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Path Level Large Deviation of Measure-Valued Processes in A Random Medium†

Isamu DÔKU (道工 勇) 埼玉大学
Department of Mathematics, Saitama University, Urawa 338-8570
Japan
E-mail: doku@post.saitama-u.ac.jp

Abstract

We consider the measure-valued processes in a super-Brownian random medium in the Dawson-Fleischmann sense (1997). We prove the full large deviation principle (LDP) of path level for a family of scaled processes of the above-mentioned class. By virtue of the general theory of LDP it suffices to show the exponential tightness of the family in question in order to derive the full LDP from the weak LDP. Our principal contribution of this paper consists in giving an easily checkable sufficient condition for the exponential tightness. Another underlying remarkable feature of this paper is an application of historical superprocess approach to analysis of specific functionals of various kinds of processes involved in the story.

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I Introduction and Main Results

The systems considered in random media are related in most cases to stochastic models which are introduced, for instance, based upon the following two distinct viewpoints in random chemical systems or in random biological systems. The first one is a microscopic view in the chemical reaction, where a molecule reveals a certain chemical reaction only in the places where exists the specific reactant. The second one is just the case where, in the macroscopic view, the chemical reaction is described by reaction-diffusion equations and the effecting of reactor enters as a spatially heterogeneous rate function. In some cases there are reactants present only in the localized regions such as networks of filaments or the surfaces of pellets.

Mathematically, such systems are modelled by the following nonlinear reaction-diffusion equations in $\mathbf{R}^d$

$$-\frac{\partial u}{\partial s} = \frac{1}{2}\Delta u + \rho_s \cdot R(u), \qquad 0 \le s \le t \tag{1}$$

with terminal condition $u|_{s=t} = \varphi$. Here R is a reaction term, and ρ_s is a spatial density of the reaction trigger at time s with continuous measure-valued path : $s \mapsto \rho_s \in \mathcal{M}(\mathbf{R}^d)$.

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Let $p(r,b)$ denote the transition density of a standard Brownian motion in $\mathbf{R}^d$. Then the above (1) can be formulated rigorously by the following integral equation [8]:

$$u(s,t,a) = \int p(t-s, b-a)\varphi(b)\mathrm{d}b + \int_s^t dr \int p(r-s, b-a)R(u(r,t,b))\rho_r(\mathrm{d}b). \qquad (2)$$

Our main concerns are firstly to formulate scaling of the stochastic process associated with the equation (2) meaningfully as generalized as possible, and secondly to investigate limiting behaviors of a family of scaled processes as the scaling parameter varies, whereby we aim at establishing the large deviation principle of the associated stochastic processes. In this paper we will treat simply the typical case $R(u) = u^2$.

Let us now introduce our main results in this paper. In connection with (1), we consider the following nonlinear parabolic equation in a random medium

$$\begin{cases} -\dfrac{\partial v}{\partial s} &= \frac{1}{2}\Delta v - \rho_s \cdot v^2, \quad 0 < s \leq t \\ v|_{s=t} &= \varphi. \end{cases} \qquad (*)$$

Then naturally there corresponds some measure-valued process X to this problem $(*)$, which we call *a super-Brownian motion in a random medium.* This type of process was originally introduced and investigated by Dawson-Fleischmann (1997) [8]. In this paper we study large deviations for such processes, and in fact establish the path level large deviation principle for a family of scaled measure-valued processes (εX_t) in a random medium.

THEOREM A. *Let $d \leq 3$ and $\mu \in \mathcal{M}_p$. For $\mathbf{P}_\nu$ -a.a. realization $X^\gamma(w)$, the distributions of $(\varepsilon X_t^{L(\gamma)})_{t\in[0,1]}$ with respect to $P^\gamma_{\mu/\varepsilon}$ satisfy the Large Deviation Principle with speed $1/\varepsilon$ and good rate function I^γ_μ as $\varepsilon \downarrow 0$.*

We are interested in large deviation principle, in particular, for measure-valued stochastic processes in a random medium in which a very singular measure is involved in the previously mentioned sense [25]. In addition to the above result, we can derive the explicit representation of our rate function for LDP.

THEOREM B. *Moreover, the good rate function I^γ_μ is given by*

$$I^\gamma_\mu(\omega) := \sup_{\substack{f\in \\ C_K([0,1]\times R^d)}} \Big(\langle\langle \omega(\cdot), f(\cdot)\rangle\rangle - \log P^\gamma_\mu \exp\langle\langle X_\cdot^{L(\gamma)}, f(\cdot)\rangle\rangle \Big)$$

for $\omega \in \mathcal{C}([0,1],\mathcal{M}_p)$. Here $\langle\langle\cdot,\cdot\rangle\rangle$ is defined by

$$\langle\langle \mu(\cdot), f(\cdot)\rangle\rangle := \int_0^1 \langle \mu(t), f(t)\rangle \mathrm{d}t \qquad \text{for} \quad \mu(t) \in \mathcal{M}_p.$$

Historically, for the cases when ρ_s in (1) are *nice* measures having mass on an open set or a hypersurface, the equation (1) has been studied via analytic method by Chadan-Yin [3], Chan-Fung [4], Bramson-Neuhauser [2], and Durrett-Swindle [29]. On the other hand, the relationship between semilinear reaction-diffusion equations, branching particle systems, and superprocesses (or measure-valued processes) has been investigated by Dynkin-Kuznetsov [33], Le Gall [41], and Gorostiza-Wakolbinger [38]. At the same time this implies that probabilistic research on analysis of this sort of equation like (1) may provide with a natural approach to the asymptotic problem, in connection with the associated superprocesses. As to the works for stochastic processes with catalytic branching, there can be found interesting and exciting new results in series of papers written by Dawson-Fleischmann [6, 7, 8], and Fleischmann-Le Gall [37].

This paper is organized as follows.

In Section II we introduce basic notations and preliminaries used in the succeeding sections through the whole paper. Section III is devoted to the construction of some measure-valued process in a super-Brownian random medium. In particular, in Subsection III.1 we shall look at a quick review of super-Brownian motion (or Dawson-Watanabe superprocess) in terms of Dynkin's formulation [32], which plays an essential role later as underlying process in construction of the superprocess in question. The useful tools called *branching rate functionals* (BRF) are provided in Subsection III.2, where we introduce several classes of BRF. Each class possesses its own peculiar feature to work effectively in the investigation of properties of the corresponding measure-valued processes, such as existence of process itself, its characterization, existence of modification with continuous sample paths, etc. Furthermore, *Brownian collision local time* (BCLT) is constructed in Subsection III.4, whereby the existence of superprocess with BCLT as its branching rate functional can be shown in Subsection III.5 as well. In Section IV we state the theorems on the path level large deviation principle for a family of scaled measure-valued processes in a random medium (Theorem 13, Theorem 13′), which are the chief results in this paper. The proof of Theorem A is given in the suceeding sections. The central argument on the proof can be attributed, in Section V, to the problem on the exponential tightness in terms of the general theory of large deviation principles. Our main contribution in this paper consists in derivation of easily checkable sufficient conditions (cf. conditions (I), (II) in Theorem 16). The main part of the proof of (I) is given in Section VI, while the principal part of the proof of (II) is stated in Section VII with functional analytical discussion. In Section VIII we indulge ourselves in the proof of Theorem B (cf. Theorem 13′). Section IX is devoted to introduction of path-valued processes and historical superprocesses. Our another contribution in this paper consists in establishing the formulation of historical version of the measure-valued processes (MPRM) $X^{L(\gamma)}$ in a random medium, which are involving the very singular measure (=BCLT). As application of those processes, we can succeed in getting various types of estimates of some functionals, which are crucial for the precise estimates of our sufficient conditions. One of peculiar features in this paper is to use historical MPRM extensively as an essential tool for stochastic analysis. Rough sketch of the above discussion is presented in the last section, namely, Section X.

II Notation and Preliminaries

Let p be a positive number such that $p > d$, where d is the space dimension parameter. φ_p is a reference function defined by

$$\varphi_p(x) := (1 + |x|^2)^{-p/2}, \qquad x \in \mathbf{R}^d.$$

We denote by $\mathcal{C}^p$ the space of continuous functions f on $\mathbf{R}^d$ such that $|f| \le C_f \varphi_p$ for some positive constant C_f depending on f. The norm $\|f\|$, $f \in \mathcal{C}^p$ is defined by

$$\|f\| := \|f/\varphi_p\|_\infty,$$

where $\|\cdot\|_\infty$ is the supremum norm. Then $(\mathcal{C}^p, \|\cdot\|)$ becomes a Banach space. $\mathcal{C}^p_+$ is the totality of positive elements of $\mathcal{C}^p$. For a time interval I in $\mathbf{R}_+$, $\mathcal{C}^{p,I}$ denotes the space of all functions $f(s,x)$ in $\mathcal{C}(I \times \mathbf{R}^d)$ such that there exists a positive constant C_f depending on f, satisfying

$$|f(s,\cdot)| \leq C_f \cdot \varphi_p \quad \text{for} \quad s \in I.$$

$\mathcal{C}_K = \mathcal{C}_K(\mathbf{R}^d)$ is the totality of all continuous functions on $\mathbf{R}^d$ with compact support. Let $\mathcal{B} \equiv \mathcal{B}(\mathbf{R}^d)$ denote the space of all Borel measurable functions on $\mathbf{R}^d$. We say that $f \in \mathcal{B}$ if $f : \mathbf{R}^d \to \mathbf{R}$ is $\mathcal{B}$-measurable. Let $\mathcal{B}^p$ denote the set of all those $f \in \mathcal{B}$ satisfying $|f| \leq C_f \varphi_p$ for some constant C_f. Moreover, $f \in b\mathcal{B}^p$ means that f is a bounded element of $\mathcal{B}^p$. As is easily imagined, the symbols $\mathcal{B}^p_+$, $\mathcal{B}^{p,I}$, etc. denote those measurable counterparts of $\mathcal{C}^p_+$, $\mathcal{C}^{p,I}$, etc. respectively. Let $\mathcal{M}_p \equiv \mathcal{M}_p(\mathbf{R}^d)$ denote the set of all locally finite non-negative measures μ on $\mathbf{R}^d$, such that

$$\|\mu\|_p := \langle \mu, \varphi_p \rangle = \int_{R^d} \varphi_p(y)\mu(\mathrm{d}y) < \infty.$$

$\mathcal{M}_p$ is also called the set of tempered measures on $\mathbf{R}^d$, endowed with the topology generated by

$$\text{the maps} : \mathcal{M}_p \ni \mu \mapsto \langle \mu, f \rangle, \quad \text{for} \quad f \in \{\varphi_p\} \bigcup \mathcal{C}_K(\mathbf{R}^d).$$

Notice that $\mathcal{M}_p$ becomes a Polish space. While, $\mathcal{M}_F = \mathcal{M}_F(\mathbf{R}^d)$ is the set of all finite measures on $\mathbf{R}^d$. We denote by

$$B = (B_t, \Pi_{s,a})$$

a d-dimensional Brownian motion. In addition, $S = (S_t)_{t \geq 0}$ denotes the Brownian semi-group.

III Super-Brownian Motion in A Random Medium

III.1 Super-Brownian Motion as The Underlying Process

We begin with definition of super-Brownian motion, which is based on the martingale problem formulation. Let Ω be the path space $\mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$, and K_0 be a special branching rate functional given by $K_0(\mathrm{d}r) := \gamma \mathrm{d}r$ for some constant $\gamma > 0$. We consider the measure-valued process $X^{K_0} \equiv X^\gamma$ with branching rate functional K_0. For each $\mu \in \mathcal{M}_p$ as initial measure, there exists a probability measure $\mathbf{P}^\gamma_\mu$ on $(\Omega, \mathcal{F})$ such that $X^\gamma_0 = \mu$, $\mathbf{P}^\gamma_\mu$- a.s., and

$$M_t(\psi) := \langle X^\gamma_t, \psi \rangle - \langle \mu, \psi \rangle - \int_0^t \langle X^\gamma_s, \frac{1}{2}\Delta\psi \rangle \mathrm{d}s, \quad (\forall t > 0, \psi \in \mathrm{Dom}(\frac{1}{2}\Delta))$$

is a continuous $\mathcal{F}_t$-martingale under $\mathbf{P}^\gamma_\mu$, where the quadratic variation process $\langle M.(\psi) \rangle_t$ is given by

$$\langle M.(\psi) \rangle_t = 2\gamma \int_0^t \int \psi(\eta)^2 X^\gamma_s(d\eta)\mathrm{d}s, \qquad \mathbf{P}^\gamma_\mu - a.s.$$

for $\forall t > 0$ (cf. [5]). We adopt this super-Brownian motion X^γ as underlying process to construct a measure-valued process in a random medium in the succeeding sections. We would rather use the symbol W as underlying process instead of X^γ for simplicity.

Next we shall present a characterization of super-Brownian motion $W \equiv (W_t)$. Actually, $W = [\,X_t^\gamma = X_t^{K_0},\, \mathbf{P}_\mu^\gamma,\, t > 0,\, \mu \in \mathcal{M}_p\,]$ with $p > d$, $\gamma > 0$ is an $\mathcal{M}_p$ -valued Markov process whose Laplace transition functional is given by

$$\mathbf{P}_{s,\mu}^\gamma \exp\langle X_t^\gamma, -\varphi\rangle = \exp\langle \mu, -v^{[\varphi]}(s,t,\cdot)\rangle, \qquad \varphi \in \mathcal{C}_{+,K} \tag{7}$$

where the solution $v(t) \equiv v^{[\varphi]}(t)$ (≥ 0) of the log-Laplace equation

$$v(s,t,x) + \Pi_{s,x}\int_s^t \gamma v^2(r,t,B_r)\mathrm{d}r = \Pi_{s,x}\varphi(B_t) \tag{8}$$

solves uniquely the nonlinear parabolic equation

$$-\frac{\partial v}{\partial s} = \frac{1}{2}\Delta v - \gamma v^2 \qquad \text{with} \qquad v|_{s=t} = \varphi. \tag{9}$$

Note that

$$\Pi_{s,x}\varphi(B_t) = \int p(s,x;t,y)\varphi(y)\mathrm{d}y,$$

where $p(s,x;t,y)$ is the probability density function associated with transition function of the Brownian motion $B = (B_t, \Pi_{s,a})$.

III.2 Branching Rate Functionals

The additive functional $K = K(w)$ of Brownian motion $B = (B_t)$ is a random measure $K = K(\omega, \mathrm{d}t)$ on $(0,\infty)$ such that for any $r \leq t$, $K(\cdot,(r,t))$ is measurable with respect to the completion of $\mathcal{F}(r,t)$ relative to $\Pi_{r,\mu}$, where $\Pi_{r,\mu}$ is defined by $\int \Pi_{r,x}\mu(\mathrm{d}x)$ for any $\mu \in M_F$. Let $\mathcal{K}$ be the set of all branching rate functionals. We say that $K \in \mathcal{K}$ if an additive functional $K = K(w)$ satisfies the following two conditions:
(a) (Continuity) $K(\mathrm{d}r)$ does not carry mass at any single point set.
(b) (Local Admissibility) For $u \geq 0$,

$$\sup_{a\in R^d} \Pi_{s,a}\int_s^t \varphi_p(B_r)K(\mathrm{d}r) \to 0 \quad \text{as} \quad s,t \to u.$$

DEFINITION 1. Let $K \in \mathcal{K}$. We say that $K \in \mathcal{K}^*$ if for each finite interval $I = [L,T] \subset \mathbf{R}_+$, there is a positive constant $C(I)$ such that

$$\sup_{s\in I} \Pi_{s,a}\int_s^T \varphi_p^2(B_r)K(\mathrm{d}r) \leq C(I)\cdot\varphi_p(a), \qquad a \in \mathbf{R}^d.$$

DEFINITION 2. We say that $K \in \mathcal{K}^\beta$ ($\beta > 0$) if for each $N > 0$, there is a positive constant $C(N)$ such that

$$\Pi_{s,a}\int_s^t \varphi_p^2(B_r)K(\mathrm{d}r) \leq C(N)|t-s|^\beta\cdot\varphi_p(a) \quad \text{for} \quad 0 \leq s \leq t \leq N,\ a \in \mathbf{R}^d.$$

Notice that we have a natural inclusion $\mathcal{K}^\beta \subset \mathcal{K}^*$.

III.3 Measure-Valued Process with Continuous Paths

Let $K \in \mathcal{K}^\beta$ for some $\beta > 0$. Then it is easy to show that there exists a probability measure $P_{s,\mu} \in \mathcal{M}_1(\mathcal{C}(\mathbf{R}_+, \mathcal{M}_p))$ (or $\in \mathcal{P}(\mathcal{C}(\mathbf{R}_+, \mathcal{M}_p))$) such that for $\varphi \in \mathcal{C}_{+,K}$

$$P_{s,\mu} \exp\langle X_t^K, -\varphi\rangle = \exp\langle \mu, -v(s,t)\rangle \tag{10}$$

and $v \equiv v^{[\varphi]}$ is the unique solution of the log-Laplace equation

$$v(s,t,a) + \Pi_{s,a} \int_s^t v^2(r,t,B_r)K(\mathrm{d}r) = \Pi_{s,a}\,\varphi(B_t). \tag{11}$$

Define the centered process

$$Z_t := P_{s,\mu} X_t^K - X_t^K \qquad \text{for} \qquad t \geq s. \tag{12}$$

Since $K \in \mathcal{K}^\beta$ for some $\beta > 0$, we can assert Hölder continuity of Z_t. As a matter of fact, we may apply the recursive scheme for moments (cf. Dawson-Fleischmann (1994) [7]) together with the Kolmogorov criterion to obtain

LEMMA 1. *For $N > 0$, $\mu \in \mathcal{M}_p$, $k \geq 1$ and $\varepsilon \in (0, \beta/2)$, there exists a modification $\tilde{Z}$ of Z such that*

$$\sup_{0\leq s\leq N} P_{s,\mu}\big[\sup_{s\leq t\leq t+h\leq N} |\langle \tilde{Z}_{t+h} - \tilde{Z}_t, \varphi\rangle|/h^\varepsilon \big]^k < +\infty \qquad for \quad \varphi \in \mathcal{D}_0 \tag{13}$$

where $\mathcal{D}_0 = \{\varphi_1, \varphi_2, \cdots\}$ is a countable subset of $\mathrm{Dom}(\frac{1}{2}\Delta)$.

For $\varphi_k \in \mathcal{D}_0$, we can define a metric d_p in $\mathcal{M}_p$ as

$$d_p(\mu,\nu) := \sum_{m=1}^{\infty} \frac{1}{2^m}(1 \wedge |\langle \mu, \varphi_m\rangle - \langle \nu, \varphi_m\rangle|) \quad \text{for} \quad \mu, \nu \in \mathcal{M}_p. \tag{15}$$

Note that $(\mathcal{M}_p, d_p)$ becomes a metric space. In particular, $\tilde{Z}$ has $P_{s,\mu}$ -a.s. locally Hölder continuous paths of order ε in the metric d_p. As a result, we obtain

PROPOSITION 2. *If $K \in \mathcal{K}^\beta$ for some $\beta > 0$, then there exists a modification $\tilde{X}$ of measure-valued process X^K with continuous paths, that is, $\tilde{X} \in \mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$.*

Proof. From the expectation formula for the measure-valued process X^K, we have $P_{s,\mu}X_t^K = S_{t-s}\mu$ for $\mu \in \mathcal{M}_p$. For the one-point compactification $\mathbf{R}_*^d$ of $\mathbf{R}^d$, we denote by $\mathcal{C}_*^p$ the subspace of all elements $f \in \mathcal{C}^p$ such that the mapping $F : x \to F(x) := f(x)/\varphi_p(x)$ can be extended to a function in $\mathcal{C}(\mathbf{R}_*^d)$. Note that $\mathcal{C}_*^p$ becomes a separable Banach space. Since $t \mapsto S_t\varphi$ is a continuous curve in $\mathcal{C}_*^p$, the map $t \mapsto S_t\mu \in \mathcal{M}_p$ can be regarded as a continuous mapping. From (12) and (13), we get $S_{t-s}\mu - \tilde{Z}_t = P_{s,\mu}X_t^K - \tilde{Z}_t = X_t^K$, implying that there can be found a continuous $\mathcal{M}_p$ -valued process if we retake the modification of X^K. Q.E.D.

III.4 Regular Paths and Brownian Collision Local Time

Let $N > 0$, $0 < \varepsilon \leq 1$ be fixed, and take $\eta \in \mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$. We define

$$R_N^\varepsilon(\eta) := \sup_{\substack{0 \leq s \leq N \\ a \in R^d}} \int_s^{s+\varepsilon} \langle \eta_r, \varphi_p \cdot p(s,a;r,\cdot)\rangle \, \mathrm{d}r. \tag{22}$$

Suggested by Dawson-Fleischmann (1997) [8], we shall give below the definition of regular paths. If the path is regular, then the existence of the corresponding collision local time as branching rate functional is able to be guaranteed.

DEFINITION 3. We say that η is a regular path if $R_N^\varepsilon(\eta) \to 0$ holds for any $N > 0$ as ε tends to zero. Then we write $\eta \in \mathcal{R}$.

For the underlying process $W = X^\gamma = X^{K_0}$ with $K_0(\mathrm{d}r) = \gamma \mathrm{d}r$, $\gamma > 0$, we know that $W \in \mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$ with probability one and moreover, $W \in \mathcal{R}$, namely, we observe that the process X^γ has a regular path in the sense of Definition 3. Indeed we have

LEMMA 6. (Dawson-Fleischmann (1997) [8]) *The realization $\rho_{\delta+(\cdot)}$ is a regular path with* $\mathbf{P}$ *-probability one.*

Then for $0 < \varepsilon \leq 1$ we define a continuous additive functional of Brownian motion W by

$$L^\varepsilon(\gamma) \equiv L^\varepsilon(\gamma)_{[B,W]}(\mathrm{d}r) := \langle W_r, p(0, B_r; \varepsilon, \cdot)\rangle \, \mathrm{d}r. \tag{24}$$

Hence a general theory for additive functionals deduces the existence of the limit $L(\gamma)$ of $\{L^\varepsilon(\gamma)\}$.

PROPOSITION 7. (Dawson-Fleischmann (1997) [8]) *There exists an additive functional* $L(\gamma) \equiv L(\gamma)_{[B,W]}(\mathrm{d}r)$ *of Brownian motion B such that for any* $\psi \in \mathcal{C}_+^{p,I}$ *with* $I = [0, N]$, $N > 0$,

$$\sup_{\substack{0 \leq s \leq N \\ a \in R^d}} \Pi_{s,a} \sup_{s \leq t \leq N} \left| \int_s^t \psi(r, B_r) L^\varepsilon(\gamma)(\mathrm{d}r) - \int_s^t \psi(r, B_r) L(\gamma)(\mathrm{d}r) \right|^2 \to 0 \quad (\varepsilon \searrow 0). \tag{25}$$

Define a continuous additive functional $A^\varepsilon = A^\varepsilon(B, \psi W)$ as

$$A^\varepsilon(B, \psi W)(\mathrm{d}r) := \langle \psi(r, B_r) W_r, p(0, B_r; \varepsilon, \cdot)\rangle \mathrm{d}r \quad \text{for} \quad \psi \in \mathcal{C}_+^{p,[0,N]}$$

in line with (24). The convergence

$$\sup_{s,a} \Pi_{s,a} \left(\sup_{0 \leq t \leq N} |A^\varepsilon(B, \psi W)(s,t) - A(B, \psi W)(s,t)|^2 \right) \to 0 \quad (\text{as } \varepsilon \downarrow 0) \quad N > 0, \tag{26}$$

for some continuous additive functional $A(B,\psi W)$ (the limit functional) of Brownian motion $B=(B_t)$ plays an essential role in the proof of Proposition 7.

Furthermore, it is possible to state a stronger result on the above convergence (26). Let h be a function : $[0,1]\to\mathbf{R}_+$ such that $h(u)\searrow 0$ as $u\to 0$. For $M\in\mathbf{N}$, $\psi\in C_+^{p,I}$, define the set $\Phi(h,M)$ as

$$\left\{\eta\in\mathcal{R}:\ \int_0^N \eta_s(1)\mathrm{d}s\le M,\ \sup_{s,a}\int_0^u \mathrm{d}r\int p(s,a;r,b)\psi(r,b)\eta_r(\mathrm{d}b)\le h(u),\ \forall u\le 1\right\}.$$

Take a sequence $\{s(k)\}$ such that $s(k)\nearrow N$ as $k\to\infty$. Set $M_t^\varepsilon:=\Pi_{s,a}[A^\varepsilon(B,\psi\eta)(s,s(\infty))\mid B_u,u\le t]$. By Markov property we can rewrite it as

$$M_t^\varepsilon=A^\varepsilon(B,\psi\eta)(s,t)+\Pi_{t,B_t}A^\varepsilon(B,\psi\eta)(t,s(\infty)). \tag{27}$$

Then notice that M_t^ε is a nonnegative $L^2(\Pi_{s,a})$ -martingale such that $\lim_{t\to N}M_t^\varepsilon=A^\varepsilon(B,\psi\eta)(s,N)$, $\Pi_{s,a}$ -a.s. Therefore, we may apply the Doob maximal L^2 inequality to get

$$\begin{aligned}
&\Pi_{s,a}(\sup_t|M_t^\varepsilon-M_t^\delta|^2)\\
\le\ & C\cdot\Pi_{s,a}|A^\varepsilon(B,\psi\eta)(s,s(\infty))-A^\delta(B,\psi\eta)(s,s(\infty))|^2\\
\le\ & 2C\cdot\Pi_{s,a}\int_s^{s(\infty)}\left(\int\{p(0,B_u;\varepsilon,b)-p(0,B_u;\delta,b)\}\psi(u,B_u)\eta_u(\mathrm{d}b)\right)\cdot\\
&\times\Pi_{u,B_u}\int_s^{s(\infty)}\left(\int\{p(0,B_r;\varepsilon,b)-p(0,B_r;\delta,b)\}\psi(r,B_r)\eta_r(\mathrm{d}b)\right)\mathrm{d}r\mathrm{d}u
\end{aligned}$$

$$\le 4C\,|||\Pi_{\cdot,\cdot}A(B,\psi\eta)(0,N)|||_\infty\cdot\left|\left|\left|\Pi_{\cdot,\cdot}\int(\psi\eta_r)*p(\varepsilon)\mathrm{d}r-\Pi_{\cdot,\cdot}\int(\psi\eta_r)*p(\delta)\mathrm{d}r\right|\right|\right|_\infty. \tag{28}$$

Combining (28) with (27) we get

$$\begin{aligned}
&\sup_{s,a}\Pi_{s,a}\left(\sup_{0\le t\le N}|A^\varepsilon(B,\psi\eta)(s,t)-A^\delta(B,\psi\eta)(s,t)|^2\right)\\
\le\ & C'\left|\left|\left|\Pi_{\cdot,\cdot}\int\int(p(\varepsilon)-p(\delta))\psi\eta_r(\mathrm{d}b)\mathrm{d}r\right|\right|\right|_\infty^2\\
&+C''\,|||\Pi_{\cdot,\cdot}A(B,\psi\eta)(0,N)|||_\infty\times\left|\left|\left|\Pi_{\cdot,\cdot}\int\int(p(\varepsilon)-p(\delta))\psi\eta_r(\mathrm{d}b)\mathrm{d}r\right|\right|\right|_\infty.
\end{aligned} \tag{29}$$

Hence it is obvious from the fact

$$\lim_{\varepsilon\downarrow 0}\left|\left|\left|\Pi_{\cdot,\cdot}A(B,\psi\eta)(0,s(\infty))-\Pi_{\cdot,\cdot}\int(\psi\eta_r)*p(\varepsilon)\mathrm{d}r\right|\right|\right|_\infty=0$$

uniformly in $\eta\in\Phi(h,M)$ that the term $|||\Pi.A(B,\psi\eta)(0,N)|||_\infty$ is uniformly bounded with respect to $\eta\in\Phi(h,M)$, because

$$\left|\left|\left|\Pi_{\cdot,\cdot}\int(\psi\eta_r)*p(\varepsilon)\mathrm{d}r\right|\right|\right|_\infty\le C(\varepsilon)\cdot\sup_{s,a}\int_0^u\mathrm{d}r\int p(s,a;r,b)\psi(r,b)\eta_r(\mathrm{d}b)$$

holds. Therefore we can deduce that (29) converges to zero as $\varepsilon, \delta \to 0$ uniformly relative to $\eta \in \Phi(h, M)$. Thus we obtain:

PROPOSITION 8. *The convergence* (26) *in the above (cf. Proposition 7) is uniform on* $\Phi(h, M)$.

On this account, we can construct the corresponding measure-valued process $X^{L(\gamma)}$ in a random medium with branching rate functional $L(\gamma)$. Actually, $L(\gamma)$ is nothing but a Brownian collision local time (BCLT) in the sense of Barlow-Evans-Perkins (1991) [1] (cf. [28]). According to Dynkin's terminology [30], it can be said that the branching phenomenon of the approaching particles is governed by the Brownian collision local time with super-Brownian particles.

PROPOSITION 9. *For any* $\mu \in \mathcal{M}_p$, *for* $\mathbf{P}_\mu$- *a.a. realization* $W(w)$, *there exists a Brownian collision local time* $L(\gamma) = L(\gamma)_{[B,W]}(\mathrm{d}r) \in \mathcal{K}^\beta$ *for some* $\beta > 0$.

Therefore we can construct a measure-valued process with $L(\gamma)$ as its branching rate functional by virtue of Proposition 2. Moreover, the existence of its continuous modification as measure-valued path is also automatically guaranteed. We shall see this in details in the next subsection.

III.5 Measure-Valued Process in A Super-Brownian Medium

Since we know that our $L(\gamma)$ lies in $\mathcal{K}^\beta$, we may resort to the general construction method for measure-valued processes with branching rate functional $K = L(\gamma)$ [8] (see also [16,22]) to obtain

THEOREM 10. *Let* $d \leq 3$. *There exists a unique* $\mathcal{M}_p$ *-valued Markov process*

$$X_t^{L(\gamma)} = [\, X_t^{L(\gamma)}, P_{s,\mu}^\gamma;\ t \geq 0, \mu \in \mathcal{M}_p\,]$$

(with branching rate functional $L(\gamma)$*) whose Laplace transition functional is given by*

$$P_{s,\mu}^\gamma \exp\langle X_t^{L(\gamma)}, -\varphi\rangle = \exp\langle \mu, -v^{[\varphi]}(s,t,\cdot)\rangle \tag{33}$$

for an element φ of $\mathcal{C}_{+,K}$, *where the function* $v \equiv v^{[\varphi]}(\cdot, t, \cdot)$ *is the unique solution of the log-Laplace equation*

$$v(s,t,a) + \Pi_{s,a} \int_s^t v^2(r,t,B_r) L(\gamma)(\mathrm{d}r) = \Pi_{s,a}\varphi(B_t) \quad (for\ \ 0 \leq s \leq t,\ \ a \in \mathbf{R}^d). \tag{34}$$

Remark 1. It can be interpreted, in fact, as the particle view that a hidden Brownian particle at position $y = B_r \in \mathbf{R}^d$ at time r branches with rate $L(\gamma)_{[B,W]}(\mathrm{d}r)$.

By virtue of the discussion in the previous sections (cf. Proposition 2 and Proposition 9), there exists a modification $\tilde{X}$ of $X^{L(\gamma)}$ such that $\tilde{X}_t \in \mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$ since $K = L(\gamma) = L(\gamma)_{[B,W]}(dr) \in \mathcal{K}^{1/2}$ (cf. [8, 18]).

III.6 Moment Formulae

We have the following moment formulae for measure-valued process $X^{L(\gamma)}$ in a random medium with branching rate functional $L(\gamma)$.

LEMMA 11. *For* $0 \le s \le t$, $\mu \in \mathcal{M}_p$, and $\varphi \in \mathcal{B}^p_+$, *we have the expectation formula*

$$P^\gamma_{s,\mu}\langle X^{L(\gamma)}_t, \varphi\rangle = \Pi_{s,\mu}\,\varphi(B_t) = \langle \mu, S_{t-s}\varphi\rangle = \langle S_{t-s}\mu, \varphi\rangle < +\infty \tag{39}$$

where $S = (S_t)_{t\ge 0}$ *is the Brownain semigroup* (cf. Section II).

Similarly we can easily show

LEMMA 12. *For* $0 \le s \le t, u$, *any* $\mu \in \mathcal{M}_p$, *and* $\varphi, \psi \in \mathcal{B}^p_+$, *we have the following covariance formula*

$$COV^{P_{s,\mu}}[\langle X^{L(\gamma)}_t, \varphi\rangle, \langle X^{L(\gamma)}_u, \psi\rangle] = 2\Pi_{s,\mu}\int_s^{t\wedge u} S_{t-r}\varphi(B_r)S_{u-r}\psi(B_r)L(\gamma)(dr). \tag{41}$$

IV Main Results

Hereafter the path space $\mathcal{C}(\mathbf{R}_+, \mathcal{M}_p)$ is assumed to be endowed with compact-open topology (cf. Remark 2 in Section V). We shall introduce the main results in this paper, which implies the establishment of path level large deviation principle for the measure-valued processes in a random medium.

THEOREM 13. *Let* $d \le 3$ *and* $\mu \in \mathcal{M}_p$. *For* $\mathbf{P}_\nu$ *-a.a. realization* $X^\gamma(w)$, *the distributions of* $(\varepsilon X^{L(\gamma)}_t)_{t\in[0,1]}$ *with respect to* $P^\gamma_{\mu/\varepsilon}$ *satisfy the Large Deviation Principle with speed* $1/\varepsilon$ *and good rate function* I^γ_μ *as* $\varepsilon \downarrow 0$. *That is to say,*
(a) (Upper bound) *for any closed subset* $A \subset \mathcal{C}([0,1], \mathcal{M}_p)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log P^\gamma_{\mu/\varepsilon}\left(\varepsilon X^{L(\gamma)}(w) \in A\right) \le -\inf_{\omega\in A} I^\gamma_\mu(\omega), \quad \mathbf{P}_\nu - a.a.\ w; \tag{42}$$

(b) (Lower bound) *for any open subset* $U \subset \mathcal{C}([0,1], \mathcal{M}_p)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log P^\gamma_{\mu/\varepsilon}\left(\varepsilon X^{L(\gamma)}(w) \in U\right) \ge -\inf_{\omega\in U} I^\gamma_\mu(\omega), \quad \mathbf{P}_\nu - a.a.\ w. \tag{43}$$

THEOREM 13′. *Moreover, the good rate function I^γ_μ is given by*

$$I^\gamma_\mu(\omega) := \sup_{\substack{f\in \\ C_K([0,1]\times R^d)}} \Big(\langle\langle \omega(\cdot), f(\cdot)\rangle\rangle - \log P^\gamma_\mu \exp \langle\langle X_\cdot^{L(\gamma)}, f(\cdot)\rangle\rangle \Big) \tag{44}$$

for $\omega \in \mathcal{C}([0,1], \mathcal{M}_p)$. Here $\langle\langle\cdot,\cdot\rangle\rangle$ is defined by

$$\langle\langle \mu(\cdot), f(\cdot)\rangle\rangle := \int_0^1 \langle \mu(t), f(t)\rangle \mathrm{d}t \qquad \text{for} \quad \mu(t) \in \mathcal{M}_p. \tag{45}$$

V Exponential Tightness

An application of the general Cramér type theorem (cf. Theorem 6.1.3, p.228 in [9]) deduces that at least a weak large deviation principle must hold as $\varepsilon \downarrow 0$ for a family of scaled measure-valued processes $\{\varepsilon X_t^{L(\gamma)},\ t \in [0,1]\}$ in a random medium. As for the weak large deviation result, we just refer to [16, 22]. In order to obtain full large deviation principle from weak large deviation principle, by virtue of Lemma 1.2.18, p.8 in [9] it is sufficient to show the exponential tightness of a family of $\{P^\gamma_{\mu/\varepsilon} \circ \varepsilon X^{L(\gamma)}\}_\varepsilon$ on $\mathcal{C}([0,1], \mathcal{M}_p)$. That is to say, we need to show the following estimate: for any M in $(0,\infty)$ given, there exists a compact subset $K \equiv K_M$ of $\mathcal{C}([0,1], \mathcal{M}_p)$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log P^\gamma_{\mu/\varepsilon} \Big(\varepsilon X^{L(\gamma)} \in (K_M)^c \Big) \le -M, \quad \mathbf{P}_\nu - a.a. \quad w, \tag{46}$$

where $(A)^c$ is a complement of the set A, i.e., $(A)^c = \Omega \setminus A$ for the whole set Ω. However, it is a task of extreme difficulty to prove (46) directly since it is an expression of measure-valued continuous paths. Our principal contribution of this paper consists in giving an easily checkable sufficient condition of the exponential tightness (46).

Suggested by the McKean-Vlasov limit argument in Djehiche-Schied (1998) [45], we can prove the following criterion for exponential tightness. Let E be some topological space, at least, being separable and metrizable. $(Y^n)_n$ denotes a sequence of stochastic processes taking values in E. Assume that for each $n \in \mathbf{N}$, the process $Y^n = (Y^n_t)$ induces a measurable mapping from a certain probability space $(\Omega, \mathcal{B}, \mathbf{P})$ into the Skorokhod space $D(I,E)$ (endowed with Skorokhod topology) with a finite interval $I \subset \mathbf{R}_+$.

PROPOSITION 14. *Let (ε_n) be a sequence of small positive real numbers satisfying that $\varepsilon_n \searrow 0$ as $n \nearrow \infty$. The sequence (Y^n) is exponentially tight in $D((I,E)$ with speed $1/\varepsilon_n$,* i.e., *for each $M > 0$ there is a compact subset K_M of $D(I,E)$ such that*

$$\limsup_{n\nearrow\infty} \varepsilon_n \log \mathbf{P}(Y^n_\cdot \in (K_M)^c) \le -M$$

if and only if (a) *for an arbitrary $L > 0$ there can be found a compact subset C_L of E such that*

$$\limsup_{n\nearrow\infty} \varepsilon_n \log \mathbf{P}\,(\exists t \in I:\ \ Y^n_t \notin C_L) \le -L, \tag{47}$$

and (b) *there is a proper additive family* $\mathcal{F} \subset \mathcal{C}(E)$ *which separates the points of* E *such that for each* $f \in \mathcal{F}$ *the sequence* $\{f(Y^n_\cdot)\}_n$ *is exponentially tight in* $D(I, \mathbf{R})$ *with speed* $1/\varepsilon_n$.

Proof. The leading philosophy of this proof is basically due to weak tightness criteria of Theorem 3.1, p.276, [46] (Jakubowski (1986)). As for sufficiency, we need the following lemma.

LEMMA 14A. (cf. Jakubowski (1986) [46, Lemma 3.2, p.277]) *For every compact subset* $K \subset E$ *there exists a countable family* $\mathcal{F}(K) \subset \mathcal{F}$ *satisfying*
(a) $\mathcal{F}(K)$ *separates points in* E; *and*
(b) $\mathcal{F}(K)$ *is closed under addition operation, i.e., if* f *and* g *are members of* $\mathcal{F}(K)$, *then* $f + g$ *is also contained in* $\mathcal{F}(K)$,
when restricted to K.

By the above lemma we may assume without loss of generality that $\mathcal{F}$ is countable, that is to say, $\mathcal{F} = \{f_1, f_2, f_3, ...\}$. The assumption of (b) allows to have that for each $k \in \mathbf{N}$ and every $R > 0$, there is a compact subset $\hat{C}^k_R$ of $D(I, \mathbf{R})$ such that

$$\limsup_{n\nearrow\infty} \varepsilon_n \log \mathbf{P}\left(f_k(Y^n_\cdot) \notin \hat{C}^k_R\right) \leq -(R + l)$$

with $l \in \mathbf{N}$. Hence there can be found some $n(0) \in \mathbf{N}$ satisfying that for all $n \geq n(0)$,

$$\mathbf{P}\left(f_k(Y^n_\cdot) \notin \hat{C}^k_R\right) \leq e^{-R/\varepsilon_n}$$

holds. Since $D(I, \mathbf{R})$ is a Polish space, we can easily enlarge the set $\hat{C}^k_R$ to a compact subset C^k_R satisfying

$$\mathbf{P}\left(f_k(Y^n_\cdot) \notin C^k_R\right) \leq e^{-R/\varepsilon_n}, \qquad \text{for all} \quad n \in \mathbf{N}.$$

Now for a given number M we define

$$K_M := \{w \in D(I, E) : \ w(t) \in C_M \quad \text{for} \quad \forall t, \quad \text{and} \quad f_k(w(\cdot)) \in C^k_{kM} \quad \text{for} \quad \forall k \in \mathbf{N}\}.$$

Then, repeating Jakubowski's argument in [46] we can show that this set K_M becomes a compact subset of $D(I, E)$. On this account, it follows immediately that

$$\begin{aligned}
&\limsup_{n\nearrow\infty} \varepsilon_n \log \mathbf{P}(Y^n_\cdot \in (K_M)^c) \\
&\leq (-M) \vee \left\{\limsup_{n\nearrow\infty} \varepsilon_n \log \sum_{k=1}^{\infty} \mathbf{P}\left(f_k(Y^n_\cdot) \notin C^k_{kM}\right)\right\} \\
&\leq (-M) \vee \left\{\limsup_{n\nearrow\infty} \varepsilon_n \log \sum_{k=1}^{\infty} e^{-kM/\varepsilon_n}\right\} \leq -M,
\end{aligned}$$

which implies establishment of the required exponential tightness. As to necessity, it is a routine work as it can be seen in the usual tightness argument (e.g. see[10]). Q.E.D.

Remark 2. Notice that the space $\mathcal{C}(I,E)$ of E-valued continuous paths endowed with compact-open topology is a closed topological subspace of $D(I,E)$ (cf. Proposition 1.6, p.267, [46]).

Naturally this implies from Jakubowski's argument (1986) [46] that our criterion (Proposition 14) remains valid even in $\mathcal{C}(I,E)$ as well. Namely,

COROLLARY 15. *Let* (ε_n) *be a sequence of small positive real numbers satisfying that* $\varepsilon_n \searrow 0$ *as* $n \nearrow \infty$. *The sequence* (Y^n) *is exponentially tight in* $\mathcal{C}(I,E)$ *with speed* $1/\varepsilon_n$, i.e., *for each* $M > 0$ *there is a compact subset* K_M *of* $\mathcal{C}(I,E)$ *such that*

$$\limsup_{n\nearrow\infty} \varepsilon_n \log \mathbf{P}(Y^n_\cdot \in (K_M)^c) \le -M$$

if and only if (a) *for an arbitrary* $L > 0$ *there can be found a compact subset* C_L *of* E *such that*

$$\limsup_{n\nearrow\infty} \varepsilon_n \log \mathbf{P}\left(\exists t \in I : \ Y^n_t \notin C_L\right) \le -L, \tag{48}$$

and (b) *there is a proper additive family* $\mathcal{F} \subset \mathcal{C}(E)$ *which separates the points of* E *such that for each* $f \in \mathcal{F}$ *the sequence* $\{f(Y^n_\cdot)\}_n$ *is exponentially tight in* $\mathcal{C}(I,\mathbf{R})$ *with speed* $1/\varepsilon_n$.

As is easily seen, we need verify two conditions (a), (b) instead, which are the payment we have to pay to compensate for this reduction. However, there are definitely some ambiguity in those statements. For instance, as to (a) of Corollary 15, it is necessary to describe what the compact set C_L is like; as to (b) of Corollary 15, we are really required to determine what a kind of functional we should prove the exponential tightness for. Otherwise, we cannot proceed any further on the proofs of our main results Theorem 13 and 13′. For each $L > 0$ given, we set

$$C_L := \Big\{\mu \in \mathcal{M}_p : \ \langle \mu, \varphi_p\rangle \le L, \exists (R_n)_n \nearrow \infty, \quad \text{and}$$
$$\langle \mu, \mathbf{I}\{|x| \ge R_n\}\cdot\varphi_p\rangle \le \frac{L}{n}, \ n \in \mathbf{N}\Big\}. \tag{49}$$

Hence, in particular, we have

THEOREM 16. (Sufficient Condition for Exponential Tightness) *If* (I) *for each* $L > 0$

$$\limsup_{n\nearrow\infty} \varepsilon_n \log P^\gamma_{\mu/\varepsilon_n}\left(\exists t \in [0,1] : \ \varepsilon_n X^{L(\gamma)}_t \notin C_L\right) \le \mathbf{E}_x\left\{\sup_{t\ge s} \varphi_p(B_s)\right\} \vee 1 - \frac{L}{2} \tag{50}$$

holds $\mathbf{P}_\nu$ -a.a. *w, and if* (II) *the distributions of the sequence* $\{\varepsilon_n\langle X^{L(\gamma)}_\cdot, f\rangle\}_n$ *is exponentially tight in* $\mathcal{C}([0,1])$ *with speed* $1/\varepsilon_n$ *under* $P^\gamma_{\mu/\varepsilon_n}$ ($\mathbf{P}_\nu$ -a.a. w), *then for given* $M > 0$ *there exists a compact subset* K_M *of* $\mathcal{C}([0,1],\mathcal{M}_p)$ *such that*

$$\limsup_{n\nearrow\infty} \varepsilon_n \log P^\gamma_{\mu/\varepsilon_n}\left(\varepsilon_n X^{L(\gamma)}_\cdot \in (K_M)^c\right) \le -M, \quad (\mathbf{P}_\nu - a.a. \quad w) \tag{51}$$

holds.

Proof. It is chiefly due to Corollary 15. We have only to apply the corollary to $P^\gamma_{\mu/\varepsilon_n}$ (resp. $\varepsilon_n X^{L(\gamma)}$) instead of $\mathbf{P}$ (resp.Y^n). Q.E.D.

N.B. The details of proof owe the precise estimates and discussions in the succeeding sections (cf. Section VI, Section VII and Section X).

VI Prokhorov Type Theorem

As for the set C_L in the first condition (I) of Theorem 16, we need to check whether C_L is compact or not, in order to complete the proof of Theorem 13. Roughly speaking, this will be taken care of the Prokhorov type argument. Here, $\mathcal{M}_p \equiv \mathcal{M}_p(\mathbf{R}^d)$ denotes the space of all p -tempered measures, consisting of all positive Radon measures μ on $\mathbf{R}^d$ that are of the form $\mu(dx) = \varphi_p(x)^{-1}\,\nu(dx)$ for some positive finite measure ν on $\mathbf{R}^d$. The space $\mathcal{M}_p$ is equipped with p -weak topology that is generated by the maps :

$$\mathcal{M}_p \ni \mu \mapsto \langle \mu, f\rangle, \qquad f \in \{\varphi_p\}\bigcup \mathcal{C}_K(\mathbf{R}^d).$$

While, we denote by $\mathcal{M}_F \equiv \mathcal{M}_F(\mathbf{R}^d)$ the space of all positive finite measures on $\mathbf{R}^d$, equipped with the topology generated by the maps :

$$\mathcal{M}_F \ni \mu \mapsto \langle \mu, f\rangle, \qquad f \in \{1\}\bigcup \mathcal{C}_K(\mathbf{R}^d).$$

It is interesting to note that this topology coincides with the usual weak topology. In addition, $\mathcal{M}_p$ is topologically isomorphic to $\mathcal{M}_F$. As a consequence, it is easy to get the following Prokhorov type theorem.

PROPOSITION 17. (A Version of Prokhorov Theorem) *Let* $K \subset \mathcal{M}_p$. K *is relatively compact if and only if the following two conditions hold* :

$$\text{(i)} \ \sup_{\mu\in K}\langle \mu, \varphi_p\rangle < \infty; \quad \text{(ii)} \ \lim_{R\to\infty}\sup_{\mu\in K}\int_{|x|\ge R}\varphi_p(x)\mu(dx) = 0.$$

Therefore we can deduce by Proposition 17 that the set C_L is relatively compact in $\mathcal{M}_p$ for each $L > 0$. Hence the validity of the statement in Theorem 16 is guaranteed.

VII Orlicz Space and Embedding Map

We have to investigate and discuss the second condition (II) of Theorem 16. Let E be a vector space (as state space) with norm $\|\cdot\|_E$ and $0 < \alpha < 1$. We denote by $H^\alpha([0,1], E)$ the space of all continuous E -valued paths w with finite Hölder norm $|w|_\alpha < +\infty$, where the norm $|\cdot|_\alpha$ is given by

$$|w|_\alpha := \sup_{t \neq s} \frac{\|w(t) - w(s)\|_E}{|t-s|^\alpha}.$$

We choose the path space as basic space, that is, $\Omega := \mathcal{C}([0,1], \mathcal{M}_p)$. For $\kappa > 0$ we define the Young function Φ_κ as

$$\Phi_\kappa(x) := (e^x - 1)/\kappa.$$

The Luxemburg norm is defined by

$$\|F\|_{\Phi_\kappa} := \inf\left\{\beta > 0 : \; P_\mu^\gamma[\Phi_\kappa(\|F\|_E/\beta)] \leq 1\right\}.$$

Furthermore, we set

$$C_*^{p,2} := \left\{f \in \mathcal{C}^p : \; \exists D^2 f \text{ is continuous, and } \Delta f \in \mathcal{C}^p\right\}.$$

$L_{\Phi_\kappa}(\Omega, E; P_\mu^\gamma)$ denotes the Orlicz space with respect to the Young function Φ_κ, consisting of all E -valued measurable functions F with $\|F\|_{\Phi_\kappa} < +\infty$. Recall that the measure-valued process $X^{L(\gamma)}$ in a random medium has a continuous modification in t since the Brownian collision local time $L(\gamma)$ belongs to $\mathcal{K}^\beta$ with $\beta = 1/2$ (cf. Subseection III.5). If f is taken from $C_*^{p,2}$, then the function

$$\langle X_\cdot^{L(\gamma)}, f\rangle \text{ lives in } H^{1/2}([0,1], L_{\Phi_2}(\Omega, \mathbf{R}; P_\mu^\gamma)), \quad \mathbf{P}_\nu - a.a.w. \tag{52}$$

On the other hand, we have the following functional space inclusion from the argument in terms of functional analysis. That is, for any $\alpha \in (0, 1/2)$

$$H^{1/2}([0,1], L_{\Phi_2}(\Omega, \mathbf{R}; P_\mu^\gamma)) \subset L_{\Phi_2}(\Omega, H^\alpha([0,1], \mathbf{R}); P_\mu^\gamma),$$

where the above embedding mapping is continuous. On this account, it is easy to see that

LEMMA 18. *There exists a positive number δ such that*

$$P_\mu^\gamma\left\{\exp(\delta|\langle X_\cdot^{L(\gamma)}, f\rangle|_\alpha)\right\} < \infty$$

holds for $\mathbf{P}_\nu$ -a.a. w.

Therefore it follows from (52) and Lemma 18 that the distributions of $\varepsilon_n\langle X_\cdot^{L(\gamma)}, f\rangle$ under $P_{\mu/\varepsilon_n}^\gamma$ on $\mathcal{C}([0,1])$ are exponentially tight ($\mathbf{P}_\nu$ -a.a. w), as far as we choose a member f of $C_*^{p,2}$.

VIII Good Rate Function

The purpose of this section is to prove the second main result (Theorem 13′) in this paper. That is to say, we will show below how the explicit representation (44) of rate function I^γ_μ can be derived. As we have seen before, when the two conditions (I), (II) in Theorem 16 for exponential tightness are fulfilled, then the full large deviation principle holds. In fact, by Lemma 1.2.18, p.8 in [9], if the exponentially tight family $\{P^\gamma_{\mu/\varepsilon_n} \circ \varepsilon_n X^{L(\gamma)}\}_n$ has the lower bound, then its rate function $J(\cdot)$ becomes a *Good Rate Function*, that is, it turns out to be that we have shown the full large deviation principle with good rate function $J(\cdot)$. From the general theory, e.g. according to the extension of Cramér's theorem [9, Theorem 6.1.3], the rate function $J(\cdot)$ is given by the Fenchel-Legendre transform of $\Lambda(\lambda) = \log \mathbf{E} \exp\langle X, \lambda\rangle$. Namely, it is given by

$$\Lambda^*(x) := \sup_{\lambda \in R^d} \{\langle \lambda, x\rangle - \Lambda(\lambda)\}, \tag{53}$$

for example, in the case of d -dimensional Euclidean space. Actually, for an arbitrary open convex subset A of a locally convex Hausdorff topological real vector space Υ

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_n(A) = -\inf_{x\in A} \Lambda^*(x) \tag{54}$$

holds. We set the new class $\mathcal{M}_p^*$ as

$$\mathcal{M}_p^* = \mathcal{M}_p - \mathcal{M}_p.$$

In our case the good rate function is given by the Legendre transform involving the topological dual of the space $\mathcal{C}([0,1], \mathcal{M}_p^*)$. Let $\mathcal{M}^+([0,1]\times \mathbf{R}^d)$ denote the space of all positive Radon measures on $[0,1]\times \mathbf{R}^d$. We define

$$I^\gamma_\mu(\omega) := \sup_{\substack{f\in \\ C_K([0,1]\times R^d)}} \{\langle\langle \omega, f\rangle\rangle - \Lambda(f)\}, \tag{55}$$

where $\langle\langle \omega, \psi\rangle\rangle$ is given by

$$\langle\langle \omega, \psi\rangle\rangle := \int_0^1 \langle \omega(t), \psi(t)\rangle \mathrm{d}t = \int_0^1 \int_{R^d} \psi(t,x)\omega(t,\mathrm{d}x)\mathrm{d}t$$

for $\omega \in \mathcal{C}([0,1], \mathcal{M}_p)$ and $\psi \in \mathcal{C}_K([0,1]\times \mathbf{R}^d)$, and Λ is given by

$$\Lambda(f) := \log P^\gamma_\mu \exp\langle\langle X^{L(\gamma)}_\cdot, f(\cdot)\rangle\rangle.$$

In order to identify our good rate function $J(\cdot)$ with $I^\gamma_\mu(\cdot)$, we need simply to embed the space $\mathcal{C}([0,1], \mathcal{M}_p)$ into $\mathcal{M}^+([0,1]\times\mathbf{R}^d)$ by choosing the form carefully and defining $\langle\langle \omega, f\rangle\rangle$ properly. The above-mentioned definition realizes a continuous embedding:

$$\mathcal{C}([0,1], \mathcal{M}_p) \subset \mathcal{M}^+([0,1]\times \mathbf{R}^d). \tag{56}$$

Recall here the *Contraction Principle* and the *Uniqueness of Rate Function* in the general theory of large deviation principles.

Contraction Principle (cf. Theorem 4.2.1 in [9]) Let Υ and Ξ be Hausdorff topological spaces and $f : \Upsilon \to \Xi$ a continuous mapping. $I : \Upsilon \to [0, \infty)$ denotes a good rate function.
(a) If we define

$$I'(y) := \inf\{I(x) : \ x \in \Upsilon, \ y = f(x)\}$$

for every $y \in \Xi$, then I' becomes a good rate function on Ξ.
(b) If I controls the large deviation principle associated with a family of probability measures $\{\mu_\varepsilon\}$ on Υ, then I' controls the large deviation principle associated with the family of probability measures $\{\mu_\varepsilon \circ f^{-1}\}$ on Ξ.

Uniqueness of the Rate Function (cf. Lemma 4.1.4 in [9]) A family of probability measures $\{\mu_\varepsilon\}$ on a regular topological space can have at most one rate function associated with its large deviation principle.

Paying attention to the aforementioned arguments, we first of all endow the space $\mathcal{M}^+([0,1] \times \mathbf{R}^d)$ with the vague topology. Next we may apply the extension of the Cramér theorem (cf. Theorem 6.1.3 in [9]) again so as to find a large deviation principle with rate function I_μ^γ. Since our embedding : $\mathcal{C}([0,1], \mathcal{M}_p) \hookrightarrow \mathcal{M}^+([0,1] \times \mathbf{R}^d)$ is injective, $J(\cdot)$ and I_μ^γ must coincide by the contraction principle and the uniqueness of the rate function. This completes the proof of our second main theorem.

IX Historical Processes

IX.1 Path Procees Associated with Brownian Motion

We introduce here the notion of historical processes (e.g. Dawson-Perkins (1991) [49]), which is a useful tool especially for estimation of some functionals of specific processes, such as measure-valued processes, and is used repeatedly later in the succeeding section. For the notation adopted in this section, we would rather recommend the readers to consult Dynkin's approach (1991) [31] to the historical superprocesses (see also [15,20]).

For $r \geq 0$ and $\tilde{x} \in \mathcal{C}(\mathbf{R}_+, \mathbf{R}^d)$, there exists a unique probability measure $\tilde{P}_{r,\tilde{x}} \in \mathcal{P}(\mathcal{C}(\mathbf{R}_+, \mathbf{R}^d))$ such that (i) for $\tilde{P}_{r,\tilde{x}}$ -a.e. $\tilde{y} \in \mathcal{C}(\mathbf{R}_+, \mathbf{R}^d)$,

$$\tilde{y}(s) = \tilde{x}(s) \qquad \text{for} \quad \text{each} \quad s \in [0, r];$$

and (ii) under $\tilde{P}_{r,\tilde{x}}$, the process : $t \mapsto \tilde{y}(r+t)$ is a Brownian motion starting from $\tilde{x}(r)$. This implies that the measure $\tilde{P}_{r,\tilde{x}}$ forces Brownian motion to follow the path $\tilde{x}$ up to time r. The path process $\tilde{B} = (\tilde{B}_t(\tilde{y}))$, $t \geq 0$ associated with Brownian motion is a path-valued stochastic process defined by

$$\tilde{B}_t(\tilde{y}) := \tilde{y}(\cdot \wedge t), \quad t \geq 0 \qquad \text{for} \quad \tilde{y} \in \mathcal{C}(\mathbf{R}_+, \mathbf{R}^d).$$

The filtration generated by $(\tilde{B}_t), t \geq 0$ coincides with the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ that is generated by the coordinate process on $\mathcal{C}(\mathbf{R}_+, \mathbf{R}^d)$. The path process $\tilde{B}$ satisfies the following equality: for $0 \leq r \leq s$ and $t \geq 0$,

$$\tilde{P}_{r,\tilde{x}}(\tilde{B}_t \in A/\mathcal{F}_s) = \tilde{P}_{s,\tilde{B}_s}(\tilde{B}_t \in A)$$

holds $\tilde{P}_{r,\tilde{x}}$ -a.s. for any $A \subset \mathcal{C}([0,\infty), \mathbf{R}^d)$, which implies the Markov property for historical processes. The strong Markov property can be also proved (cf. [49]). The collection

$$[\tilde{B}_t, \tilde{P}_{r,\tilde{x}};\ t \geq 0, \tilde{x} \in \mathcal{C}(\mathbf{R}_+, \mathbf{R}^d)]$$

forms a time-inhomogeneous strong Markov process. In other words, at time s we start with a time s -stopped path $w = \tilde{B}_s$ ($\in \mathbf{C}^s$) and let a path trajectory $\{\tilde{B}_t,\ t \in [s,T]\}$ evolve with law $\tilde{\Pi}_{s,w}$ determined by a Brownian path $\{B_t,\ s \leq t \leq T\}$ starting at time s at w_s ($= \pi_s$ $(\tilde{B}_s)$). We may regard $\tilde{\Pi}_{s,w}$ as a probability law on

$$\hat{\mathcal{C}}([s,T], \mathbf{C}) = \{\omega \in \mathcal{C}(I, \mathbf{C}):\ \omega_t \in \mathbf{C}^t,\ t \in [s,T]\}.$$

IX.2 Historical Superprocess

Roughly speaking, the superprocess $\tilde{X}^\gamma$ built on the above-mentioned path process $\tilde{B}$ is called historical super-Brownian motion (HSBM). Let $M^+(\mathcal{C}(\mathbf{R}_+, \mathbf{R}^d))$ be the set of all positive finite measures on $\mathcal{C}(\mathbf{R}_+, \mathbf{R}^d)$. The historical super-Brownian motion is a time inhomogeneous diffusion possessing the space $M^+(\mathcal{C}(\mathbf{R}_;, \mathbf{R}^d))$ as its state space. A standard theory for historical processes [49] (see also [5,20,31,37]) provides with the following characterization of historical super-Brownian motion in terms of Laplace functionals of its transition probabilities. In fact, the Laplace transition functional for HSBM is given by

$$\tilde{\mathbf{P}}_{r,\tilde{\mu}} \exp\langle \tilde{X}_t, -\varphi\rangle = \exp\langle \tilde{\mu}, -v_t(r)\rangle \tag{57}$$

for every $r \geq 0$, every $\tilde{\mu} \in M^+(\mathcal{C}(\mathbf{R}_+, \mathbf{R}^d))$ and any $\varphi \in b\mathcal{B}_+(\mathcal{C}(\mathbf{R}_+, \mathbf{R}^d))$. Here the function v_t is the unique positive solution of the nonlinear integral equation of the form:

$$v_t(r, \tilde{x}) + \gamma \int_r^t \tilde{P}_{r,\tilde{x}}[v_t(s, \tilde{B}_s)^2] \mathrm{d}s = \tilde{P}_{r,\tilde{x}}[\varphi(\tilde{B}_t)]. \tag{58}$$

In the above, $\tilde{\mathbf{P}}_{r,\tilde{\mu}}$ is a probability measure on the space $\mathcal{C}(\mathbf{R}_+,\ M^+(\mathcal{C}(\mathbf{R}_+, \mathbf{R}^d)))$. We denote by π the projection from $\mathcal{C}([0,t], \mathcal{M}_p)$ into $\mathcal{M}_p$ for each t, in other words, we have $\pi_t(\tilde{X}_t^\gamma) = X_t^\gamma \in \mathcal{M}_p$. Roughly speaking, the historical process is a process whose path gives the past history of the particle.

IX.3 Historical Superprocess in A Random Medium

Since the historical superprocess is a time-inhomogeneous process, it is convenient to work with a backward and historical setting. For brevity's sake let $I = [0, T]$, $0 < T < \infty$ in what follows. $\mathbf{C}$ denotes the Banach space $\mathcal{C}(I, \mathbf{R}^d)$. When we denote by

$$w^t := \{\, w^t(s) \equiv w(s \wedge t); \;\; s \in I\}$$

the stopped path of a path $w \in \mathbf{C}$ at time $t \in I$, then $\mathbf{C}^t$ is the whole space of those stopped paths. This stopped path is held constant after time t. For t fixed, $\mathbf{C}^t$ becomes a closed subspace of $\mathbf{C}$. Note that $\mathbf{C}^s \subset \mathbf{C}^t$ if $s \leq t$. In particular, $\mathbf{C}^T = \mathbf{C}$ and $\mathbf{C}^0$ can be identified with $\mathbf{R}$, whereas $\mathbf{C}^t$ could be considered as $\mathcal{C}([0,t], \mathbf{R})$ as well. In addition, each path $w \in \mathbf{C}$ can be associated with the so-called stopped path trajectory $\tilde{w}$ by setting

$$\tilde{w}_t := w^t, \qquad t \in I.$$

We put

$$\hat{\mathcal{C}}(I, \mathrm{C}) := \{\omega \in \mathcal{C}(I, \mathrm{C}) : \;\; \omega_t \in \mathbf{C}^t \quad \text{for} \quad \forall t \in I\}.$$

Notice that $\hat{\mathcal{C}}(I, \mathrm{C})$ becomes a closed subspace of $\mathcal{C}(I, \mathrm{C})$, because for $0 \leq s \leq t \leq T$ we have

$$\|\tilde{w}_t - \tilde{w}_s\|_\infty = \|w^t - w^s\|_\infty = \sup_{s \leq r \leq t} |w_r - w_s| \to 0$$

as the difference $t - s$ approaches to zero. Generally speaking, if A, B are sets and the map $: a \mapsto B^a$ is a mapping from A into the set of all subsets of B, then the graph of this map is written as

$$A \hat{\times} B^{\cdot} := \{\, [a, b] : \;\; a \in A, \, b \in B^a \,\} = \bigcup_{a \in A} \{a\} \times B^a.$$

Note that $A \hat{\times} B^{\cdot} \subset A \times B$. Define

$$I \hat{\times} \mathbf{C}^{\cdot} := \left\{ [t, w] : \;\; t \in I, \quad w \in \mathbf{C}^t \right\}.$$

Then, since we get

$$\|\tilde{v} - \tilde{w}\|_\infty = \sup_{t \in I} \|\tilde{v}_t - \tilde{w}_t\|_\infty = \|v^T - w^T\|_\infty = \|v - w\|_\infty$$

for every $v, w \in \mathbf{C}$, there exists a continuous mapping :

$$\mathbf{C} \ni w \mapsto \tilde{w} \in \hat{\mathcal{C}}(I, \mathbf{C}),$$

and it is easy to see that the graph of $w \in \hat{\mathcal{C}}(I, \mathbf{C})$ is a subset of $I \hat{\times} \mathbf{C}^{\cdot}$, and clearly $I \hat{\times} \mathbf{C}^{\cdot}$ is a closed subset of $I \times \mathbf{C}$.

Under these setups we are ready to consider the historical representation (HP) of measure-valued processes $\{X_t^{L(\gamma)}\}$ in a random medium. Recall that for each $z \in \mathbf{R}^d$, the symbol Π_z denotes the law of Brownian path B on $\mathbf{C}$ starting from z at time $t = 0$. Here

$$\tilde{B} = [\tilde{B}_t, \tilde{\Pi}_{s,w}, s \in I, w \in \mathbf{C}^s]$$

is a Brownian path process on I, and the semigroup $\{\tilde{S}_\cdot\}$ of $\tilde{B}$ is given by

$$\tilde{S}_{s,t}\varphi(w) = \tilde{\Pi}_{s,w}\varphi(\tilde{B}_t), \quad 0 \le s \le t \le T, \quad w \in \mathbf{C}^t, \quad \varphi \in b\mathcal{B}(\mathbf{C}).$$

The corresponding infinitesimal generator is written by $\tilde{A} = \{\tilde{A}_s;\ s \in I\}$. As a matter of fact, $\tilde{A}$ is defined by

$$\tilde{A}_s\psi(w) = \lim_{h\downarrow 0} \frac{1}{h} \left\{ \tilde{S}_{s-h,s}\psi(w^{s-h}) - \psi(w) \right\}, \quad w \in \mathbf{C}^s, \tag{58a}$$

where ψ is taken from the domain $\mathrm{Dom}(\tilde{A})$ of $\tilde{A}$, i.e., $\psi \in b\mathcal{B}(\mathbf{C})$ such that the above limit (58a) exists. Let $\mathcal{M}_F^t(\mathbf{C}^t)$ be the totality of all nonnegative finite measures μ on $\mathbf{C} = \mathcal{C}(I, \mathbf{R})$ equipped with topology of weak convergence, satisfying $\mu(\mathbf{C} \setminus \mathbf{C}^t) = 0$. The historical version of branching mechanism $\hat{\Phi}$ is given by

$$\hat{\Phi}((s,y),\lambda) := \hat{a}(s,y)\lambda + \hat{b}(s,y)\lambda^2.$$

We use below the following notations: $y^s(t) = y(t \wedge s)$ for $y \in \mathbf{C}$ and

$$y/s/w := \begin{cases} y(t) & \text{for} \quad t < s, \\ w(t-s) & \text{for} \quad t \ge s, \end{cases}$$

for $y, w \in \mathbf{C}$ and $s \ge 0$

PROPOSITION 18A. *Set*

$$P_{s,y}(A) := \tilde{\Pi}_{y(s)}(w \in \mathbf{C}: \ \ y/s/w \in A)$$

for $(s,y) \in \hat{E} := \{(s,y) \in I \times \mathbf{C}: \ \ y^s = y\}$. *Then* $P_{s,y}$ *satisfies*
(a) *the mapping* : $\hat{E} \ni (s,y) \mapsto P_{s,y}(A)$ *is* $\mathcal{B}(\hat{E})$ *-measurable for* $\forall A \in \mathcal{B}(\mathbf{C})$;
(b) *for* $Y_t(y) = y^t \in \mathbf{C}$,

$$P_{s,y}(Y_s = y) = 1 \tag{59}$$

holds for $\forall (s,y) \in \hat{E}$;
(c) *if* $(s,y) \in \hat{E}$ *and* T *is a* $\mathcal{B}_{t+}(\mathbf{C})$ *-stopping time such that* $T \ge s$ *with probability one, then for any* $\Psi \in \mathcal{B}(\mathbf{C})$

$$P_{s,y}(\Psi/\mathcal{B}_{T+}(\mathbf{C})) = P_{T,Y(T)}(\Psi) \tag{60}$$

holds $P_{s,y}$ *-a.s. on the set* $\{T < \infty\}$.

THEOREM 19. *Let* $d \le 3$. *Then for each* $t \in I$, *there exists* $\mathcal{M}_F^t(\mathbf{C}^t)$ *-valued time-inhomogeneous right Markov process*

$$\tilde{X}^{L(\gamma)} = [\tilde{X}_t^{L(\gamma)}, \tilde{P}_{s,\mu}^\gamma, s \in I, \mu \in \mathcal{M}_F^s(\mathbf{C}^s)].$$

Furthermore, its Laplace transition functional is given by

$$\tilde{P}_{s,\mu}^\gamma \exp\langle \tilde{X}_t^{L(\gamma)}, -\varphi\rangle = \exp\langle \mu, -u_\varphi(s,\cdot,t)\rangle, \tag{61}$$

$$0 \le s \le t \le T, \quad \mu \in \mathcal{M}_F^s(\mathbf{C}^s), \quad \varphi \in b\mathcal{B}_+(\mathbf{C}^t).$$

Here its log-Laplace functional $u_\varphi \equiv \tilde{u}_\varphi[A^{L(\gamma)}]$ *solves*

$$\tilde{\Pi}_{y(s)}(\varphi(y/s/Y^{t-s})) = \tilde{u}_\varphi(s,w,t)$$

$$+\int_s^t \tilde{\Pi}_{y(s)}(\hat{\Phi}((u,y/s/Y^{u-s}),\tilde{u}_\varphi(u,y/s/Y^{u-s},t))A^{L(\gamma)}(du), \tag{62}$$

where $\hat{\Phi}$ *is given as the special case of* $\hat{a} \equiv 0$, $\hat{b} \equiv 1$, *and* Y *is a process whose canonical representation is realized as* $y \in \mathbf{C}$. $A^{L(\gamma)}(dr)$ *is the additive functional of* B *which is obtained as a limit of* $\{\Pi_{0,B(r)}\ (W_\varepsilon(\mathrm{d}z)\ /\ \mathrm{d}z)\ \mathrm{d}r\}_\varepsilon$ *such that*

$$\int_L^T \Pi_{0,B(r)}\left(\frac{W_\varepsilon(\mathrm{d}z)}{\mathrm{d}z}\right)\mathrm{d}r \to A^{L(\gamma)}([L,T]) \tag{63}$$

as $\varepsilon \searrow 0$.

Proposition 20. *Moreover, the aforementioned function* $u_\varphi(\cdot,\cdot,t)$ *is the unique* $\mathcal{B}([0,t] \hat{\times} \mathbf{C}^{\cdot})$ *-measurable, bounded and nonnegative solution of formal equation of the form* :

$$-\frac{\partial}{\partial s}u_\varphi(s,w,t) = \tilde{A}_s u_\varphi(s,w,t) - \left(\frac{X_s^\gamma(\mathrm{d}x)}{\mathrm{d}z}\right)u_\varphi^2(s,w,t), \tag{64}$$

$$0 \le s \le t, \quad w \in \mathbf{C}^s, \quad \text{with} \quad u_\varphi(t,\cdot,t) = \varphi.$$

Here the term $(X_s^\gamma(\mathrm{d}z)/\mathrm{d}z)$ *is the generalized derivative of a measure.*

COROLLARY 22. *Let* $d = 1$. *There exists a jointly continuous function* $\Xi(t,z)$ *such that*

$$W_t(\mathrm{d}z) = \Xi(t,z)\mathrm{d}z \tag{65}$$

holds with $\mathbf{P}_\mu$ *-probability one. Moreover,* $A^{L(\gamma)}(\mathrm{d}r)$ *has an explicit representation, namely,*

$$A^{L(\gamma)}(\mathrm{d}r) = \Xi(r,B_r)\mathrm{d}r, \qquad \Pi_{s,z} - a.s. \quad \text{and} \quad \mathbf{P}_\mu - a.a. \quad \text{realization} \quad W_\cdot(\omega). \tag{66}$$

X Key Estimates of Random Functionals

The main theme of this section is applications of historical processes to large deviation theory for catalytic super-Brownian motions. It is left the details about how we can verify our easily checkable sufficient conditions syated in Section V. The proof of the crucial estimate for the proof is greatly due to the following three inequalities. The precise proofs of the lemmas below are omitted because the auguments are not standard and owe much to too technical computation particular to historical superprocesses, and in addition is also rather longsome and tiresome. However, rough sketches of proofs of these lemmas will be given in the succeeding subsections for readers' convenience.

LEMMA 24. *Let F be a bounded measurable functional on $\mathcal{C}([0,\infty), \mathbf{R}^d)$. Then we have the following inequality*

$$V_{t-r}\tilde{\Pi}_{r,w}[F(\tilde{B}_t)] \leq \log \tilde{P}^{\gamma}_{r,\delta_w} \exp\langle F, \tilde{X}^{L(\gamma)}_t\rangle \tag{67}$$

where V_t is an analytic extension of the special solution v_t as log-Laplace functional.

LEMMA 25. *Let Φ be a positive lower semicontinuous function on $\mathbf{R}^d$. Then the following inequality holds : for any positive number α,*

$$\tilde{P}_{0,\mu}\left\{\sup_{s\leq t}\langle \sup_{u\leq s}\Phi(B_u), \tilde{X}^{L(\gamma)}_s\rangle \geq \alpha\right\} \leq \frac{1}{\alpha}P_\mu\left\{\sup_{s\leq t}\Phi(B_s)\right\}. \tag{68}$$

LEMMA 26. *Let Φ be the same function defined as in Lemma 25. For every element ν of $M^+(\mathbf{R}^d)$,*

$$P^{\gamma}_{0,\nu}\left\{\exp\left(\frac{1}{2}\sup_{t\leq 1}\langle \Phi, X^{L(\gamma)}_t\rangle\right)\right\} \leq \tilde{P}^{\gamma}_{0,\nu}\left\{\left(\sup_{t\leq 1}\exp(\frac{1}{4}\langle \sup_{s\leq t}\Phi(B_s), \tilde{X}^{L(\gamma)}_t\rangle)\right)^2\right\}. \tag{69}$$

X.1 Proof of Lemma 24

Via branching property, Feynman-Kac argument and canonical measure, by making use of Palm representation, the proof is attributed to showing the inequality

$$P^{\gamma}_{r,\delta(w)} \exp\left\{\tilde{\Pi}_{r,w}[F(\tilde{B}_t)] \cdot \tilde{X}^{L(\gamma)}_t(\mathbf{C})\right\} \leq P^{\Gamma}_{r,\delta(w)} \exp\langle \tilde{X}^{L(\gamma)}_t, F\rangle.$$

X.2 Proof of Lemma 25

A direct computation leads to

$$\tilde{P}^{\gamma}_{0,\mu}\left\{\sup_{s\leq t}\langle \sup_{u\leq s}\Phi(B_u), \tilde{X}^{L(\gamma)}_s\rangle \geq \alpha\right\} \leq \frac{1}{\alpha}\tilde{P}^{\gamma}_{0,\mu}\langle \sup_{s\leq t}\Phi(B_u), \tilde{X}^{L(\gamma)}_t\rangle. \tag{70}$$

The assertion immediately yields from this estimate.

X.3 Proof of Lemma 26

It is easy hence omitted.

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