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Some Observations of Fixed Point Iterations

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1 Introduction

Let H be a Hilbert space, $C \subset H$ a nonempty closed convex set, $T : C \to C$ nonexpansive, and $F(T) \neq \emptyset$. We know some basic iteration processes of as follows:

$$
\begin{array}{lll}
\text{(Halpern)} & x_1 \in C, & x_{n+1} = \alpha_n x_1 + (1-\alpha_n) T x_n \quad (n = 1, 2, \ldots) \\
\text{(Mann)} & x_1 \in C, & x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n \quad (n = 1, 2, \ldots) \\
\text{(Baillon)} & x_1 \in C, & x_{n+1} = \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} T^k x \quad (n = 1, 2, \ldots) \\
\text{(Ishikawa)} & x_1 \in C, & x_{n+1} = \alpha_n T(\beta_n T x_n + (1-\beta_n) x_n) + (1-\alpha_n) x_n \ (n = 1, 2, \ldots)
\end{array}
$$

Sequence $\{x_n\}$ produced by such iterations converges strongly, or weakly, to some element of $F(T)$ under certain conditions. Also another iteration processes and these mixed process have been found and investigated in many papers; some of basic iteration processess are similar style, but these are often incompatible. It is required to establish a unified framework to discuss some iteration processes compatibly and uniformly.

In this paper we propose an idea to consider some iteration process which includes Mann's and Ishikawa's iterations in a unified framework. To observe Mann's and Ishikawa's iterations, we see that they do not have any dependency without the previous point of its generated sequence. By replacing the rule of generation from x_n to x_{n+1} as a map A_n from C into itself, we can characterize Mann's and Ishikawa's iteration processes as follows:

$$
\begin{cases}
\text{sequence } \{x_n\} \text{ constracted as} \\
\qquad x_1 \in C, x_{n+1} = A_n x_n \quad (n = 1, 2, \ldots), \\
\text{converges strongly (or weakly) to some fixed point } F(T)
\end{cases}
\tag{1.1}
$$

Conversely when we call a sequence of functions from C into itself an iteration process if it satisfies condition (1.1), the following question occures us: *if two sequences of functions* $(A_1, A_2, \ldots, A_n, \ldots)$, $(B_1, B_2, \ldots, B_n, \ldots)$ *satisfy condition (1.1),* $x_1 \in C$, *and* $\lambda \in (0, 1)$, *then does sequence* $\{x_n\}$ *constracted as*

$$x_{n+1} = (1-\lambda) A_n x_n + \lambda B_n x_n \quad (n = 1, 2, \ldots)$$

converge to any fixed points? Or, what conditions assure the convergence? The aim of the paper is to consider the question and to investage some condition assure convergence of such sequences, and characterize some of ordinary iteration processes.

2 Observation of a set including Mann's and Ishikawa's iteration processes

At first we will catch Mann's and Ishikawa's iteration process as sets of sequences of functions:

$$\mathcal{M} = \left\{ (T_1, T_2, \ldots, T_n, \ldots) \,\middle|\, \begin{array}{l} T_n = \alpha_n I + (1-\alpha_n)T, \\ \qquad\qquad \{\alpha_n\} \subset (0,1],\ \sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty \end{array} \right\}$$

$$\mathcal{I} = \left\{ (T_1, T_2, \ldots, T_n, \ldots) \,\middle|\, \begin{array}{l} T_n = \alpha_n T(\beta_n T + (1-\beta_n)I) + (1-\alpha_n)I, \\ \quad \{\alpha_n\}, \{\beta_n\} \subset [0,1),\ \lim_{n\to\infty} b_n = 0,\ \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty \end{array} \right\}$$

Clearly $\mathcal{M}$ and $\mathcal{I}$ are subsets of vector space $\{A : C \to H\}^{\infty} = \{(A_1, A_2, \ldots, A_n, \ldots) \mid A_n : C \to H\}$ on the realfield $\mathbf{R}$ with the usual addition and schalor product. We can show easily that $\mathcal{M}$ is convex but $\mathcal{I}$ is not convex generally. Actually, for each $(T_1, T_2, \ldots, T_n, \ldots), (T_1', T_2', \ldots, T_n', \ldots) \in \mathcal{M}$ there exist sequences $\{\alpha_n\}, \{\alpha_n'\} \subset [0,1]$ satisfy $T_n = \alpha_n I + (1-\alpha_n)T$, $T_n' = \alpha_n' I + (1-\alpha_n')T$, $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, and $\sum_{n=1}^{\infty} \alpha_n'(1-\alpha_n') = \infty$. For each $\lambda \in (0,1)$ and $n = 1, 2, \ldots$,

$$\begin{aligned} (1-\lambda)T_n + \lambda T_n' &= (1-\lambda)(\alpha_n I + (1-\alpha_n)T) + \lambda(\alpha_n' I + (1-\alpha_n')T) \\ &= ((1-\lambda)\alpha_n + \lambda\alpha_n')I + (1 - ((1-\lambda)\alpha_n + \lambda\alpha_n'))T \end{aligned}$$

and sequence $\{(1-\lambda)\alpha_n + \lambda\alpha_n'\}$ satisfies the Man's condition. Also we can show $\mathcal{I}$ is not a convex set to get some counter example but omitted; at least we can not use similar calculation because T is not linear or affine function generally.

Now we define the set of generalized iteration process $\mathcal{T}$ which includes Mann's and Ishikawa's iteration processes.

Definition 2.1 Let $\mathcal{T}$ be the subset of $\{A : C \to C\}^{\infty}$ satisfied the following: $(T_1, T_2, \ldots, T_n, \ldots) \in \mathcal{T}$ if and only if $x_1 \in C$ and $\{x_n\}$ be constructed by

$$x_{n+1} = T_n x_n \quad (n \in \mathbf{N}),$$

then sequence $\{x_n\}$ converges strongly to an element of $F(T)$.

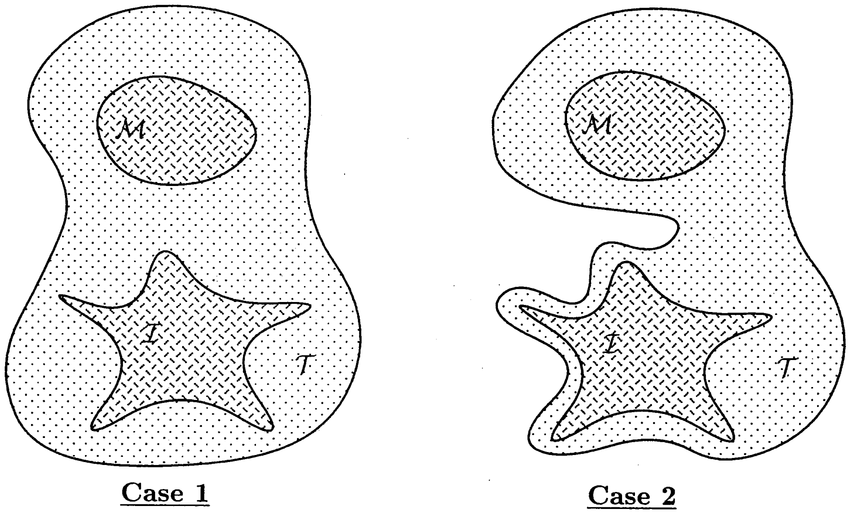

Figure 1: Which is True?

Note that if C is compact, then $\mathcal{M} \subset \mathcal{T}$ and $\mathcal{I} \subset \mathcal{T}$.

Our question is the following: is each convex combination of Mann's iteration processes and Ishikawa's iteration processes, an element of $\mathcal{T}$? Or equivalently, for each $(A_1, A_2, \ldots, A_n, \ldots)$, $(B_1, B_2, \ldots, B_n, \ldots) \in \mathcal{M} \cup \mathcal{I}$, and $\lambda \in (0,1)$,

$$(1-\lambda)(A_1, A_2, \ldots, A_n, \ldots) + \lambda(B_1, B_2, \ldots, B_n, \ldots) \in \mathcal{T}$$

is fulfilled?

To consider the question, we introduce the iteration of a generalization of both iteration processes of Mann and Ishikawa.

Definition 2.2 [Generalized Mann-Ishikawa Iteration] Let $\mathcal{GMI}$ be the subset of $\{A : C \to C\}^\infty$ satisfied the following: $(A_1, A_2, \ldots, A_n, \ldots) \in \mathcal{GMI}$ if and only if there exists a sequence $\{\delta_n\} \subset [0,1]$ such that

(i) $\|A_n x - u\|^2 \le \|x - u\|^2 - \delta_n \|Tx - x\|^2,\ \forall x \in C,\ \forall u \in F(T)$

(ii) $\sum_{n=1}^{\infty} \delta_n = +\infty$

Lemma 2.1 $\mathcal{M} \subset \mathcal{GMI}$ and $\mathcal{I} \subset \mathcal{GMI}$.

Proof. At first, we show the following formulation: for each $a, b, c \in H$ and $\lambda \in \mathbf{R}$,

$$\|\lambda a + (1-\lambda)b - c\|^2 = \lambda\|a - c\|^2 + (1-\lambda)\|b - c\|^2 - \lambda(1-\lambda)\|a - b\|^2 \tag{2.2}$$

Actually,

$$\begin{aligned}
\|\lambda a + (1-\lambda)b - c\|^2 &= \lambda^2\|a-c\|^2 + (1-\lambda)^2\|b-c\|^2 + 2\lambda(1-\lambda)\langle a-b, b-c\rangle \\
&= \lambda^2\|a-c\|^2 + (1-\lambda)^2\|b-c\|^2 + 2\lambda(1-\lambda)\langle a-c, a-c\rangle \\
&\quad + 2\lambda(1-\lambda)\langle a-c, b-a\rangle
\end{aligned}$$

From $2\langle a-c, a-b\rangle = \|a-c\|^2 + \|a-b\|^2 - \|b-c\|^2$, then we have

$$\begin{aligned}
\|\lambda a + (1-\lambda)b - c\|^2 &= \{\lambda^2 + 2\lambda(1-\lambda) - \lambda(1-\lambda)\}\|a-c\|^2 \\
&\quad + \{(1-\lambda)^2 + \lambda(1-\lambda)\}\|b-c\|^2 - \lambda(1-\lambda)\|a-b\|^2 \\
&= \lambda\|a-c\|^2 + (1-\lambda)\|b-c\|^2 - \lambda(1-\lambda)\|a-b\|^2
\end{aligned}$$

Now we show $\mathcal{M} \subset \mathcal{GMI}$. Let $(T_1, T_2, \dots, T_n, \dots) \in \mathcal{M}$ and sequence $\{\alpha_n\} \subset [0,1]$ satisfies $T_n = \alpha_n I + (1-\alpha_n)T$, and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Input $a = x$, $b = Tx$, $c = u$, and $\lambda = \alpha_n$ in equation (2.2), then we have

$$\begin{aligned}
\|T_n x - u\|^2 &= \|\alpha_n x + (1-\alpha_n)Tx - u\|^2 \\
&= \alpha_n\|x-u\|^2 + (1-\alpha_n)\|Tx-u\|^2 - \alpha_n(1-\alpha_n)\|Tx-x\|^2
\end{aligned}$$

From T is nonexpansive and $u \in F(T)$, $\|Tx-u\| \le \|x-u\|$. Then

$$\|T_n x - u\|^2 \le \|x-u\|^2 - \alpha_n(1-\alpha_n)\|Tx-x\|^2$$

and it leads to $(T_1, T_2, \dots, T_n, \dots) \in \mathcal{T}$.

Next to proof $\mathcal{I} \subset \mathcal{GMI}$, we choose $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} b_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. We set $T_n = \alpha_n T(\beta_n T + (1-\beta_n)I) + (1-\alpha_n)I$ and then

$$\begin{aligned}
\|T_n x - u\|^2 &= \|\alpha_n T(\beta_n Tx + (1-\beta_n)x) + (1-\alpha_n)x - u\|^2 \\
&= \alpha_n\|T(\beta_n Tx + (1-\beta_n)x) - u\|^2 + (1-\alpha_n)\|x-u\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|T(\beta_n Tx + (1-\beta_n)x) - x\|^2 \\
&\le \alpha_n\|T(\beta_n Tx + (1-\beta_n)x) - u\|^2 + (1-\alpha_n)\|x-u\|^2 \\
&\le \alpha_n\|\beta_n Tx + (1-\beta_n)x - u\|^2 + (1-\alpha_n)\|x-u\|^2 \\
&\le \alpha_n\{\beta_n\|Tx-u\|^2 + (1-\beta_n)\|x-u\|^2 - \beta_n(1-\beta_n)\|Tx-x\|^2\} \\
&\quad + (1-\alpha_n)\|x-u\|^2 \\
&\le \{\alpha_n\beta_n + \alpha_n(1-\beta_n) + (1-\alpha_n)\}\|x-u\|^2 - \alpha_n\beta_n(1-\beta_n)\|Tx-x\|^2 \\
&= \|x-u\|^2 - \alpha_n\beta_n(1-\beta_n)\|Tx-x\|^2
\end{aligned}$$

from equation (2.2), nonexpansiveness of T, and $u \in F(T)$. Since $\lim \beta_n = 0$, we can show that

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n) = \infty$$

and this leads to $(T_1, T_2, \dots, T_n, \dots) \in \mathcal{T}$. The proof is completed. □

Lemma 2.2 $\mathcal{GMI}$ is convex.

Proof. Let $(A_1, A_2, \ldots, A_n, \ldots), (B_1, B_2, \ldots, B_n, \ldots) \in \mathcal{GMI}$, then there exist sequences $\{\delta_n^A\}, \{\delta_n^B\} \subset [0,1]$ which satisfy conditions of Definition 2.2. For each $x \in C$, $u \in F(T)$, and $\lambda \in (0,1)$,

$$\begin{aligned}
\|(1-\lambda)A_n x + \lambda B_n x - u\|^2 &= (1-\lambda)\|A_n x - u\|^2 + \lambda\|B_n x - u\|^2 \\
&\quad -\lambda(1-\lambda)\|A_n x - B_n x\|^2 \\
&\le (1-\lambda)\|A_n x - u\|^2 + \lambda\|B_n x - u\|^2 \\
&\le \|x-u\|^2 - \{(1-\lambda)\delta_n^A + \lambda\delta_n^B\}\|Tx - x\|^2
\end{aligned}$$

from the assumption. Also we can check $\sum_{n=1}^{\infty}\{(1-\lambda)\delta_n^A + \lambda\delta_n^B\} = \infty$ easily. This completes the proof. □

Theorem 2.1 If C is compact, then $\mathcal{GMI} \subset \mathcal{T}$.

Proof. Fix $(A_1, A_2, \ldots, A_n, \ldots) \in \mathcal{GMI}$ and $x_1 \in C$. Then we can choose a sequence $\{\delta_n\} \subset [0,1]$ satisfies conditions of Definition 2.2. Let sequence $\{x_n\}$ be made by iteration process $(A_1, A_2, \ldots, A_n, \ldots)$, that is $x_{n+1} = A_n x_n$ for all $n \in \mathbf{N}$. For each $u \in F(T)$,

$$\|x_{n+1} - u\|^2 \le \|x_n - u\|^2 - \delta_n\|Tx_n - x_n\|^2$$

Let $m \in \mathbf{N}$, then

$$\sum_{n=1}^{m} \delta_n\|Tx_n - x_n\|^2 \le \|x_1 - u\|^2 - \|x_{m+1} - u\|^2$$

We see sequence $\{\|x_n - u\|\}$ is decreasing and the right hand side of the last formulation is bounded. Then we obtain $\liminf_{n\to\infty}\|Tx_n - x_n\|$, and we can choose a subsequence $\{x_{n'}\} \subset \{x_n\}$ such that $\|Tx_{n'} - x_{n'}\| \to 0$. Also C is compact, then there exist $\{x_{n''}\} \subset \{x_{n'}\}$ and $x_0 \in C$ such that $x_{n''} \to x_0$. Hence sequence $\{\|Tx_{n''} - x_{n''}\|\}$ has limits 0 and $\|Tx_0 - x_0\|$, then we obtain $x_0 \in F(T)$. Moreover, sequence $\{\|x_n - x_0\|\}$ is decreasing to 0, then $\{x_n\}$ converges strongly to x_0. This completes the proof. □

Theorem 2.2 If C is compact, then $co(\mathcal{M} \cup \mathcal{I}) \subset \mathcal{T}$. That is, $(A_1, A_2, \ldots, A_n, \ldots)$, $(B_1, B_2, \ldots, B_n, \ldots) \in \mathcal{GMI}$, $\lambda \in (0,1)$, $x_1 \in C$, and $\{x_n\}$ be constructed by

$$x_{n+1} = (1-\lambda)A_n x_n + \lambda B_n x_n \quad (n \in \mathbf{N}),$$

then the sequence $\{x_n\}$ converges strongly to an element of $F(T)$.

Proof. For each $(A_1, A_2, \ldots, A_n, \ldots)$, $(B_1, B_2, \ldots, B_n, \ldots) \in \mathcal{GMI}$, $\lambda \in (0,1)$, we obtain

$$((1-\lambda)A_1 + \lambda B_1, (1-\lambda)A_2 + \lambda B_2, \ldots, (1-\lambda)A_n + \lambda B_n, \ldots) \in \mathcal{GMI}$$

from Lemma 2.2. Also by using Theorem 2.1, we have the sequence $\{x_n\}$ converges strongly to an element of $F(T)$. □

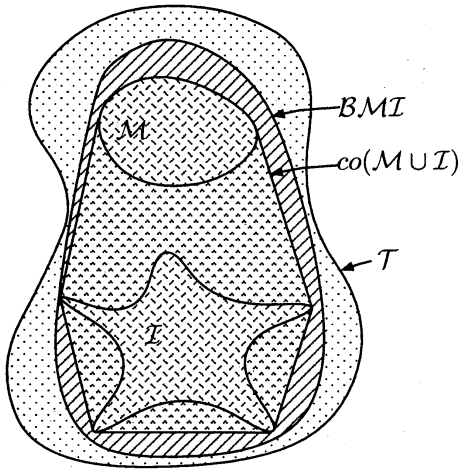

Figure 2: Case 1 is True!

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