,x})$, *sampling rate function* $c(t;x)$.

We denote the generator of independent particle system $\{w_k(t)\}$ by $\mathcal{G}^0$, which is defined as

$$\mathcal{G}^0 e^{-\langle\cdot,f\rangle}(\eta) = -\langle \eta, e^f A(1-e^{-f})\rangle e^{-\langle\eta,f\rangle}.$$

Let $\mu^{(n)} = \sum_{i=1}^n \delta_{x_i}$. The generator $\mathcal{G}_t$ of $(X_t, \mathbf{Q}_{r,\mu^{(n)}})$ is given as

$$\begin{aligned}
\mathcal{G}_t e^{-\langle \cdot, f\rangle}(\mu^{(n)}) &= \frac{1}{n}\sum_{k=1}^n \Bigg[\mathcal{G}^0 e^{-\langle\cdot,f\rangle}(\mu^{(n)}) + \sum_m c_t^g(x_m)\left(e^{-[f(x_k)-f(x_m)]}-1\right)e^{-\langle\mu^{(n)},f\rangle} \\
&\quad + \sum_{i\neq j} c_t^g(x_i,x_j,x_k)\left(e^{-[f(x_j)-f(x_i)]}-1\right)e^{-\langle\mu^{(n)},f\rangle}\Bigg] \\
&= \Bigg\{-\langle \eta, e^f A(1-e^{-f})\rangle + \frac{1}{n}\sum_{k=1}^n\Bigg[\sum_m c_t^g(x_m)\left(e^{-[f(x_k)-f(x_m)]}-1\right) \\
&\quad + \sum_{i\neq j} c_t^g(x_i,x_j,x_k)\left(e^{-[f(x_j)-f(x_i)]}-1\right)\Bigg]\Bigg\}e^{-\langle\mu^{(n)},f\rangle}.
\end{aligned}$$

Set the domain of $\mathcal{G} = (\mathcal{G}_t)$ by

$$\mathcal{D}_0(\mathcal{G}) := \text{lin span}\left\{e^{-\langle\cdot,f\rangle}; f = -\log(1-h), 0 \le h < 1, h \in D(A)\right\}.$$

Then it is easy to see that $(X_t, \mathbf{Q}_{r,\mu^{(n)}})$ is a Markov process with sample paths in $\mathbf{D}([r,\infty) \to \mathcal{M}_F(S))$ and the unique solution to the martingale problem for $(\mathcal{G}_t, \mathcal{D}_0(\mathcal{G}))$ on $\mathbf{D}([r,\infty) \to \mathcal{M}_F(S))$.

Now we consider the scaled Moran particle system $(Y_{n,t}, \mathbf{P}^{(n)}_{r,\mu_n})$, where $Y_{n,t} = X_t/n$ with $\mu_n = \mu^{(n)}/n$ and $\mathbf{P}^{(n)}_{r,\mu_n}$ is its probability law. We also denote the generator by $\mathcal{L}^g_{n,t}$. We shall show that if $\mu_n \to \mu$ in $\mathcal{M}_1$, then the scaling limit $(Y_t, \mathbf{P}^{FV}_{r,\mu})$ exists as a space-time inhomogeneous Fleming-Viot process associated with $(A, 1, c^g) = (A, g, c)$ and has the following generator $\mathcal{L}^g_t$; for $r \le t < \tau_g$,

$$\begin{aligned}
\mathcal{L}^g_t e^{-\langle\cdot,f\rangle}(\eta) &= -\langle \eta, Af\rangle e^{-\langle\eta,f\rangle} \\
&\quad -\frac{1}{g(t)}\left[\langle\eta,c\rangle\langle\eta,f\rangle - \langle\eta,cf\rangle\right]e^{-\langle\eta,f\rangle} \\
&\quad +\frac{1}{2g(t)}\left[\langle\eta,cf^2\rangle + \langle\eta,c\rangle\langle\eta,f\rangle^2 - 2\langle\eta,cf\rangle\langle\eta,f\rangle\right]e^{-\langle\eta,f\rangle}.
\end{aligned}$$

For $f \in D(A)$,

$$\langle Y_{n,t}, f\rangle - \langle Y_{n,r}, f\rangle - \int_r^t \langle Y_{n,s}, Af\rangle ds$$

is a $\mathbf{P}^{(n)}_{r,\mu_n}$-martingale and

$$\sup_n \mathbf{P}^{(n)}_{r,\mu_n}\left[\text{ess sup}_{r\le t\le T}|\langle Y_{n,t}, Af\rangle|\right] \le \|Af\|.$$

Hence by Th. 9.4 in Chap. 3 (p 145) of [3] $\{\langle Y_{n,t}, f\rangle\}$ is relatively compact, i.e., tight in $\mathbf{D}([r,T],\mathbf{R})$ (because $\mathbf{D}([r,T],\mathbf{R})$ is Polish). Moreover since S is compact and $D(A)$ is dense in $C(S)$ and closed under addition, by Th. 3.7.1 in [1] $\{Y_{n,t}\}$ is tight, i.e., relatively compact in $\mathbf{D}([r,T],\mathcal{M}_1)$. Therefore there exist a subsequence $\{(Y_{n_k,t}, \mathbf{P}^{(n)}_{r,\mu_n})\}$ and a limit point $(Y_t, \mathbf{P}_{r,\mu})$ such that $\{Y_{n_k,t}\}$ converges weakly to $\{Y_t\}$ in $\mathbf{D}([r,T],\mathcal{M}_1)$.

For each integer n, let $\mathcal{M}_1^{(n)} = \mathcal{M}_1^{(n)}(S)$ be a family of counting measures on S of the form $\eta_n = \sum_{k=1}^n \delta_{x_k}/n$. Moreover let $f_n = -n\log(1-f/n)$ for $f \in D(A)$ such that

$\|f\| < 1$ and $\inf f > 0$. It is possible to show that for each $r < T < \tau_g$,

$$\lim_{n\to\infty} \sup_{r\le t\le T} \sup_{\eta\in\mathcal{M}_1^{(n)}} |\mathcal{L}^g_{n,t} e^{-\langle\cdot,f_n\rangle}(\eta) - \mathcal{L}^g_t e^{-\langle\cdot,f\rangle}(\eta)| = 0 \tag{3.2}$$

(we shall show this at the end). Hence it is easy to that the limit point $(Y_t, \mathbf{P}_{r,\mu})$ is a solution to the martingale problem for $(\mathcal{L}^g_t, \mathcal{D}_0(\mathcal{L}))$ in $\mathbf{D}([r,T], \mathcal{M}_1)$. We need to show the continuity and the semi-martingale representation of $\{Y_t\}$. However this can be shown by the same way as in the proof of Th. 6.1.3 of [1].

Next in case of $A = 0$ the uniqueness can be shown by the same way as in [9]. In fact, for $r \le t \le T$ and $\eta \in \mathcal{M}_1$ let

$$a_t(\eta, f) = g(t)^{-1}\left[\langle\eta, c\rangle\langle\eta, f\rangle - \langle\eta, cf\rangle\right]$$

and

$$b_t(\eta, f) = g(t)^{-1}\left[\langle\eta, cf^2\rangle + \langle\eta, c\rangle\langle\eta, f\rangle^2 - 2\langle\eta, cf\rangle\langle\eta, f\rangle\right].$$

We consider the following stochastic differential equation: for $r \le t \le T$,

$$d\langle Y_t, f\rangle = a_t(Y_t, f)dt + \sqrt{b_t(Y_t, f)}dB_t, \quad Y_r = \mu, \tag{3.3}$$

where (B_t) is a one-dimensional standard Brownian motion. To show the (law) uniqueness of (Y_t) it is enough to show the pathwise uniqueness of the solution to the above equation in $\mathbf{C}_{r,T}$ (see [7]). However the pathwise uniqueness can be easily checked. Let (Y_t), $(\widetilde{Y}_t)$ be solutions for the equation (3.3) defined on th same probability space (we denote the probability measure as $\mathbf{P}_\mu$). By using the following inequality (let $g_* = \inf_{r\le t\le T} g(t)$)

$$|a_t(\eta, f) - a_t(\widetilde{\eta}, f)| \le N g_*^{-1}(\|c\| \vee \|f\|)\left(|\langle\eta - \widetilde{\eta}, c\rangle| + |\langle\eta - \widetilde{\eta}, f\rangle|\right),$$

$$|b_t(\eta, f) - b_t(\widetilde{\eta}, f)| \le N g_*^{-1}(\|cf\| \vee \|f\|^2)\left(|\langle\eta - \widetilde{\eta}, c\rangle| + |\langle\eta - \widetilde{\eta}, f\rangle|\right),$$

where N is an appropriate number, and we have

$$\mathbf{E}_\mu\left[|\langle Y_t - \widetilde{Y}_t, f\rangle|\right] \le C\int_r^t \mathbf{E}_\mu\left[|\langle Y_s - \widetilde{Y}_s, c\rangle| + |\langle Y_s - \widetilde{Y}_s, f\rangle|\right] ds,$$

where $C > 0$ is a constant depending only on $(g_*, \|c\|, \|f\|)$. Thus we get the pathwise uniqueness of $\{\langle Y_t, c\rangle\}$ and of $\{\langle Y_t, f\rangle\}$ $(f \in C(S))$. Hence the law of uniqueness holds.

Therefore if $A = 0$, then the limit process $(Y_t, \mathbf{P}^{FV}_{r,\mu})_{r\le t\le T}$ uniquely exists in $\mathbf{C}_{r,T}$.

Finally we show (3.2). Note that for $\eta_n = \sum_j \delta_{x_j}/n \in \mathcal{M}_1^{(n)}$,

$$\begin{aligned}
\mathcal{L}^g_{n,t} e^{-\langle\cdot,f_n\rangle}(\eta_n) &= \mathcal{G}_t e^{-\langle\cdot,f_n/n\rangle}(n\eta_n) \\
&= -\langle\eta_n, \frac{Af}{1 - f/n}\rangle e^{-\langle\eta_n,f_n\rangle} \\
&\quad + \frac{1}{n}\sum_{k,m=1}^{n} c^g_t(x_m)\left(e^{(f_n(x_m)-f_n(x_k))/n} - 1\right)e^{-\langle\eta_n,f_n\rangle} \\
&\quad + \frac{1}{n}\sum_{k=1}^{n}\sum_{i\ne j} c^g_t(x_i, x_j, x_k)\left(e^{(f_n(x_i)-f_n(x_j))/n} - 1\right)e^{-\langle\eta_n,f_n\rangle}.
\end{aligned}$$

It is easy to see that

$$\frac{1}{n^2}\sum_{k,m} c(x_m)(f(x_m) - f(x_k)) = -\left(\langle\eta_n, c\rangle\langle\eta_n, f\rangle - \langle\eta_n, cf\rangle\right)$$

and that by symmetry of $c_t^g(x_i, x_j, x_k)$ in (i,j)

$$\sum_{i\neq j} c_t^g(x_i, x_j, x_k)\,[f(x_i) - f(x_j)] = 0.$$

Moreover

$$\begin{aligned}\frac{1}{n^2}\sum_{i\neq j}(f(x_i) - f(x_j))^2 &= \frac{1}{n^2}\sum_{i,j}(f(x_i) - f(x_j))^2 \\ &= 2\left(\langle \eta_n, f^2\rangle - \langle \eta_n, f\rangle^2\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{n^2}\sum_{i\neq j} c(x_i)(f(x_i) - f(x_j))^2 \\ &\qquad = \frac{1}{n^2}\sum_{i,j} c(x_i)(f(x_i) - f(x_j))^2 \\ &\qquad = \langle \eta_n, cf^2\rangle + \langle \eta_n, c\rangle\langle \eta_n, f^2\rangle - 2\langle \eta_n, cf\rangle\langle \eta_n, f\rangle.\end{aligned}$$

Hence by $c_t^g(x_i, x_j, x_k) = [c_t^g(x_i) + c_t^g(x_j) - c_t^g(x_k)]/2$ we have

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n}\sum_{i\neq j} c_t^g(x_i, x_j, x_k)\frac{(f(x_i) - f(x_j))^2}{2n^2} \\ &= \frac{1}{2g(t)}\left[\langle \eta_n, cf^2\rangle + \langle \eta_n, c\rangle\langle \eta_n, f\rangle^2 - 2\langle \eta_n, cf\rangle\langle \eta_n, f\rangle\right].\end{aligned}$$

Therefore by using Taylor's expansion the equation (3.2) can be easily checked. □

Remark 3. If $A = 0$ and the starting measure $Y_r = \mu$ is pure atomic, i.e., $\mu = \sum m_i^0 \delta_{x_i^0}$, then clearly the process Y_t is also pure atomic and the corresponding generator is given as

$$\begin{aligned}G\phi(\mathbf{m}) &= \frac{1}{2}\sum_{i,j}\left[m_i c(x_i^0)\delta_{ij} + m_i m_j\left\{\sum_k m_k c(x_k^0) - c(x_i^0) - c(x_j^0)\right\}\right]\partial_{ij}^2\phi(\mathbf{m}) \\ &\quad + \sum_i b_i(\mathbf{m})\partial_i\phi(\mathbf{m}),\end{aligned}$$

where $b_i(\mathbf{m}) = \left(\sum_j c(x_j^0)m_j - c(x_i^0)\right)m_i$. However in this case our result is contained to Shiga's result [9] in 1987. He showed the result under more general conditions on $c(x)$ and $b_i(\mathbf{m})$ such that let $\beta_i = c(x_i^0)$, $\beta_i \geq 0; \sup_i \beta_i < \infty$ and for some matrix (q_{ij}); $q_{ij} \geq 0, \sup_j \sum_i q_{ij} < \infty$, $|b_i(\mathbf{m}) - b_i(\mathbf{m}')| \leq \sum_j q_{ij}|m_j - m_j'|$.

By using the same argument as in the proof of Theorem 3 we can see the following results:

Theorem 4. *In Theorem 3, if the quadratic martingale part is changed to the following*

$$\langle\langle M(f)\rangle\rangle_{r,t} = \int_r^t g(s)^{-1}\left[\langle Y_s, cf^2\rangle + \langle Y_s, c\rangle\langle Y_s, f^2\rangle - 2\langle Y_s, cf\rangle\langle Y_s, f\rangle\right]ds,$$

then the same claim holds for all bounded functions $c \in C(S); c(x) \geq 0$ without Condition 3.1. Moreover it is possible to construct the processes without the drift term $g(t)^{-1}[\langle \eta, c\rangle\langle \eta, f\rangle - \langle \eta, cf\rangle]$.

Proof. We have to consider the approximating particle systems $\{X_t\}$. However it is enough to change $c_t^g(x,y,z)$ to $c_t^g(x,y) = c^g(t;x,y) := [c^g(t,x) + c^g(t,y)]/2$, thus $c_{i,j,k}^g(t)$ to $c_{i,j}^g(t) := c_t^g(t; w_i(t), w_j(t))$. Moreover for the processes without drift term, it enough to delete the terms corresponding to $c_m^g(s)$ and $c_t^g(x_m)$. □

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