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Degree of Order regarding Multidimensional Fuzzy Sets

北九州大学経済学部　吉田祐治　(Yuji YOSHIDA)
ゲント大学応用数学科　ケアー　(E.E.Kerre)

1. Introduction and notations

Kurano et al. [4] studied a pseudo order regarding fuzzy sets on $\mathbb{R}^n$ on the basis of a set-relation in $\mathbb{R}^n$ studied by Kuroiwa et al. [2] and Kuroiwa [3] for multi-criteria crisp set-valued optimizations in mathematical programming. First, we introduce a fuzzy relation, which is fuzzy partial ordering, induced by closed convex cones. Next, a pseudo order regarding fuzzy sets on $\mathbb{R}^n$ is given by inclusions defined from the fuzzy relation, and it is also a reasonable multi-dimensional extension of the fuzzy max order regarding fuzzy numbers. For incomparable fuzzy sets on $\mathbb{R}^n$, we present a degree of order, using a subsethood degree. This method is flexible and can be applied to various types of fuzzy decision-making and optimization problems in multi-criteria. For example, we can apply the method to problems where compornents on $\mathbb{R}^n$ are related to each other in multi-criteria.

In the rest of this section, we give some notations and introduce some results regarding vector ordering on $\mathbb{R}^n$ by convex cones. Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^n$ be an n-dimensional Euclidean space, where n is a positive integer. We write fuzzy sets on $\mathbb{R}^n$ and their membership functions by $\tilde{A}$ and $\mu_{\tilde{A}} : \mathbb{R}^n \to [0,1]$ respectively. The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{A}$ on $\mathbb{R}^n$ is defined as

$$\tilde{A}_\alpha := \{x \in \mathbb{R}^n \mid \mu_{\tilde{A}}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \tilde{A}_0 := \mathrm{cl}\{x \in \mathbb{R}^n \mid \mu_{\tilde{A}}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{A}$ is called convex if the α-cut $\tilde{A}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets $\tilde{A}$ whose membership functions $\mu_{\tilde{A}} : \mathbb{R}^n \to [0,1]$ are upper-semicontinuous and normal ($\sup_{x\in\mathbb{R}^n} \mu_{\tilde{A}}(x) = 1$) and have a compact 0-cut. When one-dimensional case($n = 1$), the fuzzy sets are called fuzzy numbers and $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers. In this paper, we deal with fuzzy sets on $\mathbb{R}^n$ as a multi-dimensional extension of fuzzy numbers. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all non-empty compact convex subsets of $\mathbb{R}^n$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are as follows: For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ and a scalar $\lambda \geq 0$, the sum $\tilde{A} + \tilde{B}$ and scalar multiplication $\lambda\tilde{A}$ are given by applying Zaheh's extension principle:

$$\mu_{\tilde{A}+\tilde{B}}(x) := \sup_{y,z\in\mathbb{R}^n;\ y+z=x} \min\{\mu_{\tilde{A}}(y), \mu_{\tilde{B}}(z)\}, \tag{1.1}$$

$$\mu_{\lambda\tilde{A}}(x) := \begin{cases} \mu_{\tilde{A}}(x/\lambda) & \text{if } \lambda > 0 \\ 1_{\{\mathbf{0}\}}(x) & \text{if } \lambda = 0 \end{cases} \tag{1.2}$$

for $x \in \mathbb{R}^n$, where $1_{\{\cdot\}}(\cdot)$ is an indicator and $\{\mathbf{0}\}$ denotes the crisp set of zero in $\mathbb{R}^n$. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for

$A, B \in \mathcal{C}(\mathbb{R}^n)$, the following holds immediately:

$$(\tilde{A} + \tilde{B})_\alpha := \tilde{A}_\alpha + \tilde{B}_\alpha \quad \text{and} \quad (\lambda \tilde{A})_\alpha = \lambda \tilde{A}_\alpha \quad (\alpha \in [0,1]). \tag{1.3}$$

Let K be a non-empty convex cone of $\mathbb{R}^n$, i.e. $x + y \in K$ and $\lambda x \in K$ hold for all $\lambda \geq 0$ and all $x, y \in K$. Using a convex cone K, we can define a pseudo order $\preceq_K$ on $\mathbb{R}^n$ by

(K.1) $x \preceq_K y$ means that $y - x \in K$.

Let $\mathbb{R}^n_+ = \{x = (x^1, x^2, \cdots, x^n) \in \mathbb{R}^n \mid x^i \geq 0 (i = 1, 2, \cdots, n)\}$ be the subset of entrywise nonnegative elements in $\mathbb{R}^n$. When $K = \mathbb{R}^n_+$, the order $x \preceq_{\mathbb{R}^n_+} y$ means that $x^i \leq y^i$ for all $i = 1, 2, \cdots, n$, where $x = (x^1, x^2, \cdots, x^n)$ and $y = (y^1, y^2, \cdots, y^n) \in \mathbb{R}^n$.

2. Fuzzy partial ordering by convex cones

In this section, we discuss a fuzzy relation induced by closed convex cones. Let R_α ($\alpha \in [0,1]$) be non-empty closed convex cones on $\mathbb{R}^n$ such that $\bigcap_{\alpha' \in (0,\alpha)} R_{\alpha'} = R_\alpha$ for $\alpha \in (0,1]$. To avoid meaningless ordering, we assume R_α ($\alpha \in (0,1]$) are acute, i.e. there exists $a \in \mathbb{R}^n$ satisfying $a \cdot x > 0$ for all $x \in R_\alpha$ with $x \neq \mathbf{0}$, where $\cdot$ means the inner product of vectors on $\mathbb{R}^n$. We also put the support set by $R_0 := \mathrm{cl}(\bigcup_{\alpha \in (0,1]} R_\alpha)$. Define a fuzzy relation $\tilde{R}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by closed convex cones R_α as follows.

$$\mu_{\tilde{R}}(x, y) := \sup\{\alpha \in [0,1] \mid y - x \in R_\alpha\} \quad ((x, y) \in \mathbb{R}^n \times \mathbb{R}^n), \tag{2.1}$$

where $\sup \emptyset := 0$. Then the α-cut $\tilde{R}_\alpha$ is

$$\tilde{R}_\alpha = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \mu_{\tilde{R}}(x, y) \geq \alpha\} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y - x \in R_\alpha\} \tag{2.2}$$

for $\alpha \in (0,1]$. The fuzzy relation $\tilde{R}$ has the following properties.

Theorem 2.1. *$\tilde{R}$ is a fuzzy partial ordering ([1]), i.e., it satisfies the following (i) – (iii):*

(i) $\mu_{\tilde{R}}(x, x) = 1$ *for all* $x \in \mathbb{R}^n$.

(ii) $\mu_{\tilde{R}}(x, z) \geq \min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{R}}(y, z)\}$ *for all* $x, y, z \in \mathbb{R}^n$.

(iii) *If* $\mu_{\tilde{R}}(x, y) > 0$ *and* $\mu_{\tilde{R}}(y, x) > 0$, *then* $x = y$.

In Theorem 2.1, the property (i), (ii) and (iii) means reflexivity, transitivity and antisymmetry respectively.

3. Ordering of fuzzy quantities by the fuzzy relation $\tilde{R}$

In this section, we introduce an order for fuzzy sets on $\mathbb{R}^n$, using the fuzzy relation $\tilde{R}$ from Section 2. For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, using the sup-min composition operation with $\tilde{R}$ ([1]), we define fuzzy sets $\tilde{A} \circ \tilde{R}, \tilde{R} \circ \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ by

$$\mu_{\tilde{R}\circ\tilde{B}}(x) := \sup_{y\in\mathbb{R}^n} \min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{B}}(y)\} \quad (x \in \mathbb{R}^n), \tag{3.1}$$

$$\mu_{\tilde{A}\circ\tilde{R}}(y) := \sup_{x\in\mathbb{R}^n} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{R}}(x,y)\} \quad (y \in \mathbb{R}^n). \tag{3.2}$$

Definition 1. Consider fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. The order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ means that

$$\tilde{A} \subseteq \tilde{R} \circ \tilde{B} \quad \text{and} \quad \tilde{B} \subseteq \tilde{A} \circ \tilde{R}.$$

Further, the equivalence $\tilde{A} \sim_{\tilde{R}} \tilde{B}$ means that $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \preceq_{\tilde{R}} \tilde{A}$. We also write $\tilde{A} \prec_{\tilde{R}} \tilde{B}$ when $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{A} \not\sim_{\tilde{R}} \tilde{B}$.

Lemma 3.1. *The order $\preceq_{\tilde{R}}$ is a pseudo order: For fuzzy sets $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(\mathbb{R}^n)$, the following (i) and (ii) hold.*

(i) $\tilde{A} \preceq_{\tilde{R}} \tilde{A}$.

(ii) *If $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \preceq_{\tilde{R}} \tilde{C}$, then $\tilde{A} \preceq_{\tilde{R}} \tilde{C}$.*

In Lemma 3.1, the property (i) and (ii) means reflexivity and transitivity respectively. In the one-dimensional case($n = 1$), the order $\preceq_{\tilde{R}}$ coincides with the fuzzy max order, and then $\sim_{\tilde{R}}$ is replaced with $=$. The following example gives the order $\preceq_{\tilde{R}}$ induced from natural fuzzy relations on $\mathbb{R}^2 \times \mathbb{R}^2$ and shows that the order $\preceq_{\tilde{R}}$ is not a total order.

Example 3.1 (The order $\preceq_{\tilde{R}}$ corresponding to natural fuzzy relations on $\mathbb{R}^2 \times \mathbb{R}^2$). We consider a case when $n = 2$ and we put an acute closed convex cone $R_\alpha = \mathbb{R}^2_+ = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^i \geq 0 (i = 1,2)\}$ for all $\alpha \in [0,1]$. Then the corresponding fuzzy relation is

$$\mu_{\tilde{R}}((x^1,x^2),(y^1,y^2)) = \begin{cases} 1 & \text{if } y^1 \geq x^1 \text{ and } y^2 \geq x^2 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

For $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, the definition of the order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ is reduced to the following conditions (3.4) and (3.5):

$$\mu_{\tilde{A}}(x^1,x^2) \leq \sup_{(y^1,y^2):y^1\geq x^1, y^2 \geq x^2} \mu_{\tilde{B}}(y^1,y^2) \quad \text{for all } (x^1,x^2) \in \mathbb{R}^2 \times \mathbb{R}^2; \tag{3.4}$$

$$\mu_{\tilde{B}}(y^1,y^2) \leq \sup_{(x^1,x^2):y^1\geq x^1, y^2 \geq x^2} \mu_{\tilde{A}}(x^1,x^2) \quad \text{for all } (y^1,y^2) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{3.5}$$

Take pyramid-type fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined by

$$\mu_{\tilde{A}}(x^1,x^2) = \max\{\min\{1 - |x^1+1|, 1 - |x^2 - 1|\}, 0\}, \tag{3.6}$$

$$\mu_{\tilde{B}}(x^1,x^2) = \max\{\min\{1 - |x^1-1|, 1 - |x^2 + 1|\}, 0\} \tag{3.7}$$

for $(x^1,x^2) \in \mathbb{R}^2$. Then we can easily check that neither $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ nor $\tilde{B} \preceq_{\tilde{R}} \tilde{A}$ hold.

4. Degree of the fuzzy order $\preceq_{\tilde{R}}$

In this section, by using a subsethood degree, we present a method to evaluate the degree of satisfaction of the fuzzy order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ for all fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. The fuzzy order $\preceq_{\tilde{R}}$ is not a total order(Example 3.1), however we can apply this method to fuzzy sets which are incomparable by the order $\preceq_{\tilde{R}}$. For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, the subsethood degree is defined by([5],[1])

$$\mathrm{Sub}(\tilde{A} \subseteq \tilde{B}) := \frac{|\tilde{A} \cap \tilde{B}|}{|\tilde{A}|} \tag{4.1}$$

if $|\tilde{A}| > 0$, where

$$|\tilde{A}| := \int_{\mathbb{R}^n} \mu_{\tilde{A}}(x)\,dx \quad (\tilde{A} \in \mathcal{F}(\mathbb{R}^n)). \tag{4.2}$$

For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, in spirit of Definition 1 we define the degree of order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ by

$$D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) := \min\{\mathrm{Sub}(\tilde{A} \subseteq \tilde{R} \circ \tilde{B}), \mathrm{Sub}(\tilde{B} \subseteq \tilde{A} \circ \tilde{R})\}. \tag{4.3}$$

Then, the degree of order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ is written as

$$\begin{aligned} D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) &= \min\left\{ \frac{|\tilde{A} \cap (\tilde{R} \circ \tilde{B})|}{|\tilde{A}|}, \frac{|\tilde{B} \cap (\tilde{A} \circ \tilde{R})|}{|\tilde{B}|} \right\} \\ &= \min\left\{ \frac{\int_{\mathbb{R}^n} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{R}\circ\tilde{B}}(x)\}\,dx}{\int_{\mathbb{R}^n} \mu_{\tilde{A}}(x)\,dx}, \frac{\int_{\mathbb{R}^n} \min\{\mu_{\tilde{B}}(x), \mu_{\tilde{A}\circ\tilde{R}}(x)\}\,dx}{\int_{\mathbb{R}^n} \mu_{\tilde{B}}(x)\,dx} \right\}. \end{aligned} \tag{4.4}$$

The following lemma implies a correspondence between the order $\preceq_{\tilde{R}}$ in Definition 1 and the degree of order $\preceq_{\tilde{R}}$ in (4.4).

Theorem 4.1. *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. The order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ holds if and only if $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 1$.*

The degree of order $\preceq_{\tilde{R}}$ has the following properties.

Lemma 4.1. *Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \geq 0$. Then, the following (i) – (ii) holds:*

(i) $D(\tilde{A} \preceq_{\tilde{R}} \tilde{A}) = 1$.

(ii) *If $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 1$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{C}) = 1$, then $D(\tilde{A} \preceq_{\tilde{R}} \tilde{C}) = 1$.*

The following results are related to Theorem 3.1.

Theorem 4.2. *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, $z \in \mathbb{R}^n$ and $\lambda > 0$. Then*

$$D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = D(\tilde{A} + \{z\} \preceq_{\tilde{R}} \tilde{B} + \{z\}) = D(\lambda\tilde{A} \preceq_{\tilde{R}} \lambda\tilde{B}). \tag{4.5}$$

The following theorem is useful to calculate fuzzy sets $\tilde{R} \circ \tilde{B}$ and $\tilde{A} \circ \tilde{R}$ in the order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and the degree of order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ (Definition 1 and (4.4)).

Theorem 4.3. *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in (0,1]$. Then, the following (i) – (ii) holds:*

(i) $(\tilde{R} \circ \tilde{B})_\alpha = \tilde{B}_\alpha - R_\alpha$;

(ii) $(\tilde{A} \circ \tilde{R})_\alpha = \tilde{A}_\alpha + R_\alpha$,

where $-R_\alpha = \{-x \mid x \in R_\alpha\}$.

We consider an example in the two-dimensional case to illustrate the meaning of the degree of order $\preceq_{\tilde{R}}$.

Example 4.1 (Degree of order $\preceq_{\tilde{R}}$ on $\mathcal{F}(\mathbb{R}^2)$). We consider the case when $n = 2$ and the acute closed convex cone $R_\alpha = \mathbb{R}^2_+$ for all $\alpha \in [0,1]$ in Example 3.1. First, for fuzzy sets $\tilde{A}$ and $\tilde{B}$ given by (3.6) and (3.7), we have $\tilde{A}_\alpha = [-2+\alpha, -\alpha] \times [\alpha, 2-\alpha]$ and $\tilde{B}_\alpha = [\alpha, 2-\alpha] \times [-2+\alpha, -\alpha]$. From Theorem 4.3, $(\tilde{R} \circ \tilde{B})_\alpha = \tilde{B}_\alpha - R_\alpha = (-\infty, 2-\alpha] \times (-\infty, -\alpha]$ and $(\tilde{A} \circ \tilde{R})_\alpha = \tilde{A}_\alpha + R_\alpha = [-2+\alpha, \infty) \times [\alpha, \infty)$. Clearly, $\tilde{A}_\alpha \cap (\tilde{R} \circ \tilde{B})_\alpha = \tilde{B}_\alpha \cap (\tilde{A} \circ \tilde{R})_\alpha = \emptyset$ for all $\alpha \in (0,1]$, and so $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 0$. Similarly we can check $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) = 0$. Next, we take polyhedral cone-type fuzzy sets $\tilde{A}$ and $\tilde{B}$ by

$$\mu_{\tilde{A}}(x^1, x^2) = \max\{\min\{1 - |x^1 + 1|, 1 - |x^2 - 1|\}, 0\}, \tag{4.6}$$

$$\mu_{\tilde{B}}(x^1, x^2) = \max\{\min\{-\sqrt{3}x^2/2, (3x^1 + \sqrt{3}x^2 + 6)/4, (-3x^1 + \sqrt{3}x^2 + 6)/4\}, 0\} \tag{4.7}$$

for $(x^1, x^2) \in \mathbb{R}^2$. Then similarly we can easily check $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 0$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) = \min\{\mathrm{Sub}(\tilde{B} \subseteq \tilde{R} \circ \tilde{A}), \mathrm{Sub}(\tilde{A} \subseteq \tilde{B} \circ \tilde{R})\} = \min\{7/9, 1/3\} = 1/3$.

5. Approximation for numerical calculation

Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. In this section, using discrete cases, we discuss a method to approximate the degree of fuzzy order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ numerically. For simplicity, we deal with the case when $n = 2$ and fuzzy sets $\tilde{A}$, $\tilde{B}$ such that membership functions $\mu_{\tilde{A}}$, $\mu_{\tilde{B}}$, $\mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}$ and $\mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}$ are continuous. In the discrete case, the subsethood degree ([1, p.28]) is defined by (4.1) with scalar cardinality

$$|\tilde{A}| := \sum_x \mu_{\tilde{A}}(x), \tag{5.1}$$

where the sum is taken over some finite set. Then, the degree of order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ in the discrete case is given by

$$\min\left\{ \frac{\sum_x \mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}(x)}{\sum_x \mu_{\tilde{A}}(x)}, \frac{\sum_x \mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}(x)}{\sum_x \mu_{\tilde{B}}(x)} \right\}. \tag{5.2}$$

We approximate (4.4), using (5.2) which are easy to calculate numerically.

Let a region $C := [-c,c]^2$ such that $\tilde{A}_0 \cup \tilde{B}_0 \subseteq C = [-c,c]^2$ with $c > 0$. For $m = 1,2,\cdots$, put a mesh $C^{(m)}$ of C by $C^{(m)} := \{(x^1,x^2) \mid x^1 = ic/m, x^2 = jc/m, i,j = -m,-(m-1),\cdots,-1,0,1,\cdots,m-1,m\}$. Let $I_{\tilde{A}} := \int_{\mathbb{R}^2} \mu_{\tilde{A}}(x^1,x^2)\,dx^1dx^2$ and define

$$I_{\tilde{A}}^{(m)} := \sum_{(x^1,x^2)\in C^{(m)}} \mu_{\tilde{A}}(x^1,x^2)\left(\frac{c}{m}\right)^2 = \sum_{i=-m}^{m}\sum_{j=-m}^{m} \mu_{\tilde{A}}(x_i^{1,(m)},x_j^{2,(m)})\left(\frac{c}{m}\right)^2 \tag{5.3}$$

for $m = 1,2,\cdots$, where $(x_i^{1,(m)},x_j^{2,(m)}) = (ic/m, jc/m) \in C^{(m)}$. Then, the integrand $I_{\tilde{A}} := \int_{\mathbb{R}^2} \mu_{\tilde{A}}(x^1,x^2)\,dx^1dx^2$ is approximated:

$$I_{\tilde{A}}^{(m)} \to I_{\tilde{A}} = \int_C \mu_{\tilde{A}}(x^1,x^2)\,dx^1dx^2 = \int_{\mathbb{R}^2} \mu_{\tilde{A}}(x^1,x^2)\,dx^1dx^2$$

as $m \to \infty$. Here, we put an error $\varepsilon_{\tilde{A}}^{(m)} := |I_{\tilde{A}} - I_{\tilde{A}}^{(m)}|$ for $m = 1,2,\cdots$. For fuzzy sets $\tilde{B}$, $\tilde{A}\cap(\tilde{R}\circ\tilde{B})$ and $\tilde{B}\cap(\tilde{A}\circ\tilde{R})$, we also define integrands, discrete approximations and errors similarly. Then we obtain the following estimation of errors. The degree of order $D(\tilde{A} \preceq_{\tilde{R}}^{(m)} \tilde{B})$ in the discrete case on $C^{(m)}$ is given by

$$D(\tilde{A} \preceq_{\tilde{R}}^{(m)} \tilde{B}) := \min\left\{\frac{\sum_{x\in C^{(m)}} \mu_{\tilde{A}\cap(\tilde{R}\circ\tilde{B})}(x)}{\sum_{x\in C^{(m)}} \mu_{\tilde{A}}(x)}, \frac{\sum_{x\in C^{(m)}} \mu_{\tilde{B}\cap(\tilde{A}\circ\tilde{R})}(x)}{\sum_{x\in C^{(m)}} \mu_{\tilde{B}}(x)}\right\} \tag{5.4}$$

for $m = 1,2,\cdots$.

Theorem 5.1. *For $m = 1,2,\cdots$, it holds that*

$$|D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) - D(\tilde{A} \preceq_{\tilde{R}}^{(m)} \tilde{B})| \le \max\left\{\frac{\varepsilon_{\tilde{A}}^{(m)} + \varepsilon_{\tilde{A}\cap(\tilde{R}\circ\tilde{B})}^{(m)}}{I_{\tilde{A}} - \varepsilon_{\tilde{A}}^{(m)}}, \frac{\varepsilon_{\tilde{B}}^{(m)} + \varepsilon_{\tilde{B}\cap(\tilde{A}\circ\tilde{R})}^{(m)}}{I_{\tilde{B}} - \varepsilon_{\tilde{B}}^{(m)}}\right\} \tag{5.5}$$

if $0 < \varepsilon_{\tilde{A}}^{(m)} < I_{\tilde{A}}$ and $0 < \varepsilon_{\tilde{B}}^{(m)} < I_{\tilde{B}}$.

Theorem 5.1 implies the convergence $D(\tilde{A} \preceq_{\tilde{R}}^{(m)} \tilde{B}) \to D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ as $m \to \infty$ and also gives an estimation of errors. Finally, we give an example to approximate the degree of order on $\mathcal{F}(\mathbb{R}^2)$ by Theorem 5.1.

Example 5.1 (Approximation of the degree of order $\preceq_{\tilde{R}}$ on $\mathcal{F}(\mathbb{R}^2)$). We consider the case when $n = 2$ and the acute closed convex cone $R_\alpha = \mathbb{R}^2_+$ for all $\alpha \in [0,1]$. Take a helmet-type fuzzy set $\tilde{A}$ and a polyhedron-type fuzzy set $\tilde{B}$ by

$$\mu_{\tilde{A}}(x^1,x^2) = \max\{1 - 0.25(x^1+1)^2 - 0.25(x^2-1)^2, 0\}, \tag{5.6}$$

$$\mu_{\tilde{B}}(x^1,x^2) = \max\{\min\{-\sqrt{3}x^2, (3x^1+\sqrt{3}x^2+6)/2, (-3x^1+\sqrt{3}x^2+6)/2, 1\}, 0\} \tag{5.7}$$

for $(x^1,x^2) \in \mathbb{R}^2$. Then, we can approximate $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) \approx \min\{0.0903, 0.2331\} = 0.0903$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) \approx \min\{0.5890, 0.5714\} = 0.5714$.

Table 5.1. Approximation of degree of order $\preceq_{\tilde{R}}$ in Example 5.1 ($n = 2$).

m	10	20	30	40	50	60	70	80
$D(\tilde{A} \preceq_{\tilde{R}}^{(m)} \tilde{B})$	.08102	.08781	.08905	.08962	.08985	.08999	.09008	.09014
$D(\tilde{B} \preceq_{\tilde{R}}^{(m)} \tilde{A})$	.56905	.57100	.57066	.57139	.57134	.57125	.57136	.57136

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