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Recent Topics in Finite Difference Methods for Boundary Value Problems

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1 Introduction

Although finite difference method (FDM) is one of central numerical techniques for solving boundary value problems, it appears that the method has not so extensively been studied as compared with finite element method (FEM).

For example, consider the Swartztrauber-Sweet algorithm [14] for solving the Dirichlet problem

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right]+c(r,\theta)u=f(r,\theta),\quad 0<r<R,\ 0\le\theta<2\pi$$
$$u=g(\theta),\quad r=R,\ 0\le\theta<2\pi,$$

which is described as follows:

$$h=\Delta r=\frac{R}{m+1},\quad r_i=ih,\ i=0,\frac{1}{2},1,\dots,m+\frac{1}{2},m+1,$$
$$k=\Delta\theta=\frac{2\pi}{n},\quad \theta_j=jk,\ j=0,1,2,\dots,n-1,n$$
$$-\Big[\frac{1}{r_ih^2}\{r_{i+\frac{1}{2}}(U_{i+1\,j}-U_{ij})-r_{i-\frac{1}{2}}(U_{ij}-U_{i-1\,j})\}+\frac{1}{r_i^2k^2}(U_{i\,j+1}-2U_{ij}+U_{i\,j-1})\Big]$$
$$+c_{ij}U_{ij}=f_{ij},\quad i=1,2,\dots,m,\ j=0,1,2,\dots,n-1,$$
$$U_{in}=U_{i0}\ (\forall i),\quad U_{0j}=U_{00}\ (\forall j),\quad U_{m+1\,j}=g_j\ (\forall j),$$
$$(1+\frac{c_{00}}{4}h^2)U_{00}-\frac{1}{n}\sum_{j=0}^{n-1}U_{1j}=\frac{h^2}{4}f_{00},$$

where U_{ij} stand for the approximations at $P_{ij}=(r_i,\theta_j)$, $c_{ij}=c(r_i,\theta_j)$, $f_{ij}=f(r_i,\theta_j)$ and $g_j=g(\theta_j)$.

Then a question arises: Does it converge at a neighbor of the origin? The algorithm was proposed in 1973 for the case $c=0$ with no convergence analysis. In 1986, Strikwerda-Nagel [13] remarked in that case $(c=0)$ that if $u\in C^4(\overline{\Omega})$, then the local truncation error τ_{00} at the origin was $\mathrm{O}(h^4)+\mathrm{O}(k^4)$ and showed by numerical experiment that the scheme had the second order accuracy at the origin. However, no proof was given there. In 1998 the author proved its convergence, and published joint papers [10,11] with N. Matsunaga, where not only the convergence but also a superconvergence property of FDM is proved for Dirichlet problems.

Since then, the author has establised several new results on FDM together with his colleagues and students (cf. [2, 3, 5, 6, 11, 15–19]). In this paper, we shall review those results.

2 Superconvergence and Nonsuperconvergence of FD Solutions

Let Ω be a bounded domain of $\mathbb{R}^2$ and consider the boundary value problem

$$-\Delta u + \boldsymbol{b}(x,y) \cdot \nabla u + c(x,y)u = f(x,y) \quad \text{in } \Omega \tag{2.1}$$
$$u = g(x,y) \quad \text{on } \Gamma = \partial\Omega, \tag{2.2}$$

where $\boldsymbol{b} = (b_1(x,y), b_2(x,y))$ is bounded in $\overline{\Omega} = \Omega \cup \Gamma$.

We construct a net over $\overline{\Omega}$ by the grid points $P_{ij} = (x_i, y_j)$ in $\overline{\Omega}$ with the equal mesh size h in the x and y directions. We denote by Ω_h and Γ_h the set of grid points in Ω and the set of points of intersection of grid lines with Γ. Let $\hat{\Gamma}$ be a part or the whole of Γ and K a constant with $K > 1$ (say $K = 2, 5, 10$, etc.), which is arbitrarily chosen independently of h. We define

$$\mathscr{S}_h(K, \hat{\Gamma}) = \{P \in \Omega_h \mid \operatorname{dist}(P, \hat{\Gamma}) \leq Kh\}.$$

If $\hat{\Gamma} = \Gamma$, then we write $\mathscr{S}_h(K)$ in place of $\mathscr{S}_h(K, \Gamma)$. Furthermore, we define the neighbors of $P \in \Omega_h$ to be four points in $\overline{\Omega}_h = \Omega_h \cup \Gamma_h$ on horizontal and vertical grid lines through P. These points are denoted by P_E, P_W, P_S, P_N and their distances to P by h_E, h_W, h_S, h_N, respectively (cf. Figs. 1 and 2). We denote by $U(P)$ the approximate solution to $u(P)$ at $P \in \Omega_h$. Then the Shortley-Weller (S-W) formula

$$\begin{aligned} -\Delta_h u(P) \equiv {} & \left(\frac{2}{h_E h_W} + \frac{2}{h_S h_N}\right) U(P) - \frac{2}{h_E(h_E + h_W)} U(P_E) \\ & - \frac{2}{h_W(h_E + h_W)} U(P_W) - \frac{2}{h_S(h_S + h_N)} U(P_S) \\ & - \frac{2}{h_N(h_S + h_N)} U(P_N) \end{aligned} \tag{2.3}$$

is used to approximate $-\Delta u(P)$. The term $\boldsymbol{b}(P) \cdot \nabla u(P)$ is approximated by

$$b_1(P) \frac{u(P_E) - u(P_W)}{h_E + h_W} + b_2(P) \frac{u(P_N) - u(P_S)}{h_N + h_S}. \tag{2.4}$$

Then the problem (2.1)–(2.2) is discretized by

$$\begin{aligned} \mathscr{L}_h U(P) &= f(P), \quad P \in \Omega_h, \\ U(P) &= g(P), \quad P \in \Gamma_h, \end{aligned}$$

where

$$
\begin{aligned}
\mathscr{L}_h U(P) = &\left(\frac{2}{h_E h_W} + \frac{2}{h_S h_N} + c(P)\right) U(P) \\
&- \frac{1}{h_E(h_E + h_W)}\{2 - h_E b_1(P)\}U(P_E) \\
&- \frac{1}{h_W(h_E + h_W)}\{2 + h_W b_1(P)\}U(P_W) \\
&- \frac{1}{h_N(h_S + h_N)}\{2 - h_N b_2(P)\}U(P_N) \\
&- \frac{1}{h_S(h_S + h_N)}\{2 + h_S b_2(P)\}U(P_S).
\end{aligned} \tag{2.5}
$$

This leads to a system of linear equations

$$A\boldsymbol{U} = \tilde{\boldsymbol{f}},$$

with respect to the unknown vector $\boldsymbol{U} = (U(P))$, $P \in \Omega_h$, where h is sufficiently small so as to satisfy

$$\sup_{P \in \Omega} h|b_i(P)| < 2, \quad i = 1, 2,$$

so that A is an irreducibly diagonally dominant L-matrix (hence, A is an M-matrix). The vector $\tilde{\boldsymbol{f}}$ is determined by $f(P)$ and the boundary condition (2.2).

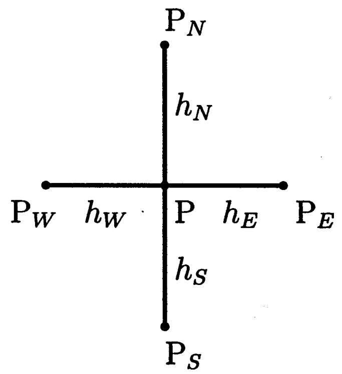

Fig. 1.

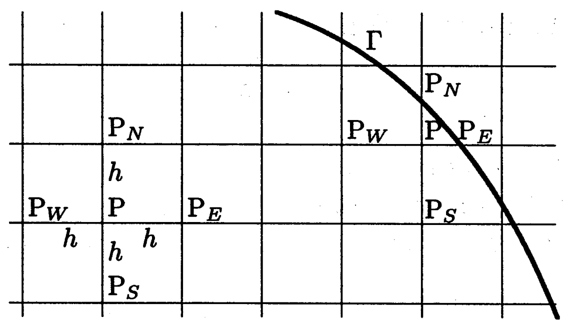

Fig. 2.

If $u \in C^4(\overline{\Omega})$, then the local truncation error $\tau(P)$ for $\mathscr{L}_h$ is given (cf. [12]) by

$$
\begin{aligned}
\tau(P) &\equiv \mathscr{L}_h u(P) - f \\
&= \mathscr{L}_h u(P) - \mathscr{L} u(P) \\
&= (h_E - h_W)\big[\frac{1}{2} b_1(P) u_{xx}(P) + \frac{1}{3} u_{xxx}(P)\big] \\
&\quad + (h_N - h_S)\big[\frac{1}{2} b_2(P) u_{yy}(P) + \frac{1}{3} u_{yyy}(P)\big] \\
&\quad + \frac{1}{6}\frac{1}{h_E + h_W}\big[h_E^3\{b_1(P) u_{xxx}(Q_E) + \frac{1}{2} u_{xxxx}(Q_E)\} \\
&\qquad + h_W^3\{b_1(P) u_{xxx}(Q_W) + \frac{1}{2} u_{xxxx}(Q_W)\}\big] \\
&\quad + \frac{1}{6}\frac{1}{h_S + h_N}\big[h_S^3\{b_2(P) u_{yyy}(Q_S) + \frac{1}{2} u_{yyyy}(Q_S)\} \\
&\qquad + h_N^3\{b_2(P) u_{yyy}(Q_N) + \frac{1}{2} u_{yyyy}(Q_N)\}\big],
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
&Q_E = (x + \theta h_E, y),\ Q_W = (x - \theta h_W, y), \\
&Q_N = (x, y + \theta h_N),\ Q_S = (x, y - \theta h_S), \quad 0 < \theta < 1.
\end{aligned}
$$

We thus obtain

$$
\tau(P) = \begin{cases} \mathrm{O}(h^2) & \text{if } h_E = h_W = h_S = h_N = h \\ \mathrm{O}(h) & \text{otherwise.} \end{cases} \tag{2.7}
$$

Then, the Bramble-Hubbard result [1] asserts that

$$
u(P) - U(P) = \mathrm{O}(h^2) \quad \forall P \in \Omega_h, \tag{2.8}
$$

even if a grid point P exists such that $(h_E, h_W, h_S, h_N) \neq (h, h, h, h)$. A matrix theoretic proof of this result can be found in Gorenflo [7], Meis-Marcowitz [12], Hackbusch [8], etc. Recently, Matsunaga-Yamamoto [11] further sharpened (2.8) as

$$
u(P) - U(P) = \mathrm{O}(h^3) \quad \forall P \in \mathscr{S}_h(K),
$$

for the case $\boldsymbol{b} = \mathbf{0}$. The same proof can be used to derive the following:

Theorem 2.1. *Let $u \in C^4(\overline{\Omega})$ be the solution of* (2.1)–(2.2). *Then*

$$
|u(P) - U(P)| = \begin{cases} \mathrm{O}(h^3) & P \in \mathscr{S}_h(K) \\ \mathrm{O}(h^2) & \textit{otherwise.} \end{cases} \tag{2.9}
$$

Similar results have been obtained for nonsmooth Dirichlet problem

$$
\begin{aligned}
-\Delta u + \max(0, q(u)) &= f(x, y) \quad \text{in } \Omega &(2.10)\\
u &= g(x, y) \quad \text{on } \Gamma &(2.11)
\end{aligned}
$$

where f, g are given functions and q is a continuously differentiable function with $q'(u) \geq 0$, provided that (2.10)–(2.11) has a solution $u \in C^2(\overline{\Omega})$ (cf. Chen-Matsunaga-Yamamoto [2]). For convection-diffusion problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \operatorname{div}\{-\kappa(x,y)\nabla u + u\boldsymbol{a}\} &= f(x,y) && \text{in } \Omega \times (0,T), && (2.12)\\
\frac{\partial u}{\partial n} &= \varphi(x,y,t) && \text{on } \Gamma_1, && (2.13)\\
u &= \psi(x,y,t) && \text{on } \Gamma_2, && (2.14)\\
u(x,y,0) &= u^0(x,y) && \text{in } \Omega, && (2.15)
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^2$ wiht the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, we assume that $\kappa \in C^{1,\alpha}(\overline{\Omega})$, $\boldsymbol{a} = (a^1(x,y), a^2(x,y))$ with $a^i \in C^{1,\alpha}(\overline{\Omega})$ and $\operatorname{div}(\boldsymbol{a}) \geq 0$, $f \in C^{\alpha}(\overline{\Omega})$, $u^0(x,y) \in C^{2,\alpha}(\overline{\Omega})$ and that there exists a positive constant κ_0 such that $\kappa(x,y) > \kappa_0$ in Ω. It is then known that there exists a unique solution $u(x,y,t)$ of (2.10)–(2.11) with $u \in C^{2+\alpha,1}(Q_T)$, where $Q_T = \Omega \times [0,T]$ and that $u \in C^{4,1}(Q_T)$ if κ, $\boldsymbol{a}$, f and the boundary value are sufficiently smooth as well as the boundary. Furthermore, we assume that Γ_1 is paralleled to x-axis or y-axis. Then the following result is shown in Fang-Yamamoto [6]:

Theorem 2.2. *To solve* (2.12)–(2.15), *apply the implicit scheme corresponding to* (2.12)–(2.15)

$$
\begin{aligned}
\frac{U_i^k - U_i^{k-1}}{\Delta t} + L_h U_i^k &= f_i, && (2.16)\\
D_n U_i^k &= \varphi_i^k \ \textit{on } \Gamma_h^1 && (2.17)\\
U_i^k &= \psi_i^k \ \textit{on } \Gamma_h^2 && (2.18)\\
U_i^0 &= u_i^0, \quad 1 \leq i \leq N && (2.19)
\end{aligned}
$$

where Δt is the increment of time t, $f_i = f(P_i)$, $\varphi_i^k = \varphi(P_i, k\Delta t)$, $\psi_i^k = \psi(P_i, k\Delta t)$, and $u_i^0 = u^0(P_i)$. L_h is the usual discretization of the operator L defined by

$$Lu = \operatorname{div}\{-\kappa(x,y)\nabla u + u\boldsymbol{a}\}.$$

D_n is the discretization of Neumann boundary condition by the method which uses the line of the fictitious nodes. Then (2.16)–(2.19) *can be written in the matrix-vector form*

$$(I + \Delta t A)\boldsymbol{U}^k = \boldsymbol{U}^{k-1} + \tilde{\boldsymbol{f}} \tag{2.20}$$

$$(\boldsymbol{U}^k = (U_1^k, \ldots, U_N^k)^t, \ A \text{ is an } N \times N \text{ matrix}).$$

If $u \in C^{4,1}(Q_T)$, then

$$|u_i^k - U_i^k| \leq |u_i^0 - U_i^0| + \begin{cases} O((\Delta t + h)h), & P_i \in \mathscr{S}_h(K, \Gamma^2) \\ O(\Delta t + h), & \textit{otherwise.} \end{cases} \tag{2.21}$$

To prove the theorem, the estimate

$$(I + \Delta t A)^{-k}\boldsymbol{v} \leq \boldsymbol{v} \quad \forall \boldsymbol{v} \in \mathbb{R}^N, \ \boldsymbol{v} \geq \boldsymbol{0} \tag{2.22}$$

was used. However, in a workshop held in February 21–22, 2001 at Ehime University, I. Marek of Charlse University was pointed out that (2.22) was not true. In fact, (2.22) should be corrected to

$$(I + \Delta t A)^{-k} \boldsymbol{v} \le \|\boldsymbol{v}\|_\infty \boldsymbol{e} \quad \forall \boldsymbol{v} \ge \boldsymbol{0},$$

where $\boldsymbol{e} = (1, \dots, 1)^t \in \mathbb{R}^N$. Therefore in (2.21), $|u_i^0 - U_i^0|$ should be read for $\max_j |u_j^0 - U_j^0|$, so that we have a corrected estimate

$$|u_i^k - U_i^k| \le \max_j |u_j^0 - U_j^0| + \begin{cases} \mathrm{O}((\Delta t + h)h), & P_i \in \mathscr{S}_h(K, \Gamma^2) \\ \mathrm{O}(\Delta t + h), & \text{otherwise,} \end{cases} \tag{2.23}$$

in place of (2.21). The author are grateful to him. The property like (2.9), (2.23), etc. are called "superconvergence property".

More precisely, we define the superconvergence property for discretized solution $\{U(P)\}$ for (2.1)–(2.2) as

Definition 2.1 (Yamamoto-Fang-Chen [19]). We say that a discretized solution $\{U(P)\}$ has a superconvergence property near $\hat{\Gamma} \subseteq \Gamma$, if, for some constants $\sigma > 0$ and $K > 1$,

$$|u(P) - U(P)| = \begin{cases} \mathrm{O}(h^{\sigma+1}), & P_i \in \mathscr{S}_h(K, \hat{\Gamma}) \\ \mathrm{O}(h^{\sigma}), & \text{otherwise.} \end{cases}$$

Theorems 2.1 and 2.2 are proved under the assumptions $u \in C^4(\overline{\Omega})$ and $u \in C^{4,1}(Q_T = \overline{\Omega} \times [0, T])$, respectively. We are now interested in the case where $u \notin C^4(\overline{\Omega})$ for the S-W approximation to the problem (2.1)–(2.2). This case has been discussed in Yamamoto-Fang-Chen [19] for the centered five point FDM applied to the problem

$$\begin{gathered} -\Delta u = f \text{ in } \Omega = (0,1) \times (0,1), u = g \text{ on } \Gamma, \\ u \in C(\overline{\Omega}) \cap C^\infty(\Omega) \text{ but } u \notin C^4(\overline{\Omega}). \end{gathered} \tag{2.24}$$

It is shown that different situations occur: no superconvergence case near any $\hat{\Gamma} \subseteq \Gamma$, a superconvergence case near a side $\hat{\Gamma}$ of Γ, etc.

3 Convergence of Inconsistent Schemes

The results in Yamamoto-Fang-Chen [19] can be extended to a slightly general problem:

$$-\Delta u + c(x,y)u = f \quad \text{in } \Omega = (0,1) \times (0,1) \tag{3.1}$$

$$u = g(x,y) \quad \text{on } \Gamma, \tag{3.2}$$

where solution u belong to $C(\overline{\Omega}) \cap C^4(\Omega)$ and has singular derivatives near Γ such that

$$\sup_{x \in (0,1)} \frac{x^j (1-x)^j |\frac{\partial^j u}{\partial x^j}(x,y)|}{x^\alpha (1-x)^\beta} \le K_1 < \infty \tag{3.3}$$

$$\sup_{y \in (0,1)} \frac{y^j (1-y)^j |\frac{\partial^j u}{\partial y^j}(x,y)|}{y^\gamma (1-y)^\delta} \le K_2 < \infty, \quad j = 2,3,4 \tag{3.4}$$

with constants $\alpha, \beta, \gamma, \delta \in (0,2)$ and positive constants K_1, $K_2 > 0$ independent of x and y. We apply the centered five point formula

$$h = \frac{1}{n+1}, \quad x_i = ih,\ i = 0,1,2,\ldots,n+1, \quad y_j = jh,\ j = 0,1,2,\ldots,n+1$$

to solve (3.1)–(3.2). Then it is easy to see that

$$|\tau(P)| = \begin{cases} \mathrm{O}(h^{\min(\alpha,\beta,\gamma)-2}) & (\text{near } \Gamma_1 = \{(x,0) \mid 0 \le x \le 1\}) \\ \mathrm{O}(h^{\min(\beta,\gamma,\delta)-2}) & (\text{near } \Gamma_2 = \{(1,y) \mid 0 \le y \le 1\}) \\ \mathrm{O}(h^{\min(\alpha,\beta,\delta)-2}) & (\text{near } \Gamma_3 = \{(x,1) \mid 0 \le x \le 1\}) \\ \mathrm{O}(h^{\min(\alpha,\gamma,\delta)-2}) & (\text{near } \Gamma_4 = \{(0,y) \mid 0 \le y \le 1\}) \end{cases}$$
$$\to \infty$$

as $h \to 0$. However, we can prove the following:

Theorem 3.1 (Fang-Matsubara-Shogenji-Yamamoto [3]). *In addition to the conditions* (3.3)–(3.4) *we assume*

$$\sup_{\mathrm{dist}(P,Q)\le d} |u(P) - U(Q)| \le K_0 d^\sigma \quad \text{at } P, Q \text{ near } \Gamma, \tag{3.5}$$

where K_0 is a positive constant and $\sigma = \min(\alpha, \beta, \gamma, \delta)$. Then

$$|u(P) - U(P)| \le \mathrm{O}(h^\sigma) \quad \forall P \in \Omega_h.$$

4 Acceleration Techniques

We can improve the accuracy $\mathrm{O}(h^\sigma)$ in Theorem 3.1 by a coordinate transformation under the conditions (3.3)–(3.5). Let $\varphi(t)$ be the function defined by

$$\varphi(t) = c_\rho \int_0^t \{s(1-s)\}^\rho ds, \quad c_\rho = \left[\int_0^1 \{s(1-s)\}^\rho ds\right]^{-1},$$

where $\rho \ge 0$. Observe that $\varphi(t) = t$ if $\rho = 0$. We then put

$$h = \frac{1}{n+1}, \quad t_i = ih,$$
$$x_i = \varphi(t_i),\ y_j = \varphi(t_j), \quad i,j = 0,1,2,\ldots,n+1$$

and generate non-equidistant grid points $P_{ij} = (x_i, y_j)$. Then we can prove the following result:

Theorem 4.1 (Yamamoto [18]). *Under the conditions* (3.3)–(3.5), *apply the S-W approximation to the problem* (3.1)–(3.2). *Put $r = \sigma(\rho + 1)$. Then, at every $P \in \Omega_h$, we have*

$$|u(P) - U(P)| = \begin{cases} \mathrm{O}(h^r) & (r < 2) \\ \mathrm{O}(h^2 \log \frac{1}{h}) & (r = 2). \end{cases}$$

Furthermore, we have

$$|u(P) - U(P)| = \lambda(\rho) h^r + \mu(\rho) h^2,$$

where $\lambda(\rho)$ and $\mu(\rho)$ are increasing functions with polynomial orders as $\rho \to \infty$.

Another transformation

$$\psi(t) = \frac{\exp(at) - 1}{\exp(a) - 1}, \quad 0 \le t \le 1$$

is known as a stretching function, where a is a positive constant. We can also prove that with the constant σ defined as in Theorem 3.1

$$|u(P) - \hat{U}(P)| = \hat{\lambda}(a)h^{\sigma} + \hat{\mu}(a)h^2, \quad \forall P \in \Omega_h$$

where $\hat{\lambda}(a)$ and $\hat{\mu}(a)$ are monotonically decreasing and increasing functions, respectively, with exponential order as $a \to \infty$. Therefore, the stretching function ψ works by letting the parameter a large.

5 Unified Understanding of FDM, FEM, FVM for Two-point Boundary Value Problems

We can understand three methods FDM, FEM and FVM (finite volume method) through the simple two-point boundary value problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = f(x), \quad a < x < b \tag{5.1}$$

$$u(a) = u(b) = 0. \tag{5.2}$$

Let

$$a = x_0 < x_1 < \cdots < x_i < \cdots < x_{n+1} = b, \quad h_i = x_i - x_{i-1}, \tag{5.3}$$

$$h = \max_i h_i, \quad x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1}) \tag{5.4}$$

and discretize (5.1)–(5.2) with the use of three methods:

(i) FDM

$$-\frac{p_{i+\frac{1}{2}}\frac{(U_{i+1}-U_i)}{h_{i+1}} - p_{i-\frac{1}{2}}\frac{(U_i-U_{i-1})}{h_i}}{\frac{h_{i+1}+h_i}{2}} = f_i, \quad i = 1, 2, \ldots, n \tag{5.5}$$

(ii) FEM
The FE approximation $v_h = \sum_{i=1}^n \hat{U}_i\varphi_i(x)$ with piecewise linear polynomials is determined by solving

$$\sum_{j=1}^n \left(\int_a^b p(x)\varphi_i'\varphi_j'(x)dx\right)\hat{U}_j = \int_a^b f(x)\varphi_i(x), \quad i = 1, 2, \ldots, n \tag{5.6}$$

with respect to $\{\hat{U}_j\}$.

(iii) FVM

The FV approximation $w_h(x) = \sum_{i=1}^{n} \bar{U}_i\varphi_i$ is obtained (cf. Li-Chen-Wu [9]) by solving the linear system

$$\sum_{j=1}^{n}\left(\int_a^b p(x)\varphi_j'(x)\psi_i'\right)\bar{U}_j = \int_a^b f(x)\psi_i(x)dx, \quad i = 1, 2, \ldots, n \tag{5.7}$$

with respect to $\{\bar{U}_j\}$, where

$$\psi_i(x) = \begin{cases} 1 & (x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}) \\ 0 & (\text{otherwise}). \end{cases}$$

Then (5.5)–(5.7) can be written in the tridiagonal linear systems

$$A\boldsymbol{U} = \boldsymbol{f}, \quad \hat{A}\hat{\boldsymbol{U}} = \hat{\boldsymbol{f}}, \quad A\bar{\boldsymbol{U}} = \bar{\boldsymbol{f}},$$

or

$$\boldsymbol{U} = A^{-1}\boldsymbol{f}, \quad \hat{\boldsymbol{U}} = \hat{A}^{-1}\hat{\boldsymbol{f}}, \quad \bar{\boldsymbol{U}} = A^{-1}\bar{\boldsymbol{f}},$$

where $\boldsymbol{f} = (f_1, \ldots, f_n)^t$, $\hat{\boldsymbol{f}} = (\hat{f}_1, \ldots, \hat{f}_n)^t$, $\bar{\boldsymbol{f}} = (\bar{f}_1, \ldots, \bar{f}_n)^t$ with

$$f_i = f(x_i), \quad \hat{f}_i = \int_a^b f(x)\varphi_i(x)dx, \quad \bar{f}_i = \int_a^b f(x)\psi_i(x)dx,$$

$A = HA_0$,

$$A_0 = \begin{pmatrix} a_1 + a_2 & -a_2 & & & \\ -a_2 & a_2 + a_3 & -a_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{n-1} & a_{n-1} + a_n & -a_n \\ & & & -a_n & a_n + a_{n+1} \end{pmatrix}, \quad a_i = \frac{1}{h_i}p_{i-\frac{1}{2}}$$

$$H = \operatorname{diag}\left\{\frac{2}{h_1 + h_2}, \ldots, \frac{2}{h_n + h_{n+1}}\right\},$$

and $\hat{A} = \hat{A}_0$ is obtained by putting $H = I$ in the expression $A = HA_0$ and replacing the elements a_i of A_0 by

$$\hat{a}_i = \frac{1}{h_i}\hat{p}_i, \qquad \hat{p}_i = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} p(x)dx.$$

It was shown in Yamamoto [17] that the matrix $A^{-1} = (g_{ij})$ is given by

$$g_{ij} = \begin{cases} \left(\sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right)^{-1} \left(\sum_{k=1}^{i} \frac{h_k}{p_{k-\frac{1}{2}}}\right) \left(\sum_{k=j+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right) & (i \leq j) \\ \left(\sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right)^{-1} \left(\sum_{k=1}^{j} \frac{h_k}{p_{k-\frac{1}{2}}}\right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right) & (i \geq j). \end{cases} \tag{5.8}$$

The element $\hat{g}_{ij}$ of the matrix $\hat{A}^{-1} = (\hat{g}_{ij})$ are obtained by replacing $p_{k-\frac{1}{2}}$ in (5.8) by

$$\hat{p}_{k-\frac{1}{2}} = \frac{1}{h_{k-\frac{1}{2}}} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} p(x)dx.$$

Let $G(x, \xi)$ be the Green function for the problem (5.1)–(5.2). Denoting $G(x_i, x_j)$ by G_{ij} and noting that

$$u_i = \int_a^b G(x_i, \xi) f(\xi) d\xi \tag{5.9}$$

we can conclude the following (cf. Fang-Tsuchiya-Yamamoto [5]):

(a) $U_i = \frac{1}{2}\left(\sum_{j=1}^n g_{ij} f_j h_j + \sum_{j=1}^n g_{ij} f_j h_{j+1}\right)$ (the mean of two Riemann's sums)

$$u_i - U_i = \begin{cases} \mathrm{o}(h) & (p \in C^1[a,b]) \\ \mathrm{O}(h^2) & (p \in C^{1,1}[a,b]) \end{cases} \quad f \in C^{1,1}[a,b]$$

(b) $\hat{U}_i = \int_a^b \left(\sum_{j=1}^n \hat{g}_{ij} \varphi_j(x)\right) f(x) dx$

$$u_i - \hat{U}_i = \begin{cases} \mathrm{o}(h) & (p \in C^1[a,b]) \\ \mathrm{O}(h^2) & (p \in C^{1,1}[a,b]) \end{cases} \quad f \in C[a,b]$$

(c) $\bar{U}_i = \int_a^b \left(\sum_{j=1}^n g_{ij} \psi_j(x)\right) f(x) dx$

$$u_i - \bar{U}_i = \begin{cases} \mathrm{o}(h) & (p \in C^1[a,b]) \\ \mathrm{O}(h^2) & (p \in C^{1,1}[a,b]) \end{cases} \quad f \in C^{0,1}[a,b].$$

Numerical experiments show that there is no remarkable difference among the accuracy of three methods if f is sufficiently smooth (i.e., $f \in C^{1,1}[a,b]$). However, the above results show that FEM has a slight advantage over other methods, especially over FDM if $f \notin C^{1,1}[a,b]$. Finally we remark that numerical experiments by Q. Fang showed that FDM with nodes (5.3)–(5.4) applied to the problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x), \quad a < x < b \tag{5.10}$$

$$u(a) = \alpha, \quad u(b) = \beta \tag{5.11}$$

has also the $\mathrm{O}(h^2)$ accuracy, provided that p, q and f are sufficiently smooth (cf. [4,5]). This suggests a possibility of generalizing the error estimates in (a)–(c) to (5.10)–(5.11).

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