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On Overcompleteness of Coherent Vectors

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Introduction

The appellation "coherent vector" appears in a large number of references in various contexts with wide applications (see e.g., [5], [13]). In this paper we focus on the so-called "canonical" coherent vectors generated by annihilation and creation operators of Boson field. For the simplest case let a and a^* denote respectively the annihilation and creation operators satisfying the canonical commutation relation $[a, a^*] = I$. The Boson Fock space $\Gamma(\mathbf{C})$ is constructed from the Fock vacuum $|0\rangle$ with the action of the creation operator and is decomposed into a direct sum of 1-dimensional subspaces spanned by the number vectors $|n\rangle \in \Gamma(\mathbf{C})$, $n = 0, 1, 2, \cdots$. Then, a (normalized) *coherent vector* or a *coherent state* is defined by

$$|z\rangle = e^{za^*-\bar{z}a}\,|0\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle, \qquad z \in \mathbf{C}.$$

It is well known that $\{|z\rangle\,;\, z \in \mathbf{C}\}$ is linearly independent and is overcomplete, i.e., each $\phi \in \Gamma(\mathbf{C})$ admits an expression of the form:

$$\phi = \frac{1}{\pi}\int_{\mathbf{C}} \rho(z)|z\rangle\, d^2z, \tag{1}$$

but the density function $\rho(z)$ is not uniquely specified. The above expression (1) follows also by the resolution of the identity

$$I = \frac{1}{\pi}\int_{\mathbf{C}} |z\rangle\langle z|\, d^2z, \tag{2}$$

where $|z\rangle\langle z|$ stands for the one-dimensional projection onto the subspace spanned by $|z\rangle$ and d^2z is the Lebesgue measure on $\mathbf{C}$. The above formula (2), due to Klauder [12], has extensive applications in quantum physics, see e.g., [7], [14], [24].

The main purpose of this paper is to look at a role of coherent vectors systematically on the basis of white noise theory initiated by Hida [9], for a recent framework of white noise analysis see e.g., [16], [18]. In fact, we introduce a white noise triple over the complex Gaussian space by modifying the idea of CKS-space [4] and establish an isomorphism, which will turn out to be a natural extension of the Segal–Bargmann transform (see [8] for a concise survey), between the white noise distributions on the real Gaussian space and the holomorphic distributions on the complex Gaussian space. Then we discuss coherent state represenations of white noise functions and of white noise operators. The characterization theorems of S-transform (white noise version of the Segal–Bargmann transform) and of symbols of white noise operators [16], [18], [23] are essential tools during our discussion. Then the overcompleteness of coherent vectors is discussed along with the inverse S-transform and the Wick multiplication.

1 White Noise Functions

1.1 Weighted Fock space For a Hilbert space H and a sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ of positive numbers we put

$$\Gamma_\alpha(H) = \left\{ \phi = (f_n)_{n=0}^{\infty}\,;\; f_n \in H^{\widehat{\otimes} n},\;\; \|\phi\|^2 \equiv \sum_{n=0}^{\infty} n!\,\alpha(n)|f_n|^2 < \infty \right\},$$

where $H^{\widehat{\otimes} n}$ is the n-fold symmetric tensor power of H. In an obvious manner $\Gamma_\alpha(H)$ becomes a Hilbert space and is called the *weighted Fock space* over H. The Boson Fock space is by definition the special case of $\alpha(n) \equiv 1$ and is denoted by $\Gamma(H)$.

Throughout the paper the weight sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ is assumed to satisfy the following conditions in order to guarantee basic properties of white noise operators as well as the characterization theorems for S-transform and operator symbols:

(A1) $1 = \alpha(0) \le \alpha(1) \le \alpha(2) \le \cdots$;

(A2) $G_\alpha(t) \equiv \displaystyle\sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n$ has an infinite radius of convergence;

(A3) $\widetilde{G}_\alpha(t) \equiv \displaystyle\sum_{n=0}^{\infty} t^n \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\}$ has a positive radius of convergence;

(A4) there exists a constant $C_1 > 0$ such that $\alpha(n)\alpha(m) \le C_1^{n+m}\alpha(n+m)$ for any n, m;

(A5) there exists a constant $C_2 > 0$ such that $\alpha(n+m) \le C_2^{n+m}\alpha(n)\alpha(m)$ for any n, m;

Examples of such a weight sequence are $(n!)^\beta$ with $0 \le \beta < 1$, the Bell numbers of order k, and so on, see [4]. For later use we record some essential properties of $G_\alpha(t)$, whose proofs are straightforward.

Proposition 1.1 *Let $\alpha = \{\alpha(n)\}$ be a positive sequence satisfying conditions (A1)–(A5) above and $G_\alpha(t)$ the generating function defined therein. Then, for $s, t \ge 0$ we have:*

(1) $G_\alpha(0) = 1$ *and* $G_\alpha(s) \le G_\alpha(t)$ *for* $s \le t$.

(2) $G_\alpha(s)G_\alpha(t) \le G_\alpha(C_1(s+t))$.

(3) $G_\alpha(s+t) \le G_\alpha(C_2 s)G_\alpha(C_2 t)$.

(4) $e^s G_\alpha(t) \le G_\alpha(s+t)$.

(5) $e^t \le G_\alpha(t)$.

In fact, conditions (A1)–(A5) are not minimum prerequisites and an almost ultimate (but somehow implicit) description has been investigated in terms of a function G_α, see [1], [6].

1.2 CKS-Space We start with the real Gelfand triple:

$$E = \mathcal{S}(\mathbf{R}) \subset H = L^2(\mathbf{R}) \subset E^* = \mathcal{S}'(\mathbf{R}), \tag{3}$$

where the canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$ and the norm of $L^2(\mathbf{R})$ by $|\cdot|_0$. For $p \geq 0$ let $E_{\pm p}$ be the Hilbert space obtained by completing $E = \mathcal{S}(\mathbf{R})$ with respect to the norm $|\xi|_{\pm p} = |A^{\pm p}\xi|_0$, where $A = 1 + t^2 - d^2/dt^2$. We then have

$$E \cong \operatorname*{proj\,lim}_{p\to\infty} E_p, \qquad E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p}.$$

In general, for a real vector space $\mathfrak{X}$ the complexification is denoted by $\mathfrak{X}_{\mathbf{C}}$. For notational convenience, the $\mathbf{C}$-bilinear extension on $E^*_{\mathbf{C}} \times E_{\mathbf{C}}$ is denoted by the same symbol so that $|\xi|_0^2 = \langle \bar{\xi}, \xi \rangle$ holds for $\xi \in H_{\mathbf{C}}$.

Let $\Gamma_\alpha(E_p)$ be the weighted Fock space over E_p. Then

$$\mathcal{W} = \Gamma_\alpha(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\alpha(E_p) \tag{4}$$

becomes a nuclear space and we obtain a Gelfand triple:

$$\mathcal{W} = \Gamma_\alpha(E) \subset \Gamma(H_{\mathbf{C}}) \subset \Gamma_\alpha(E)^* = \mathcal{W}^*, \tag{5}$$

which is called the *Cochran–Kuo–Sengupta space* or the *CKS-space* for short [4]. By definition the topology of $\mathcal{W}$ is given by the family of norms:

$$\|\phi\|_{p,+}^2 = \sum_{n=0}^{\infty} n!\,\alpha(n)|f_n|_p^2, \qquad \phi = (f_n), \quad p \geq 0.$$

We see by a standard argument that

$$\Gamma_\alpha(E)^* \cong \operatorname*{ind\,lim}_{p\to\infty} \Gamma_{\alpha^{-1}}(E_{-p}),$$

where $\Gamma_\alpha(E)^*$ carries the strong dual topology and $\cong$ stands for a topological linear isomorphism. The canonical $\mathbf{C}$-bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Then

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n!\,\langle F_n, f_n \rangle, \qquad \Phi = (F_n) \in \mathcal{W}^*, \quad \phi = (f_n) \in \mathcal{W},$$

and it holds that

$$|\langle\!\langle \Phi, \phi \rangle\!\rangle| \leq \|\Phi\|_{-p,-}\|\phi\|_{p,+},$$

where

$$\|\Phi\|_{-p,-}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} |F_n|_{-p}^2, \qquad \Phi = (F_n) \in \mathcal{W}^*.$$

1.3 Wiener–Itô–Segal Isomorphism and Coherent Vectors On the basis of the Gelfand triple (3) we introduce the Gaussian measure on E^* with variance σ^2, denoted by μ_{σ^2}, by the characteristic function:

$$e^{-\sigma^2|\xi|_0^2/2} = \int_{E^*} e^{i\langle x, \xi\rangle} \mu_{\sigma^2}(dx), \qquad \xi \in E. \tag{6}$$

We denote simply by μ the Gaussian measure on E^* with variance $\sigma^2 = 1$ and by $L^2(E^*, \mu)$ the Hilbert space of $\mathbf{C}$-valued L^2-functions on E^*. The celebrated Wiener-Itô-Segal isomorphism is the unitary isomorphism between $L^2(E^*, \mu)$ and $\Gamma(H_{\mathbf{C}})$ uniquely determined by the correspondence

$$\phi_\xi(x) = e^{\langle x, \xi\rangle - \langle \xi, \xi\rangle/2} \quad \leftrightarrow \quad \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right), \qquad \xi \in E_{\mathbf{C}}. \tag{7}$$

The right hand side in (7) will be denoted by the same symbol ϕ_ξ. We call ϕ_ξ an *exponential vector* or a *coherent vector*. Note that

$$\langle\langle \phi_\xi, \phi_\eta \rangle\rangle = e^{\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E_{\mathbf{C}},$$

and hence ϕ_ξ is not normalized in general.

The next result is easily verified.

Proposition 1.2 $\{\phi_\xi\,;\, \xi \in E_{\mathbf{C}}\}$ *spans a dense subspace of* $\mathcal{W} = \Gamma_\alpha(E_{\mathbf{C}})$.

2 Complex White Noise

2.1 Complex Gaussian Space Let μ' be the Gaussian measure on E^* with variance $\sigma^2 = 1/2$, see (6). In view of the topological isomorphism $E^*_{\mathbf{C}} \cong E^* \times E^*$, we define a probability measure $\nu = \mu' \times \mu'$ on $E^*_{\mathbf{C}}$ by

$$\nu(dz) = \mu'(dx)\mu'(dy), \qquad z = x + iy \in E^*_{\mathbf{C}}.$$

The probability space $(E^*_{\mathbf{C}}, \nu)$ is called the *complex Gaussian space* [10, Chapter 6]. We write $\bar{z} = x - iy$ for $z = x + iy \in E^* + iE^*$. Here are basic formulae:

$$\int_{E^*_{\mathbf{C}}} e^{\langle \bar{z}, \xi\rangle + \langle z, \eta\rangle}\, \nu(dz) = e^{\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}, \tag{8}$$

$$\int_{E^*_{\mathbf{C}}} \langle z^{\otimes m}, \xi_1 \otimes \cdots \otimes \xi_m\rangle \langle \bar{z}^{\otimes n}, \eta_1 \otimes \cdots \otimes \eta_n\rangle\, \nu(dz)$$
$$= \delta_{mn}\, m!\, \langle \xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_m, \eta_1 \widehat{\otimes} \cdots \widehat{\otimes} \eta_m\rangle, \qquad \xi_i, \eta_j \in E_{\mathbf{C}}. \tag{9}$$

The proofs are straightforward computation.

2.2 CKS-Space over Complex Gaussian Space We see by modifying the Wiener–Itô–Segal isomorphism (7) that the correspondence

$$\psi_\xi(x) \equiv e^{\sqrt{2}\langle x, \xi\rangle - \langle \xi, \xi\rangle/2} \leftrightarrow \phi_\xi \equiv \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right), \quad \xi \in E_{\mathbf{C}}, \tag{10}$$

determines a unitary isomorphism:

$$L^2(E^*, \mu') \cong \Gamma(H_{\mathbf{C}}). \tag{11}$$

Then,

$$L^2(E^*_{\mathbf{C}}, \nu) \cong L^2(E^*, \mu') \otimes L^2(E^*, \mu') \cong \Gamma(H_{\mathbf{C}}) \otimes \Gamma(H_{\mathbf{C}}), \tag{12}$$

where functions on $E^*_{\mathbf{C}}$ and on $E^* \times E^*$ are identified in a canonical manner:

$$\phi \otimes \psi(x + iy) = \phi(x)\psi(y), \qquad x, y \in E^*, \quad \phi, \psi \in L^2(E^*, \mu'). \tag{13}$$

Finally, duplicating the Gelfand triple (5) and using (12) we obtain the CKS-space over the complex Gaussian space:

$$\mathcal{D} \subset L^2(E^*_{\mathbf{C}}, \nu) \subset \mathcal{D}^*, \tag{14}$$

where $\mathcal{D} \cong \mathcal{W} \otimes \mathcal{W}$.

By construction $\mathcal{W} \subset L^2(E^*, \mu)$ and $\mathcal{D} \subset L^2(E^*_{\mathbf{C}}, \nu)$ consist of equivalence classes of L^2-functions on E^* and those on $E^*_{\mathbf{C}}$, respectively. However, each equivalence class contains a unique continuous function (continuous version theorem, see e.g., [18, Chapter 1.4]), and moreover, further analytic properties of the continuos function are examined in a similar manner as in [17], [22].

Lemma 2.1 *For any $\phi = (f_m) \in \Gamma(H_{\mathbf{C}})$*

$$\omega_\phi(z) = \sum_{m=0}^{\infty} \langle z^{\otimes m}, f_m\rangle$$

is defined in L^2-sense and $\phi \mapsto \omega_\phi$ is an isometric map from $\Gamma(H_{\mathbf{C}})$ into $L^2(E^*_{\mathbf{C}}, \nu)$.

PROOF. By the sdandard argument with (9). First, for $f_m \in H_{\mathbf{C}}^{\widehat{\otimes} m}$ the function $\omega_m(z) = \langle z^{\otimes m}, f_m\rangle$ is defined in the L^2-sense. Then, so is $\omega(z) = \sum_{m=0}^{\infty} \omega_m(z)$. ∎

Let $L^2(E^*_{\mathbf{C}}, \nu)_{\mathrm{HOL}} \subset L^2(E^*_{\mathbf{C}}, \nu)$ be the image of the isometric map $\phi \mapsto \omega_\phi$, $\phi \in \Gamma(H_{\mathbf{C}})$, introduced in Lemma 2.1. Then, in view of the Wiener–Itô–Segal isomorphism $L^2(E^*, \mu) \cong \Gamma(H_{\mathbf{C}})$ we obtain a unitary isomorphism $L^2(E^*, \mu) \cong L^2(E^*_{\mathbf{C}}, \nu)_{\mathrm{HOL}}$. This is the famous Segal–Bargmann transform [8].

For later use we prove the following

Lemma 2.2 *Under the identification $L^2(E^*_{\mathbf{C}}, \nu) \cong \Gamma(H_{\mathbf{C}}) \otimes \Gamma(H_{\mathbf{C}})$ described in (12) we have the following correspondence:*

$$\langle z, \zeta\rangle^m \leftrightarrow \sum_{k=0}^{m} \binom{m}{k} \left(0, \cdots, \left(\frac{\zeta}{\sqrt{2}}\right)^{\otimes k}, 0, \cdots\right) \otimes \left(0, \cdots, \left(\frac{i\zeta}{\sqrt{2}}\right)^{\otimes (m-k)}, 0, \cdots\right), \tag{15}$$

$$\langle \bar{z}, \zeta\rangle^m \leftrightarrow \sum_{k=0}^{m} \binom{m}{k} \left(0, \cdots, \left(\frac{\zeta}{\sqrt{2}}\right)^{\otimes k}, 0, \cdots\right) \otimes \left(0, \cdots, \left(\frac{\zeta}{i\sqrt{2}}\right)^{\otimes (m-k)}, 0, \cdots\right), \tag{16}$$

where $\zeta \in E_{\mathbf{C}}$.

PROOF. Since the identification $L^2(E_{\mathbf{C}}^*, \nu) \cong \Gamma(H_{\mathbf{C}}) \otimes \Gamma(H_{\mathbf{C}})$ is characterized by (10) and (13), we have the correspondence:

$$\psi_\xi \otimes \psi_\eta(z) = e^{\langle z, \zeta' \rangle + \langle \bar{z}, \zeta \rangle - \langle \zeta, \zeta' \rangle} \quad \leftrightarrow \quad \phi_\xi \otimes \phi_\eta = \left(\frac{\xi^{\otimes n}}{n!} \right) \otimes \left(\frac{\eta^{\otimes n}}{n!} \right), \tag{17}$$

where $\zeta = (\xi + i\eta)/\sqrt{2}$, $\zeta' = (\xi - i\eta)/\sqrt{2}$, and $\xi, \eta \in E_{\mathbf{C}}$. Then the assertion follows from (17) by Taylor expansion. ∎

Note that the right hand sides of (15) and (16) are orthogonal sums with respect to the norm of $\Gamma(E_p) \otimes \Gamma(E_p)$ for any $p \geq 0$. Then we have

Lemma 2.3 *Let $\phi = (f_m) \in \Gamma(H_{\mathbf{C}})$ and $\omega = \omega_\phi \in L^2(E_{\mathbf{C}}^*, \nu)$ be related as in Lemma 2.1. Then for any $p \geq 0$,*

$$\| \phi \|_p^2 = \sum_{m=0}^{\infty} m! \, | f_m |_p^2 = \| \omega \|_p^2 ,$$

where $\| \omega \|_p$ stands for the norm of $\Gamma(E_p) \otimes \Gamma(E_p)$.

3 Representation of White Noise Functions

3.1 S-transform and Characterization Theorem The S-transform of $\Phi \in \mathcal{W}^*$ is defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle , \qquad \xi \in E_{\mathbf{C}}.$$

It follows from Proposition 1.2 that the S-transform determines a white noise function uniquely. In fact, for $\Phi = (F_n)$ we have

$$S\Phi(\xi) = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n} \rangle , \qquad \xi \in E_{\mathbf{C}}. \tag{18}$$

Moreover, we have the following fundamental result known as the characterization theorem for S-transform.

Theorem 3.1 [4] (Characterization for S-transform) *A $\mathbf{C}$-valued function F defined on $E_{\mathbf{C}}$ is the S-transform of a white noise distribution $\Phi \in \mathcal{W}^*$ if and only if*

(F1) *for any $\xi, \xi_1 \in E_{\mathbf{C}}$, $z \mapsto F(z\xi + \xi_1)$ is entire holomorphic on $\mathbf{C}$;*

(F2) *there exist some $C \geq 0$ and $p \geq 0$ such that*

$$|F(\xi)|^2 \leq C G_\alpha(| \xi |_p^2), \qquad \xi \in E_{\mathbf{C}}.$$

In that case,

$$\| \Phi \|_{-(p+q),-}^2 \leq C \widetilde{G}_\alpha(\|A^{-q}\|_{\mathrm{HS}}^2),$$

for any $q > 1/2$ such that $\widetilde{G}_\alpha(\|A^{-q}\|_{\mathrm{HS}}^2) < \infty$. (This choice is always possible since $\|A^{-q}\|_{\mathrm{HS}} \to 0$ as $q \to \infty$.)

For $\phi \in \mathcal{W}$ the S-transform $S\phi$ is naturally extended to a function defined on $E^*_{\mathbf{C}}$ by

$$S\phi(z) = \langle\!\langle \phi_z, \phi \rangle\!\rangle , \qquad z \in E^*_{\mathbf{C}},$$

where

$$\phi_z = \left(1, z, \frac{z^{\otimes 2}}{2!}, \cdots, \frac{z^{\otimes n}}{n!}, \cdots \right) \in \mathcal{W}^*$$

and is also referred as a coherent vector. Then, for $\phi = (f_n)$ we have

$$S\phi(z) = \sum_{n=0}^{\infty} \left\langle z^{\otimes n}, f_n \right\rangle , \qquad z \in E^*_{\mathbf{C}}.$$

From this with Lemma 2.1 it follows that the S-transform restricted to $L^2(E^*, \mu)$ is nothing but the Segal–Bargmann transform. Moreover, for $\Phi = (F_n) \in \mathcal{W}^*$ consider a formal notation

$$\Omega(z) = \sum_{n=0}^{\infty} \left\langle z^{\otimes n}, F_n \right\rangle . \tag{19}$$

With the help of Lemmas 2.1–2.3 it is easily verifed that Ω gives rise to a distribution in $\mathcal{D}^*$. Let $\mathcal{D}^*_{\mathrm{HOL}}$ be the subspace of $\Omega \in \mathcal{D}^*$ of the form (19). Thus the S-transform $S\Phi$ is extended to a holomorphic distribution on the complex Gaussian space, which will be denoted by the same symbol. Summing up,

Theorem 3.2 *The S-transform (in the above sense) extends the Segal–Bargmann transform and is a topological isomorphism from $\mathcal{W}^*$ onto $\mathcal{D}^*_{\mathrm{HOL}}$.*

A holomorphic distribution $\Omega \in \mathcal{D}^*_{\mathrm{HOL}}$ feels only anti-holomorphic test functions. Let $L^2(E^*_{\mathbf{C}}, \nu)_{\mathrm{AH}}$ be the subspace of anti-holomorphic L^2-functions, i.e., of all $\phi \in L^2(E^*_{\mathbf{C}}, \nu)$ such that $\bar{\phi} \in L^2(E^*_{\mathbf{C}}, \nu)_{\mathrm{HOL}}$. Then for $\omega \in \mathcal{D}_{\mathrm{AH}} = \mathcal{D} \cap L^2(E^*_{\mathbf{C}}, \nu)_{\mathrm{AH}}$ given by $\omega(z) = \sum_{n=0}^{\infty} \langle \bar{z}^{\otimes n}, f_n \rangle$ we have

$$\langle\!\langle \Omega, \omega \rangle\!\rangle = \sum_{n=0}^{\infty} n! \left\langle F_n, f_n \right\rangle .$$

3.2 Coherent State Representation and Inverse S-transform For any $\xi \in E_{\mathbf{C}}$ we consider

$$\epsilon_\xi(z) = e^{\langle z, \xi \rangle} = \psi_{\xi/\sqrt{2}}(x)\psi_{i\xi/\sqrt{2}}(y), \qquad z = x + iy \in E^*_{\mathbf{C}}.$$

Then, $\epsilon_\xi = \psi_{\xi/\sqrt{2}} \otimes \psi_{i\xi/\sqrt{2}}$ and by definition $\epsilon_\xi \in \mathcal{D} \cong \mathcal{W} \otimes \mathcal{W}$.

Lemma 3.3 *For any $\rho \in \mathcal{D}^*$ there exists a unique $\Phi \in \mathcal{W}^*$ such that*

$$S\Phi(\xi) = \langle\!\langle \rho, \epsilon_\xi \rangle\!\rangle , \qquad \xi \in E_{\mathbf{C}}. \tag{20}$$

PROOF. Denote by $F(\xi)$ the right hand side of (20). It is obvious that F satisfies condition (F1) in Thoerem 3.1. We shall prove (F2). Choosing $p \ge 0$ such that $\| \rho \|_{-p,-} < \infty$, we observe

$$\begin{aligned} |F(\xi)|^2 &\le \| \rho \|^2_{-p,-} \| \epsilon_\xi \|^2_{p,+} = \| \rho \|^2_{-p,-} \| \psi_{\xi/\sqrt{2}} \|^2_{p,+} \| \psi_{i\xi/\sqrt{2}} \|^2_{p,+} \\ &= \| \rho \|^2_{-p,-} G_\alpha\!\left(\left| \frac{\xi}{\sqrt{2}} \right|^2_p \right) G_\alpha\!\left(\left| \frac{i\xi}{\sqrt{2}} \right|^2_p \right). \end{aligned}$$

In view of Proposition 1.1 (2) we have

$$G_\alpha\Big(\Big|\frac{\xi}{\sqrt{2}}\Big|_p^2\Big)G_\alpha\Big(\Big|\frac{i\xi}{\sqrt{2}}\Big|_p^2\Big) \le G_\alpha(C_1\,|\,\xi\,|_p^2) \le G_\alpha(C_1\rho^{2q}\,|\,\xi\,|_{p+q}^2),$$

where $\rho = \|A^{-1}\|_{\mathrm{OP}} = 1/2$. Choose $q \ge 0$ such that $C_1\rho^{2q} \le 1$. Then we come to

$$|F(\xi)|^2 \le \|\,\rho\,\|_{-p,-}^2\, G_\alpha(|\,\xi\,|_{p+q}^2),$$

and condition (F2) is fulfilled. Thus, by Thoerem 3.1, F is the S-transform of a white noise distribution in $\mathcal{W}^*$. ∎

The white noise distribution Φ defined as in (20) is denoted by

$$\Phi = \int_{E_{\mathbf{C}}^*} \rho(z)\phi_z \nu(dz)$$

and is called a *coherent state representation.*

Theorem 3.4 (Inverse S-transform) *For any $\Phi \in \mathcal{W}^*$, the S-transform $S\Phi$ being regarded as a holomorphic distribution on the complex Gaussian space, it holds that*

$$\Phi = \int_{E_{\mathbf{C}}^*} S\Phi(\bar{z})\phi_z \nu(dz).$$

In particular, every white noise distribution Φ admits a coherent state representation.

PROOF. It follows from Theorem 3.2 that $\rho(z) = S\Phi(\bar{z}) = \sum_{n=0}^\infty \langle \bar{z}^{\otimes n}, F_n\rangle$ belongs to $\mathcal{D}^*$ and by Lemma 3.3 there exists a unique $\Psi \in \mathcal{W}^*$ satisfying $S\Psi(\xi) = \langle\langle \rho, \epsilon_\xi\rangle\rangle$ for $\xi \in E_{\mathbf{C}}$. Now, in view of

$$\begin{aligned} S\Psi(\xi) &= \langle\langle \rho, \epsilon_\xi\rangle\rangle \\ &= \sum_{n=0}^\infty \int_{E_{\mathbf{C}}^*} \langle \bar{z}^{\otimes n}, F_n\rangle \Big\langle z^{\otimes n}, \frac{\xi^{\otimes n}}{n!}\Big\rangle \nu(dz) \\ &= \sum_{n=0}^\infty \langle F_n, \xi^{\otimes n}\rangle = S\Phi(\xi), \end{aligned}$$

we see that $\Phi = \Psi$. ∎

The inverse S-transform was discussed in [15], [17] by means of the real Gaussian integral, and in [2, Chapter 2.5.3] based on a similar idea of holomorphic distributions in a different context.

4 Representation of White Noise Operators

4.1 White Noise Operators and Characterization of Symbols In general, a continuous operator from $\mathcal{W}$ into $\mathcal{W}^*$ is referred to as a *white noise operator.* Let $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ denote the space of white noise operators. The *symbol* of $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is by definition a $\mathbf{C}$-valued function on $E_{\mathbf{C}} \times E_{\mathbf{C}}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle, \qquad \xi, \eta \in E_{\mathbf{C}}.$$

The symbol uniquely specifies a white noise operator by Proposition 1.2.

Theorem 4.1 [3] (Characterization for operator symbols) *A function* $\Theta : E_{\mathbf{C}} \times E_{\mathbf{C}} \to \mathbf{C}$ *is the symbol of an operator* $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ *if and only if*

(O1) *for any* $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbf{C}}$, $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ *is entire holomorphic on* $\mathbf{C} \times \mathbf{C}$;

(O2) *there exist constant numbers* $C \geq 0$ *and* $p \geq 0$ *such that*

$$|\Theta(\xi, \eta)|^2 \leq C G_\alpha(|\xi|_p^2) G_\alpha(|\eta|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

In that case

$$\|\Xi\phi\|^2_{-(p+q),-} \leq C \widetilde{G}^2_\alpha(\|A^{-q}\|^2_{\mathrm{HS}}) \|\phi\|^2_{p+q,+}, \qquad \phi \in \mathcal{W}, \tag{21}$$

where $q > 1/2$ *is taken as* $\widetilde{G}_\alpha(\|A^{-q}\|^2_{\mathrm{HS}}) < \infty$.

4.2 Diagonal Coherent State Representation With each $z \in E^*_{\mathbf{C}}$ we associate $Q_z \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ by the formula:

$$Q_z \phi = \langle\!\langle \phi_{\bar{z}}, \phi \rangle\!\rangle \, \phi_z, \qquad \phi \in \mathcal{W}.$$

Note here that both maps $z \mapsto \phi_z \in \mathcal{W}^*$ and $z \mapsto Q_z \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ are continuous. The symbol of Q_z is given by

$$\widehat{Q}_z(\xi, \eta) = q_{\xi,\eta}(z) = e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle}, \qquad z \in E^*_{\mathbf{C}}, \quad \xi, \eta \in E_{\mathbf{C}}. \tag{22}$$

Then, for $z = x + iy$ we obtain

$$q_{\xi,\eta}(x + iy) = e^{\langle x, \xi + \eta \rangle} e^{\langle y, i(-\xi + \eta) \rangle} = e^{\langle \xi, \eta \rangle} \psi_{(\xi+\eta)/\sqrt{2}}(x) \psi_{i(-\xi+\eta)/\sqrt{2}}(y). \tag{23}$$

In other words, $q_{\xi,\eta} = e^{\langle \xi, \eta \rangle} \psi_{(\xi+\eta)/\sqrt{2}} \otimes \psi_{i(-\xi+\eta)/\sqrt{2}}$ and hence $q_{\xi,\eta} \in \mathcal{W} \otimes \mathcal{W} \cong \mathcal{D}$.

Lemma 4.2 *For* $\rho \in \mathcal{D}^*$ *there is a unique operator* $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ *such that*

$$\langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle = \langle\!\langle \rho, q_{\xi,\eta} \rangle\!\rangle, \qquad \xi, \eta \in E_{\mathbf{C}}. \tag{24}$$

The proof is a simple application of Theorem 4.1 and a similar argument as in the proof of Lemma 3.3, see also [20]. The operator Ξ defined by (24) is denoted by

$$\Xi = \int_{E^*_{\mathbf{C}}} \rho(z) Q_z \, \nu(dz) \tag{25}$$

and is called the *diagonal coherent state representation*.

Theorem 4.3 [20] *Every white noise operator in* $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ *admits a unique diagonal coherent state representation.*

PROOF. Here is an outline for the sake of the readers' convenience, for more details see [20]. Suppose we are given $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. Then, by Theorem 4.1 there exists a white noise operator $W \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that

$$\langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle = \langle\!\langle W\phi_{(\xi+\eta)/\sqrt{2}}, \phi_{i(-\xi+\eta)/\sqrt{2}} \rangle\!\rangle \, e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}. \tag{26}$$

Using the canonical isomorphism $(\mathcal{W}\otimes\mathcal{W})^* \cong \mathcal{L}(\mathcal{W},\mathcal{W}^*)$, we choose $\rho\in(\mathcal{W}\otimes\mathcal{W})^*$ corresponding to $W\in\mathcal{L}(\mathcal{W},\mathcal{W}^*)$. Then (26) becomes

$$\langle\!\langle \Xi\phi_\xi,\, \phi_\eta\rangle\!\rangle = \langle\!\langle \rho,\, \phi_{(\xi+\eta)/\sqrt{2}}\otimes\phi_{i(-\xi+\eta)/\sqrt{2}}\rangle\!\rangle\, e^{\langle\xi,\,\eta\rangle}. \tag{27}$$

By (10), (22) and (23) the functional realization of $\phi_{(\xi+\eta)/\sqrt{2}}\otimes\phi_{i(-\xi+\eta)/\sqrt{2}}\, e^{\langle\xi,\,\eta\rangle}$ is equal to

$$\psi_{(\xi+\eta)/\sqrt{2}}(x)\psi_{i(-\xi+\eta)/\sqrt{2}}(y)e^{\langle\xi,\,\eta\rangle} = q_{\xi,\eta}(x+iy) = \langle\!\langle Q_z\phi_\xi,\, \phi_\eta\rangle\!\rangle\,,$$

where $z=x+iy\in E^*_{\mathbf{C}}$. Then (27) is written in a formal integral

$$\langle\!\langle \Xi\phi_\xi,\, \phi_\eta\rangle\!\rangle = \int_{E^*_{\mathbf{C}}} \rho(z)\,\langle\!\langle Q_z\phi_\xi,\, \phi_\eta\rangle\!\rangle\; \nu(dz),$$

which means that Ξ admits a diagonal coherent state representation as in (25). The uniqueness follows from the fact that w is uniquely specified essentially by the symbol of Ξ. ∎

Corollary 4.4 (Resolution of the identity) *It holds that*

$$I = \int_{E^*_{\mathbf{C}}} Q_z\,\nu(dz).$$

For the proof we need only to compute the symbols of both sides. We refer to [13] for the prototype of the above formula and various developments. Some applications of Corollary 4.4 are found in [20], [21].

5 Wick Product and Overcompleteness of Coherent Vectors

5.1 Integral Kernel Operators With each $\kappa_{l,m}\in(E_{\mathbf{C}}^{\otimes(l+m)})^*$ we associate a white noise operator by a formal integral expression:

$$\Xi_{l,m}(\kappa_{l,m}) =$$
$$= \int_{\mathbf{R}^{l+m}} \kappa_{l,m}(s_1,\cdots,s_l,t_1,\cdots,t_m)a^*_{s_1}\cdots a^*_{s_l}a_{t_1}\cdots a_{t_m}ds_1\cdots ds_l dt_1\cdots dt_m. \tag{28}$$

This is called an *integral kernel operator.* The precise definition is as follows, for more details see [3], [18, Chapter 4]. Let $\phi=(f_n)\in\mathcal{W}$. Then $\Xi_{l,m}(\kappa_{l,m})\phi=(g_n)$ is defined by

$$g_n=0,\quad 0\le n<l;\qquad g_n=\frac{(n-l+m)!}{(n-l)!}\,\kappa\otimes_m f_{n-l+m},\quad n\ge l,$$

where $\otimes_m$ denotes the right contraction of tensor products. It is known that for any $\kappa_{l,m}\in(E_{\mathbf{C}}^{\otimes(l+m)})^*$, the integral kernel operator $\Xi_{l,m}(\kappa_{l,m})$ always belongs to $\mathcal{L}(\mathcal{W},\mathcal{W}^*)$. The symbol is easily obtained:

$$\Xi_{l,m}(\kappa)\widehat{\ }(\xi,\eta) = \left\langle\kappa,\, \eta^{\otimes l}\otimes\xi^{\otimes m}\right\rangle\, e^{\langle\xi,\,\eta\rangle}.$$

Moreover, it is proved [3] that any $\Xi\in\mathcal{L}(\mathcal{W},\mathcal{W}^*)$ admits an infinite series expansion:

$$\Xi = \sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m}), \tag{29}$$

where the right hand side converges in $\mathcal{L}(\mathcal{W},\mathcal{W}^*)$.

Lemma 5.1 *For $F \in (E_{\mathbf{C}}^{\otimes m})^*$ we have*

$$\Xi_{0,m}(F) = \int_{E_{\mathbf{C}}^*} \langle z^{\otimes m}, F \rangle Q_z \, \nu(dz), \qquad \Xi_{m,0}(F) = \int_{E_{\mathbf{C}}^*} \langle \bar{z}^{\otimes m}, F \rangle Q_z \, \nu(dz). \tag{30}$$

PROOF. We put $\rho(z) = \langle z^{\otimes m}, F \rangle$. As is easily verified by definition, $\rho \in \mathcal{D}^*$ and a white noise operator

$$\Xi = \int_{E_{\mathbf{C}}^*} \langle z^{\otimes m}, F \rangle Q_z \, \nu(dz)$$

is defined. Then the operator symbol is given by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \rho, q_{\xi,\eta} \rangle\rangle = \int_{E_{\mathbf{C}}^*} \langle z^{\otimes m}, F \rangle \, e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle} \, \nu(dz).$$

Then using the orthogonal relation (9), we obtain with no difficulty

$$\widehat{\Xi}(\xi, \eta) = \langle F, \xi^{\otimes m} \rangle \, e^{\langle \xi, \eta \rangle} = \Xi_{0,m}(F)\widehat{\ }(\xi, \eta).$$

The second identity in (30) is obtained by duality. ∎

For $F = \delta_t \in E^*$ we write naturally $z(t) = \langle z, \delta_t \rangle$. Then $\{z(t)\}$ is the complex white noise [10]. As a special case of Lemma 5.1, we obtain the diagonal coherent state representation of the quantum white noise:

$$a_t = \int_{E_{\mathbf{C}}^*} z(t) \, Q_z \nu(dz), \qquad a_t^* = \int_{E_{\mathbf{C}}^*} \overline{z(t)} \, Q_z \nu(dz).$$

5.2 Wick Product of White Noise Functions For $\Phi_1, \Phi_2 \in \mathcal{W}^*$ there exists a unique $\Psi \in \mathcal{W}^*$ such that $S\Psi(\xi) = S\Phi_1(\xi) \cdot S\Phi_2(\xi)$ for $\xi \in E_{\mathbf{C}}$. The verification is simple with the help of Theorem 3.1. In that case we write $\Psi = \Phi_1 \diamond \Phi_2$.

Lemma 5.2 *For $\Phi \in \mathcal{W}^*$ fixed, the map $W_\Phi : \phi \mapsto \Phi \diamond \phi$ is a continuous linear operator from $\mathcal{W}$ into $\mathcal{W}^*$.*

PROOF. We compute the symbol.

$$\begin{aligned} \langle\langle W_\Phi \phi_\xi, \phi_\eta \rangle\rangle &= \langle\langle \Phi \diamond \phi_\xi, \phi_\eta \rangle\rangle = S(\Phi \diamond \phi_\xi)(\eta) \\ &= S\Phi(\eta) \cdot S\phi_\xi(\eta) = \langle\langle \Phi, \phi_\eta \rangle\rangle \, e^{\langle \xi, \eta \rangle}. \end{aligned} \tag{31}$$

Choosing $p \geq 0$ such that $\| \Phi \|_{-p,-} < \infty$, we come to

$$\begin{aligned} |\langle\langle W_\Phi \phi_\xi, \phi_\eta \rangle\rangle|^2 &\leq \| \Phi \|_{-p,-}^2 \, \| \phi_\eta \|_{p,+}^2 \, e^{2|\langle \xi, \eta \rangle|} \\ &\leq \| \Phi \|_{-p,-}^2 \, G_\alpha(|\eta|_p^2) \, e^{|\xi|_p^2 + |\eta|_p^2}, \end{aligned} \tag{32}$$

where we used a simple inequality: $2|\langle \xi, \eta \rangle| \leq |\xi|_0^2 + |\eta|_0^2 \leq |\xi|_p^2 + |\eta|_p^2$. Now, with the help of Proposition 1.1 we see that (32) becomes

$$|\langle\langle W_\Phi \phi_\xi, \phi_\eta \rangle\rangle|^2 \leq \| \Phi \|_{-p,-}^2 \, G_\alpha(2|\eta|_p^2) G_\alpha(|\xi|_p^2) \leq \| \Phi \|_{-p,-}^2 \, G_\alpha(|\eta|_{p+q}^2) G_\alpha(|\xi|_{p+q}^2),$$

where $q \geq 0$ is taken in such a way that $2\,|\,\eta\,|_p^2 \leq 2\rho^{2q}\,|\,\eta\,|_{p+q}^2 \leq |\,\eta\,|_{p+q}^2$. Then the assertion follows from the characterization theorem for operator symbols (Theorem 4.1). ∎

The operator $W_\Phi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ defined in Lemma 5.2 is called the *Wick multiplication operator* associated with $\Phi \in \mathcal{W}^*$.

Theorem 5.3 *For any $\Phi \in \mathcal{W}^*$,*

$$W_\Phi = \int_{E_{\mathbf{C}}^*} S\Phi(\bar{z}) Q_z\, \nu(dz). \tag{33}$$

PROOF. Let $\Phi = (F_m)$. Then

$$S\Phi(\bar{z}) = \sum_{m=0}^{\infty} \langle \bar{z}^{\otimes m},\, F_m \rangle$$

as an element of $\mathcal{D}^*$, see also the proof of Theorem 3.4. Consider the diagonal coherent state representation:

$$\Xi = \int_{E_{\mathbf{C}}^*} S\Phi(\bar{z}) Q_z\, \nu(dz).$$

It then follows from Lemma 5.1 that

$$\Xi = \sum_{m=0}^{\infty} \Xi_{m,0}(F_m),$$

and therefore,

$$\widehat{\Xi}(\xi, \eta) = \sum_{m=0}^{\infty} \langle F_m,\, \eta^{\otimes m} \rangle\, e^{\langle \xi,\, \eta \rangle} = S\Phi(\eta) e^{\langle \xi,\, \eta \rangle} = \langle\!\langle \Phi,\, \phi_\eta \rangle\!\rangle\, e^{\langle \xi,\, \eta \rangle}.$$

This coincides with (31) and hence $\Xi = W_\Phi$ as desired. ∎

The inversion formula for the S-transform is obtained also from Theorem 5.3 and a simple relation: $W_\Phi \phi_0 = \Phi$.

5.3 Overcompleteness of Coherent Vectors Let $\mathcal{D}_{\mathrm{AH}}^\perp$ denote the space of all $\rho \in \mathcal{D}^*$ which annihilate $\mathcal{D}_{\mathrm{AH}}$, that is, all $\rho \in \mathcal{D}^*$ such that $\langle\!\langle \rho,\, \phi \rangle\!\rangle = 0$ for all $\phi \in \mathcal{D}_{\mathrm{HOL}} \equiv \mathcal{D} \cap L^2(E_{\mathbf{C}}^*, \nu)_{\mathrm{HOL}}$. Recall that $\langle\!\langle \cdot,\, \cdot \rangle\!\rangle$ is a $\mathbf{C}$-bilinear form by our convention.

Theorem 5.4 *For $\rho \in \mathcal{D}^*$ put*

$$\Phi = \int_{E_{\mathbf{C}}^*} \rho(z)\phi_z\, \nu(dz), \qquad \Xi = \int_{E_{\mathbf{C}}^*} \rho(z) Q_z\, \nu(dz).$$

Then the following four conditions are equivalent:

(i) $\rho \in \mathcal{D}_{\mathrm{AH}}^\perp$;

(ii) $\Phi = 0$;

(iii) Ξ *annihilates the vacuum:* $\Xi\phi_0 = 0$;

(iv) Ξ *is a quantum stochastic integral against the annihilation process, i.e., there exists* $L \in E_{\mathbf{C}}^* \otimes \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ *such that*

$$\Xi = \int_{\mathbf{R}} L(t) a_t \, dt.$$

PROOF. (i) $\Longrightarrow$ (ii). By Lemma 3.3,

$$S\Phi(\xi) = \langle\!\langle \rho, \epsilon_\xi \rangle\!\rangle, \qquad \xi \in E_{\mathbf{C}}.$$

Since $\epsilon_\xi \in \mathcal{D}_{\mathrm{HOL}}$, the assertion follows immediately.

(ii) $\Longrightarrow$ (i). It is easily verified that $t \mapsto \langle\!\langle \rho, \epsilon_{t\xi} \rangle\!\rangle$ is entire holomorphic on $\mathbf{C}$. Then from $0 = S\Phi(t\xi) = \langle\!\langle \rho, \epsilon_{t\xi} \rangle\!\rangle$ for all $z \in \mathbf{C}$ it follows that

$$\langle\!\langle \rho, \omega_m \rangle\!\rangle = 0, \qquad \omega_m(z) = \left\langle z^{\otimes m}, \xi^{\otimes m} \right\rangle.$$

This being valid for all $\xi \in E_{\mathbf{C}}$, we conclude that $\rho \in \mathcal{D}_{\mathrm{AH}}^{\perp}$.

(ii) $\Longleftrightarrow$ (iii) is obvious from $Q_z \phi_0 = \phi_z$.

(iii) $\Longleftrightarrow$ (iv). Let $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$ be the expansion of the form (29). Then condition (iii) is equivalent to $\langle\!\langle \Xi\phi_0, \phi_\eta \rangle\!\rangle = \widehat{\Xi}(0,\eta) = 0$ for all $\eta \in E_{\mathbf{C}}$, that is,

$$\widehat{\Xi}(0,\eta) = \sum_{l=0}^{\infty} \left\langle \kappa_{l,0}, \eta^{\otimes l} \right\rangle = 0, \qquad \eta \in E_{\mathbf{C}}.$$

This is equivalent to $\kappa_{l,0} = 0$ for all $l \geq 0$. Therefore Ξ is a sum of integral kernel operators involving one or more annihilation operators. Then by a similar argument as in [19, Section 6] we see that such an operator is a quantum stochastic integral (in a broad sense) against the annihilation process. ∎

Corollary 5.5 *Let* $\Phi \in \mathcal{W}^*$. *Then for* $\rho \in \mathcal{D}^*$ *a coherent state representation:*

$$\Phi = \int_{E_{\mathbf{C}}^*} \rho(z) \phi_z \, \nu(dz)$$

holds if and only if

$$\rho = \rho_1 + \rho_2, \qquad \rho_1 \in \mathcal{D}_{\mathrm{AH}}^*, \quad \rho_2 \in \mathcal{D}_{\mathrm{AH}}^{\perp},$$

with

$$\rho_1(z) = S\Phi(\bar{z}).$$

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