数理解析研究所講究録
1198 巻 2001 年 21-27

Laplace 方程式一般境界値問題の直接近似解法

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We propose in this paper a unified treatment of conventional boundary value problem, the Cauchy problem, and under- or over-determined problems of the Laplace equation in two-dimensional domain enclosed by the smooth curve. The Dirichlet data can be prescribed on any part of the boundary, while the Neumann data can be prescribed on any other part of the boundary. This problem is reformulated in terms of the variational problem with a least-square functional, which is then recast into primary and adjoint boundary value problems of the Laplace equation. A non-iterative numerical method of solution using the BEM is presented. Numerical examples suggest that our treatment is effective.

Key Words: *Inverse problem, Boundary value identification, Direct method*

1. Introduction

Let Ω be a simply connected bounded domain with its smooth boundary Γ in $\boldsymbol{R}^2$. Let $\boldsymbol{n}$ be the exterior normal to the boundary.

We consider the Laplace equation;

$$-\Delta u(\boldsymbol{x})=0\,,\quad \boldsymbol{x}\in\Omega \tag{1}$$

subject to Dirichlet and Neumann data;

$$u|_{\Gamma_u}=\bar{u}\quad\text{and}\quad \frac{\partial u}{\partial n}=q|_{\Gamma_q}=\bar{q} \tag{2}$$

given on respective non-zero measure parts of the boundary Γ_u and Γ_q. Here we notice that the components Γ_u and Γ_q can be taken arbitrarily to some extent. This problem setting encompasses the conventional mixed boundary value problem, the Cauchy problem, under- and over-determined problems of the Laplace equation. From this reason we call the problem the general or inverse boundary value problem.

If the solution of the problem eqns (1), (2) exists, the solution u at internal points of the domain can be expressed by Green's formula;

$$u(\boldsymbol{\xi})=\int_\Gamma G(\boldsymbol{x};\boldsymbol{\xi})q(\boldsymbol{x})d\Gamma(\boldsymbol{x}) -\int_\Gamma \frac{\partial G}{\partial n}(\boldsymbol{x};\boldsymbol{\xi})u(\boldsymbol{x})d\Gamma(\boldsymbol{x})\,,\quad \boldsymbol{\xi}\in\Omega \tag{3}$$

where $G(\boldsymbol{x};\boldsymbol{\xi})$ is the fundamental solution to the Laplacian;

$$-\Delta G(\boldsymbol{x};\boldsymbol{\xi})=\delta(\boldsymbol{x}-\boldsymbol{\xi}) \tag{4}$$

with the Dirac measure δ at the point $\boldsymbol{\xi}$. In two dimensions we know

$$G(\boldsymbol{x};\boldsymbol{\xi})=\frac{1}{2\pi}\ln\frac{1}{\|\,\boldsymbol{x}-\boldsymbol{\xi}\,\|}\,. \tag{5}$$

The boundary values $u|_\Gamma$ and $q|_\Gamma$ should satisfy the boundary integral equation;

$$\frac{1}{2}u(\boldsymbol{\xi})+\int_\Gamma \frac{\partial G}{\partial n}(\boldsymbol{x};\boldsymbol{\xi})u(\boldsymbol{x})d\Gamma(\boldsymbol{x}) =\int_\Gamma G(\boldsymbol{x};\boldsymbol{\xi})q(\boldsymbol{x})d\Gamma(\boldsymbol{x})\,,\quad \boldsymbol{\xi}\in\Gamma\,. \tag{6}$$

In preceding papers(1),(2) the authors presented an iterative method for numerical solution of the problem eqns (1), (2). However, our problem is essentially linear. The authors feel that linear problems should be solved in principle without iteration. In this paper an attempt is presented at an approximate solution of the problem using the boundary element method without the iteration.

2. Variational Problem

Let Γ_u^c and Γ_q^c be complement sets of Γ_u and Γ_q, respectively. We recast the problem eqns (1), (2) into the following variational problem: Find $u|_{\Gamma_u^c}=\omega$ that minimizes the functional

$$J(\omega)=\int_{\Gamma_q}|q(\boldsymbol{x};\omega)-\bar{q}(\boldsymbol{x})|^2d\Gamma(\boldsymbol{x}) +\eta\int_\Gamma|q(\boldsymbol{x};\omega)|^2d\Gamma(\boldsymbol{x}) \tag{7}$$

subject to

$$-\Delta u(\boldsymbol{x};\omega)=0\,,\quad \boldsymbol{x}\in\Omega \tag{8}$$

$$u|_{\Gamma_u}=\bar{u}\quad\text{and}\quad u|_{\Gamma_u^c}=\omega\,. \tag{9}$$

The second term on the right hand side of eqn (7) is the Tikhonov regularizer with the regularization parameter $\eta>0$ in order to make the problem well-posed. Here we assume $J:H^{1/2}(\Gamma_u^c)\ni\omega\longmapsto \boldsymbol{R}_+=[0,+\infty)$.

We discuss some mathematical questions about the existence and the uniqueness of the solution ω of the variational problem in which the functional $J(\omega)$ attains its minimum. The first theorem states that our under-determined problem is quasi-controlable [(3)].

Theorem 1 The convex set

$$\left\{ q(\omega) = \left.\frac{\partial u}{\partial n}\right|_{\Gamma_u^c} \;/\; \Delta u = 0 \quad \text{in} \quad \Omega, \qquad u \in H^{1/2}(\Gamma) \right.$$

$$\left. \text{s.t.} \quad u|_{\Gamma_u} = 0, \qquad u|_{\Gamma_u^c} = \omega \in H^{1/2}(\Gamma_u^c) \right\}$$

is dense in $H^{-1/2}(\Gamma_u^c)$.

Proof We consider a bounded linear operator K by definition:

$$K : H^{1/2}(\Gamma_u^c) \ni \omega \mapsto \frac{\partial u}{\partial n}(x;\omega) \in H^{-1/2}(\Gamma_u^c)\,.$$

In order to prove that the range of K is dense in $H^{-1/2}(\Gamma_u^c)$, it suffices us to show that the adjoint operator K^* is an injection (one-to-one map).

We will find K^* from the definition:

$$\langle K\omega, \varphi\rangle = \langle \omega, K^*\varphi\rangle \quad \text{for} \quad \forall\varphi \in H_0^{1/2}(\Gamma_u^c)\,.$$

For given $\varphi \in H_0^{1/2}(\Gamma_u^c)$, we consider the boundary value problem:

$$\Delta\psi(x) = 0\,, \quad x \in \Omega$$
$$\text{subject to} \quad \psi|_{\Gamma_u} = 0, \quad \psi|_{\Gamma_u^c} = \varphi\,.$$

The solution $\psi(x)$ exists uniquely in $H^{3/2}(\Omega)$. From Green's integral theorem, we have

$$\begin{aligned} 0 &= \int_\Omega (\Delta u)\psi d\Omega \\ &= \int_\Gamma \frac{\partial u}{\partial n}\psi d\Gamma - \int_\Gamma u\frac{\partial\psi}{\partial n} d\Gamma + \int_\Omega u\Delta\psi d\Omega \\ &= \int_{\Gamma_u^c} \frac{\partial u}{\partial n}\varphi d\Gamma - \int_{\Gamma_u^c} \omega\frac{\partial\psi}{\partial n} d\Gamma\,. \end{aligned}$$

This implies that

$$\int_{\Gamma_u^c} \frac{\partial u}{\partial n}\varphi d\Gamma = \int_{\Gamma_u^c} \omega\frac{\partial\psi}{\partial n} d\Gamma \quad \text{for} \quad \forall\varphi \in H_0^{1/2}(\Gamma_u^c)\,.$$

We know now that

$$K^* : H_0^{1/2}(\Gamma_u^c) \ni \varphi \mapsto \frac{\partial\psi}{\partial n} \in L^2(\Gamma_u^c)\,.$$

We will show that K^* is injective. Let $K^*\varphi = \frac{\partial\psi}{\partial n} = 0$. To this end, we consider the boundary value problem:

$$\Delta\psi(x) = 0\,, \quad x \in \Omega$$
$$\text{subject to} \quad \psi|_{\Gamma_u} = 0\,, \quad \frac{\partial\psi}{\partial n}|_{\Gamma_u^c} = 0\,.$$

This problem is uniquely solvable with the solution $\psi(x) = 0$ in Ω. Therefore we obtain $\varphi = \psi = 0$ on Γ_u^c. □

This theorem guarantees that our variational problem is solvable for almost all $u|_\Gamma$ and $q|_\Gamma$.

Theorem 2 The Fréchet derivative $J'(\omega)$ in $L^2(\Gamma_u^c)$-sense is given by

$$J'(\omega)|_{\Gamma_u^c} = \frac{\partial v}{\partial n}(x)\,. \tag{10}$$

Proof We see

$$\begin{aligned} &J(\omega+\delta\omega) - J(\omega) \\ &= \int_{\Gamma_q} \left\{|q(x;\omega+\delta\omega) - \bar{q}(x)|^2 - |q(x;\omega) - \bar{q}(x)|^2\right\} d\Gamma \\ &\quad + \eta\int_\Gamma \left\{|q(x;\omega+\delta\omega)|^2 - |q(x;\omega)|^2\right\} d\Gamma \\ &= \int_{\Gamma_q} \{q(x;\omega+\delta\omega) + q(x;\omega) - 2\bar{q}(x)\} \\ &\qquad \{q(x;\omega+\delta\omega) - q(x;\omega)\}\, d\Gamma \\ &\quad + \eta\int_\Gamma \{q(x;\omega+\delta\omega) + q(x;\omega)\} \\ &\qquad \{q(x;\omega+\delta\omega) - q(x;\omega)\}\, d\Gamma \\ &= \int_{\Gamma_q} \{q(x;\omega+\delta\omega) - q(x;\omega) \\ &\qquad +2\,[q(x;\omega) - \bar{q}(x)]\}\, \delta q(x;\omega) d\Gamma \\ &\quad + \eta\int_\Gamma \{q(x;\omega+\delta\omega) - q(x;\omega) \\ &\qquad +2q(x;\omega)\}\, \delta q(x;\omega) d\Gamma \\ &= \int_{\Gamma_q} 2[q(x;\omega) - \bar{q}(x)]\delta q(x;\omega) d\Gamma + \int_{\Gamma_q} |\delta q(x;\omega)|^2 d\Gamma \\ &\quad + \eta\int_\Gamma 2q(x;\omega)\delta q(x;\omega) d\Gamma + \eta\int_\Gamma |\delta q(x;\omega)|^2 d\Gamma \\ &= \int_{\Gamma_q} 2[(1+\eta)q(x;\omega) - \bar{q}(x)]\delta q(x;\omega) d\Gamma \\ &\quad + \int_{\Gamma_q^c} 2\eta q(x;\omega)\delta q(x;\omega) d\Gamma \\ &\quad + \int_{\Gamma_q} (1+\eta)|\delta q(x;\omega)|^2 d\Gamma + \int_{\Gamma_q^c} \eta|\delta q(x;\omega)|^2 d\Gamma\,. \end{aligned}$$

In the above, we put $\delta u(x;\omega) = u(x;\omega+\delta\omega) - u(x;\omega)$, and accordingly $\delta q(x;\omega) = q(x;\omega+\delta\omega) - q(x;\omega)$. We notice that $\Delta(\delta u) = 0$ in Ω, $\delta u = 0$ on Γ_u, and $\delta u = \delta\omega$ on Γ_u^c.

We now consider $v \in H^2(\Omega)$ as a solution of the Laplace equation

$$-\Delta v(x;\omega) = 0\,, \qquad x \in \Omega \tag{11}$$

subject to the boundary conditions

$$\begin{aligned} &v|_{\Gamma_q} = 2\{(1+\eta)q(x;\omega) - \bar{q}(x)\} \\ &\text{and} \quad v|_{\Gamma_q^c} = 2\eta q(x;\omega)\,. \end{aligned} \tag{12}$$

From Green's integral theorem;

$$\int_\Omega (\Delta v)\delta u d\Omega = \int_\Gamma \frac{\partial v}{\partial n}\delta u d\Gamma - \int_\Gamma v\frac{\partial\delta u}{\partial n} d\Gamma + \int_\Omega v\Delta(\delta u) d\Omega\,,$$

we have

$$\begin{aligned} 0 &= \int_{\Gamma_u^c} \frac{\partial v}{\partial n}\delta\omega d\Gamma - \int_{\Gamma_q} 2[(1+\eta)q(x;\omega) - \bar{q}(x)]\delta q d\Gamma \\ &\quad - \int_{\Gamma_q^c} 2\eta q(x;\omega)\delta q d\Gamma\,. \end{aligned}$$

Consequently we obtain

$$\begin{aligned} &J(\omega+\delta\omega)-J(\omega) \\ &=\int_{\Gamma_u^c}\frac{\partial v}{\partial n}\delta\omega d\Gamma \\ &+\int_{\Gamma_q}(1+\eta)|\delta q(\boldsymbol{x};\omega)|^2 d\Gamma+\int_{\Gamma_q^c}\eta|\delta q(\boldsymbol{x};\omega)|^2 d\Gamma \\ &=\left(\frac{\partial v}{\partial n},\delta\omega\right)_{L^2(\Gamma_u^c)}+o(\|\,\delta\omega\,\|)\,. \end{aligned}$$

Corollary The second-order derivative $J''(\omega)$ is given by

$$J''(\omega)\delta\omega|_{\Gamma_u^c}=2\frac{\partial w}{\partial n}(\boldsymbol{x};\delta q) \tag{13}$$

with $w\in H^2(\Omega)$ as a solution of the Laplace equation

$$-\Delta w(\boldsymbol{x};\delta q)=0\,,\quad \boldsymbol{x}\in\Omega \tag{14}$$

subject to the boundary conditions;

$$w|_{\Gamma_q}=(1+\eta)\delta q \quad\text{and}\quad w|_{\Gamma_q^c}=\eta\delta q\,. \tag{15}$$

Proof We start with the expression

$$\begin{aligned} J(\omega+\delta\omega)=J(\omega)&+(J'(\omega),\delta\omega)_{L^2(\Gamma_u^c)} \\ &+\int_{\Gamma_q}(1+\eta)|\delta q(\boldsymbol{x};\omega)|^2 d\Gamma \\ &+\int_{\Gamma_q^c}\eta|\delta q(\boldsymbol{x};\omega)|^2 d\Gamma\,. \end{aligned} \tag{16}$$

From Green's integral theorem;

$$\begin{aligned} \int_\Omega(\Delta w)\delta u d\Omega=&\int_\Gamma\frac{\partial w}{\partial n}\delta u d\Gamma \\ &-\int_\Gamma w\frac{\partial\delta u}{\partial n}d\Gamma+\int_\Omega w\Delta(\delta u)d\Omega\,, \end{aligned}$$

we have

$$\begin{aligned} 0=&\int_{\Gamma_u^c}\frac{\partial w}{\partial n}\delta w d\Gamma \\ &-\int_{\Gamma_q}(1+\eta)|\delta q|^2 d\Gamma-\int_{\Gamma_q^c}\eta|\delta q|^2 d\Gamma\,. \end{aligned}$$

Consequently we obtain

$$\begin{aligned} J(\omega+\delta\omega)&=J(\omega)+(J'(\omega),\delta\omega)_{L^2(\Gamma_u^c)} \\ &\quad+\frac{1}{2}\int_{\Gamma_u^c}2\frac{\partial w}{\partial n}\delta\omega d\Gamma \\ &=J(\omega)+(J'(\omega),\delta\omega)_{L^2(\Gamma_u^c)} \\ &\quad+\frac{1}{2}(J''(\omega)\delta\omega,\delta\omega)_{L^2(\Gamma_u^c)}\,. \end{aligned}$$

The next theorem states the unique existence of the minimizer of the functional $J(\omega)$ [4].

Theorem 3 The functional $J: H^{1/2}(\Gamma_u^c)\ni\omega\mapsto \boldsymbol{R}_+$ is strictly convex.

Proof For ω_k $(k=1,2)$, let $u(\boldsymbol{x};\omega_k)$ be the solution of the boundary value problem:

$$\Delta u=0\quad\text{in}\quad\Omega,\qquad u|_{\Gamma_u}=\overline{u},\quad u|_{\Gamma_u^c}=\omega_k.$$

For any number θ $(0<\theta<1)$, we note that $\theta u(\boldsymbol{x};\omega_1)+(1-\theta)u(\boldsymbol{x};\omega_2)$ is also the solution of the boundary value problem with $u|_{\Gamma_u^c}=\theta\omega_1+(1-\theta)\omega_2$ due to the linearity of the problem. Using the convexity of the parabola $\{\theta\tau_1+(1-\theta)\tau_2\}^2<\theta\tau_1^2+(1-\theta)\tau_2^2$ for any $\tau_1,\tau_2\in\boldsymbol{R}$, we see

$$\begin{aligned} &J(\theta\omega_1+(1-\theta)\omega_2) \\ &=\int_{\Gamma_q}|\theta q(\boldsymbol{x};\omega_1)+(1-\theta)q(\boldsymbol{x};\omega_2)-\overline{q}(\boldsymbol{x})|^2 d\Gamma \\ &\quad+\eta\int_\Omega|\nabla\{\theta u(\boldsymbol{x};\omega_1)+(1-\theta)u(\boldsymbol{x};\omega_2)\}|^2 d\Omega \\ &=\int_{\Gamma_q}|\theta\{q(\boldsymbol{x};\omega_1)-\overline{q}(\boldsymbol{x})\} \\ &\qquad+(1-\theta)\{q(\boldsymbol{x};\omega_2)-\overline{q}(\boldsymbol{x})\}|^2 d\Gamma \\ &\quad+\eta\int_\Omega|\theta\nabla u(\boldsymbol{x};\omega_1)+(1-\theta)\nabla u(\boldsymbol{x};\omega_2)|^2 d\Omega \\ &\le\int_{\Gamma_q}\{\theta|q(\boldsymbol{x};\omega_1)-\overline{q}(\boldsymbol{x})|^2 \\ &\qquad+(1-\theta)|q(\boldsymbol{x};\omega_2)-\overline{q}(\boldsymbol{x})|^2\}d\Gamma \\ &\quad+\eta\int_\Omega\{\theta|\nabla u(\boldsymbol{x};\omega_1)|^2+(1-\theta)|\nabla u(\boldsymbol{x};\omega_2)|^2\}d\Omega \\ &=\theta J(\omega_1)+(1-\theta)J(\omega_2). \end{aligned}$$

This implies that $J(\omega)$ is convex. To show that $J(\omega)$ is strictly convex, we can see that

$$\begin{aligned} &\frac{1}{2}(J''(\omega)\delta\omega,\delta\omega)_{L^2(\Gamma_u^c)} \\ &=\int_{\Gamma_u^c}\frac{\partial w}{\partial n}\delta\omega d\Gamma \\ &=\int_{\Gamma_q}(1+\eta)|\delta q|^2 d\Gamma+\int_{\Gamma_q^c}\eta|\delta q|^2 d\Gamma>0 \end{aligned}$$

if and only if $\delta\omega\neq 0$ in $H^{1/2}(\Gamma_u^c)$. □

3. Boundary Element Method

We divide the whole boundary Γ into the series of n boundary elements as $\Gamma\simeq\Gamma^h=\cup_{j=1}^n\Gamma_j$ for its approximation, where h stands for some representative size of the boundary elements. Here the boundary element subdivision should be in accordance with the boundary components Γ_u and Γ_q.

We approximate the boundary values $u|_\Gamma$ and $q|_\Gamma$ by introducing the interpolation functions $N_j(\boldsymbol{x})$ in the form;

$$u|_\Gamma\simeq u^h(\boldsymbol{x})=\sum_{j=1}^n N_j(\boldsymbol{x})u_j\,, \tag{17}$$

$$q|_\Gamma\simeq q^h(\boldsymbol{x})=\sum_{j=1}^n N_j(\boldsymbol{x})q_j\,,\quad \boldsymbol{x}\in\Gamma \tag{18}$$

with approximate nodal values u_j and q_j to the exact nodal values $u(\boldsymbol{x}_j)$ and $q(\boldsymbol{x}_j)$, respectively, at the nodes $\boldsymbol{x}_j$ $(j=1,2,\cdots,n)$ on the boundary Γ. We approximate the boundary values $v|_\Gamma$ and $r|_\Gamma=\dfrac{\partial v}{\partial n}$ also in the form;

$$v|_\Gamma \simeq v^h(x) = \sum_{j=1}^{n} N_j(x)v_j\,, \tag{19}$$

$$r|_\Gamma \simeq r^h(x) = \sum_{j=1}^{n} N_j(x)r_j\,, \quad x \in \Gamma \tag{20}$$

with approximate nodal values v_j and r_j to the exact $v(x_j)$ and $r(x_j)$, respectively, at x_j on Γ.

The exact boundary values $u|_\Gamma$ and $q|_\Gamma$ in the boundary integral equation (6) are replaced by the approximations $u^h|_\Gamma$ and $q^h|_\Gamma$ of eqns (17), (18) respectively. This yields

$$\frac{1}{2}u(\xi) + \sum_{j=1}^{n}\int_\Gamma \frac{\partial G}{\partial n}(x;\xi)N_j(x)d\Gamma(x)u_j = \sum_{j=1}^{n}\int_\Gamma G(x;\xi)N_j(x)d\Gamma(x)q_j\,, \quad \xi \in \Gamma. \tag{21}$$

We take those n nodes x_j again as collocation points in order to fully discretize the boundary integral equation (21). Put $\xi = x_i \quad (i = 1, 2, \cdots, n)$. Then we have

$$\frac{1}{2}u_i + \sum_{j=1}^{n}\int_\Gamma \frac{\partial G}{\partial n}(x;x_i)N_j(x)d\Gamma(x)u_j = \sum_{j=1}^{n}\int_\Gamma G(x;x_i)N_j(x)d\Gamma(x)q_j\,, \tag{22}$$

which results in the system of linear equations in the matrix form;

$$[H]\{u\} = [G]\{q\} \tag{23}$$

with each entity

$$h_{ij} = \frac{1}{2}\delta_{ij} + \int_\Gamma \frac{\partial G}{\partial n}(x;x_i)N_j(x)d\Gamma(x)\,, \tag{24}$$

$$g_{ij} = \int_\Gamma G(x;x_i)N_j(x)d\Gamma(x)\,, \tag{25}$$

for $i, j = 1, 2, \cdots, n$. Here, δ_{ij} is the Kronecker symbol.

We apply the discretizing procedure again to the boundary integral equation corresponding to the adjoint problem eqns (11), (12) to obtain

$$[H]\{v\} = [G]\{r\} \tag{26}$$

with the same $n \times n$ coefficient matrices $[H]$ and $[G]$ as in eqn(23).

We denote by n_1 the number of nodes on Γ_u, and by n_2 the number of nodes on Γ_q, respectively. Let $n_1^c = n - n_1$ and $n_2^c = n - n_2$, being the respective numbers of nodes on Γ_u^c and Γ_q^c. According to the respective boundary components Γ_u and Γ_q we write the column vectors $\{u\}$ and $\{q\}$ in the form;

$$\{u\} = \left\{ \begin{array}{ccc} u_1 & \text{on} & \Gamma_u \\ u_2 & \text{on} & \Gamma_u^c \end{array} \right\} \begin{array}{c} n_1 \\ n_1^c \end{array}\,,$$

$$\{q\} = \left\{ \begin{array}{ccc} q_1 & \text{on} & \Gamma_q^c \\ q_2 & \text{on} & \Gamma_q \end{array} \right\} \begin{array}{c} n_2^c \\ n_2 \end{array}\,, \tag{27}$$

where n_1 nodal values u_j on Γ_u are collected in $\{u_1\}$, and the n_1^c nodal values on Γ_u^c in $\{u_2\}$, whereas n_2 nodal values q_j on Γ_q are collected in $\{q_2\}$, and the n_2^c nodal values on Γ_q^c in $\{q_1\}$.

In the similar way we write

$$\{v\} = \left\{ \begin{array}{ccc} v_1 & \text{on} & \Gamma_q^c \\ v_2 & \text{on} & \Gamma_q \end{array} \right\} \begin{array}{c} n_2^c \\ n_2 \end{array}\,,$$

$$\{r\} = \left\{ \begin{array}{ccc} r_1 & \text{on} & \Gamma_u \\ r_2 & \text{on} & \Gamma_u^c \end{array} \right\} \begin{array}{c} n_1 \\ n_1^c \end{array}\,. \tag{28}$$

Then the systems eqns (23) and (26) can be written respectively in the partitioned form;

$$\begin{array}{c} \\ n \end{array} \overset{\begin{array}{cc} n_1 & n_1^c \end{array}}{\left[H_1^{(1)}\ H_2^{(1)} \right]} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \overset{\begin{array}{cc} n_2^c & n_2 \end{array}}{\left[G_1^{(1)}\ \ G_2^{(1)} \right]} \left\{ \begin{array}{c} q_1 \\ q_2 \end{array} \right\} \tag{29}$$

and

$$\begin{array}{c} \\ n \end{array} \overset{\begin{array}{cc} n_2 & n_2^c \end{array}}{\left[H_1^{(2)} H_2^{(2)} \right]} \left\{ \begin{array}{c} v_1 \\ v_2 \end{array} \right\} = \overset{\begin{array}{cc} n_1 & n_1^c \end{array}}{\left[G_1^{(2)}\ \ G_2^{(2)} \right]} \left\{ \begin{array}{c} r_1 \\ r_2 \end{array} \right\}, \tag{30}$$

where numbers of rows and columns of the coefficient matrices are indicated.

4. Direct Method of Solution

We insert boundary conditions of primary and adjoint problems into the partitioned systems of eqns (29), (30): From eqn (9) we have

$$\{u_1\} = \{\bar{u}_1\}, \quad \{u_2\} = \{\omega\}\,. \tag{31}$$

From eqn (12) we have

$$\{v_1\} = 2\eta\{q_1\}, \quad \{v_2\} = 2((1+\eta)\{q_2\} - \{\bar{q}_2\})\,, \tag{32}$$

and from eqn (10) as the necessary condition that $J(\omega)$ is minimal we have

$$\{r_2\} = \{0\}\,. \tag{33}$$

Therefore the systems eqns (29), (30) reduce to the form;

$$\left[\begin{array}{cc} H_1^{(1)} & H_2^{(1)} \end{array}\right] \left\{ \begin{array}{c} \bar{u}_1 \\ \omega \end{array} \right\} = \left[\begin{array}{cc} G_1^{(1)} & G_2^{(1)} \end{array}\right] \left\{ \begin{array}{c} q_1 \\ q_2 \end{array} \right\} \tag{34}$$

and

$$\left[\begin{array}{cc} H_1^{(2)} & H_2^{(2)} \end{array}\right] \left\{ \begin{array}{c} 2\eta q_1 \\ 2(1+\eta)(q_2 - \bar{q}_2) \end{array} \right\} = \left[\begin{array}{cc} G_1^{(2)} & G_2^{(2)} \end{array}\right] \left\{ \begin{array}{c} r_1 \\ 0 \end{array} \right\}, \tag{35}$$

respectively.

We combine eqns (34) and (35). We take unknown nodal values to the left of the equation to obtain

$$\begin{array}{c} \\ n \\ n \end{array}\begin{array}{c} \begin{array}{cccc} n_2^c & n_1^c & n_1 & n_2 \end{array} \\ \left[\begin{array}{cccc} -G_1^{(1)} & H_2^{(1)} & O & -G_2^{(1)} \\ 2\eta H_1^{(2)} & O & -G_1^{(2)} & 2(1+\eta)H_2^{(2)} \end{array}\right] \end{array}\left\{\begin{array}{c} q_1 \\ \omega \\ r_1 \\ q_2 \end{array}\right\}$$

$$= \begin{array}{c} n \\ n \end{array}\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left[\begin{array}{cc} -H_1^{(1)} & O \\ O & 2H_2^{(2)} \end{array}\right] \end{array}\left\{\begin{array}{c} \bar{u}_1 \\ \bar{q}_2 \end{array}\right\}. \qquad (36)$$

We notice that the coefficient matrix on the left hand side of the augmented new system of linear eqns (36) is square of order $2n$.

5. Numerical Examples

Suppose that the harmonic function

$$u(x_1, x_2) = x_1^2 - x_2^2 = r^2 \cos(2\vartheta) \qquad (37)$$

with the polar coordinates $x_1 = r\cos\vartheta$, $x_2 = r\sin\vartheta$ serves as a solution of the inverse boundary value problem eqns (1), (2) in the unit circle;

$$\Omega = \{(r,\vartheta) \mid 0 \le r < 1,\ 0 \le \vartheta < 2\pi\} \qquad (38)$$

as shown in Fig. 1

The collocation boundary element method with C^0 linear elements is used. The boundary $\Gamma = \partial\Omega$ is uniformly divided into 48 and 96 boundary elements as shown in Fig. 2. The double nodes are taken at the edges of the boundary components Γ_u and Γ_q, so that discontinuity of q and r at the edges is admitted in the computation.

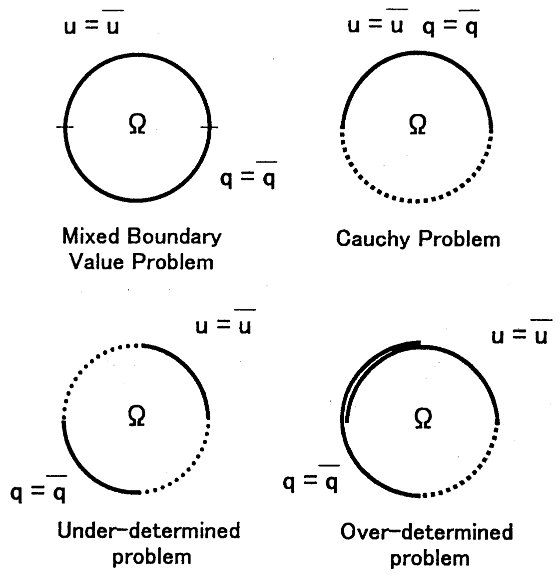
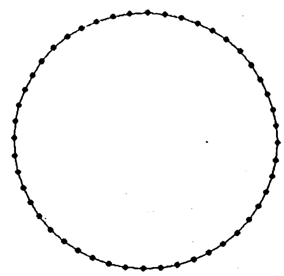
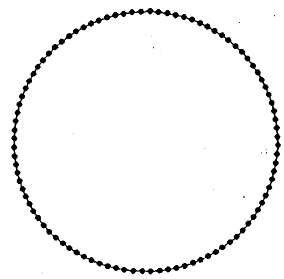


Fig. 1 Problem statement



(a) 48 boundary nodes



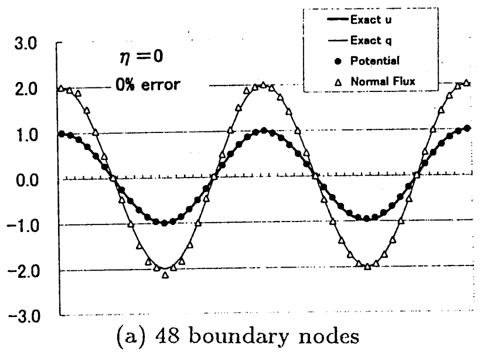
(b) 96 boundary nodes

Fig. 2 Boundary elements

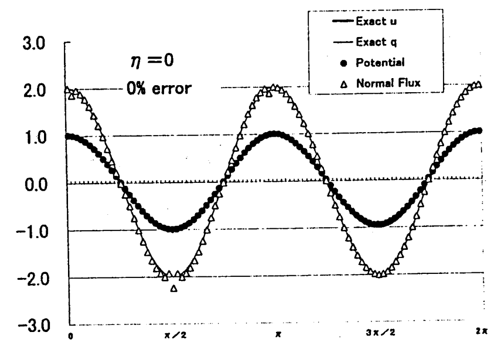
5.1 Mixed boundary value problem

The Dirichlet data $\bar{u} = \cos(2\vartheta)$ on $\Gamma_u = \{(1,\vartheta) \mid 0 \le \vartheta \le \pi\}$ and the Neumann data $\bar{q} = 2\cos(2\vartheta)$ on $\Gamma_q = \{(1,\vartheta) \mid \pi < \vartheta < 2\pi\}$ are given as shown in Fig. 1.

Calculated profiles of u^h and q^h against the central angle ϑ $(0 \le \vartheta < 2\pi)$ are depicted in Fig. 3 with reference to the exact u along the boundary Γ. The approximate u^h is in good agreement with the exact u.



(a) 48 boundary nodes



(b) 96 boundary nodes

Fig. 3 Exact u and approximate u^h, q^h on Γ

5.2 Cauchy problem

The Cauchy data $\bar{u} = \cos(2\vartheta)$ and $\bar{q} = 2\cos(2\vartheta)$ on $\Gamma_u = \Gamma_q = \{(1, \vartheta) \mid 0 \le \vartheta \le \pi\}$ are given as shown in Fig. 1.

Calculated profiles of u^h and q^h against the central angle ϑ $(0 \le \vartheta < 2\pi)$ are depicted in Fig. 4 with reference to the exact u and q along the boundary Γ. Both of the approximate u^h and q^h are in good agreement respectively with the exact u and q.

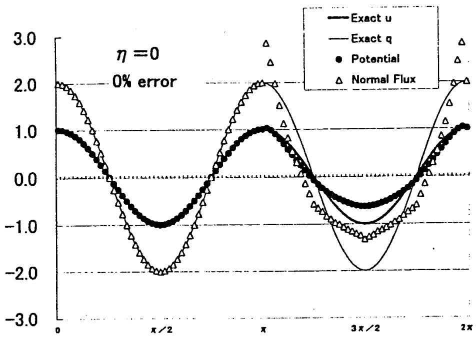
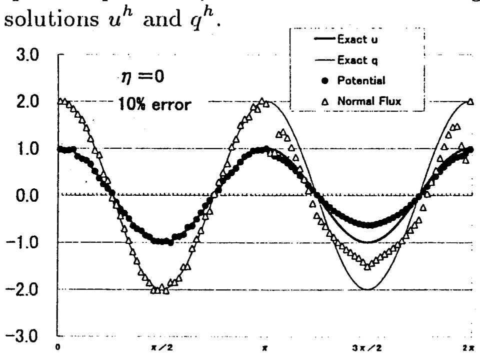


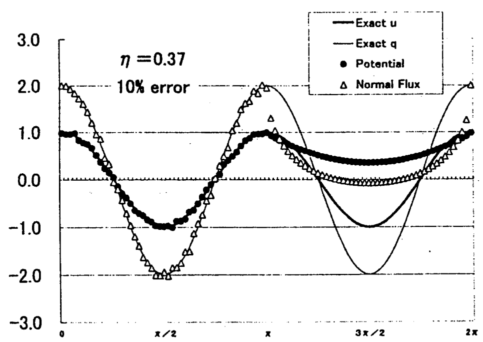
Fig. 4 Exact u, q and approximate u^h, q^h on Γ

We add uniformly distributed random errors with the magnitude of 10% to the Cauchy data. Figure 6 shows Hansen's L-curve diagram [(5)] for $0 \le \eta \le 1$. The optimum value is $\eta = 0.37$.

Calculated profiles of approximate u^h and q^h against the central angle ϑ $(0 \le \vartheta < 2\pi)$ are depicted in Fig. 5 with reference to the exact u and q along the boundary Γ for $\eta = 0$ and 0.37. Both of u^h and q^h for $\eta = 0$ are in fairly good agreement respectively with the exact u and q. The optimum $\eta = 0.37$ is seen to over-regularize the solutions u^h and q^h.



(a) $\eta = 0$



(b) $\eta = 0.37$

Fig. 5 Exact u, q and approximate u^h, q^h on Γ

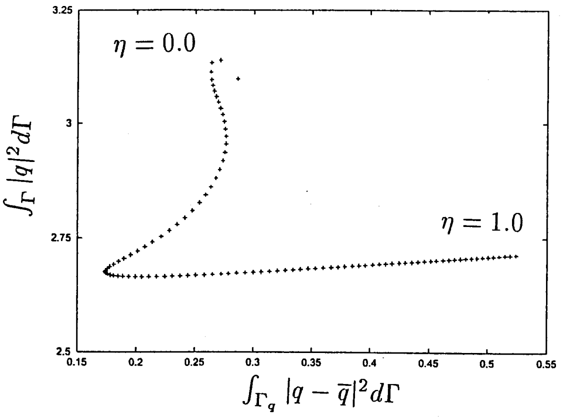
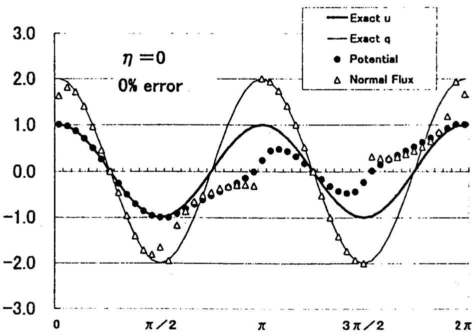


Fig. 6 Hansen's L-curve

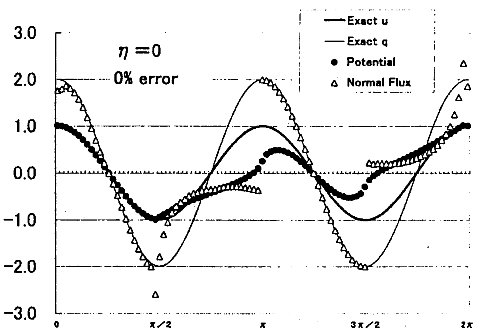
5.3 Under-determined problem

The Dirichlet data $\bar{u} = \cos(2\vartheta)$ on $\Gamma_u = \{(1, \vartheta) \mid 0 \le \vartheta \le \pi/2\}$ and the Neumann data $\bar{q} = 2\cos(2\vartheta)$ on $\Gamma_q = \{(1, \vartheta) \mid \pi < \vartheta < 3\pi/2\}$ are given as shown in Fig. 1.

Calculated profiles of u^h and q^h against the central angle ϑ $(0 \le \vartheta < 2\pi)$ are depicted in Fig. 7 with reference to the exact u and q along the boundary Γ. The approximate u^h is in fairly good agreement on Γ_q, and the approximate q^h is in fairly good agreement on Γ_u.



(a) 48 boundary nodes



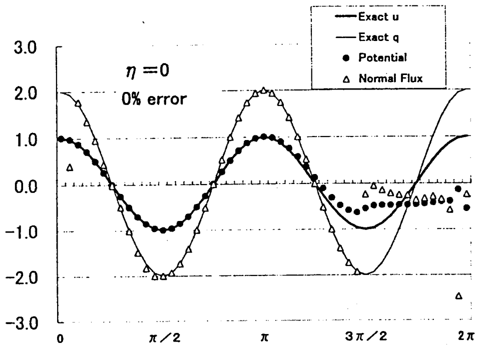
(b) 96 boundary nodes

Fig. 7 Exact u, q and approximate u^h, q^h on Γ

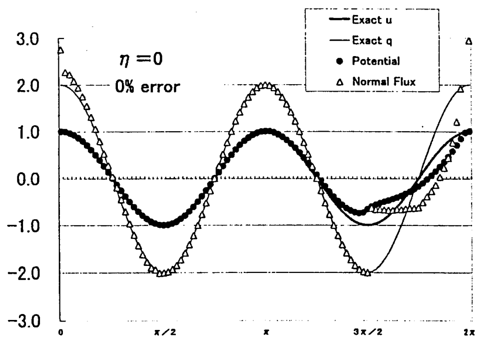
5.4 Over-determined problem

The Dirichlet data $\overline{u} = \cos(2\vartheta)$ on $\Gamma_u = \{(1,\vartheta) \mid 0 \le \vartheta \le \pi\}$ and the Neumann data $\overline{q} = 2\cos(2\vartheta)$ on $\Gamma_q = \{(1,\vartheta) \mid \pi/2 < \vartheta < 3\pi/2\}$ are given as shown in Fig. 1.

Calculated profiles of u^h and q^h against the central angle ϑ $(0 \le \vartheta < 2\pi)$ are depicted in Fig. 8 with reference to the exact u and q along the boundary Γ. The approximate u^h is in good agreement on $\Gamma_q \setminus \Gamma_u$, and the approximate q^h is in good agreement on $\Gamma_u \setminus \Gamma_q$.



(a) 48 boundary nodes



(b) 96 boundary nodes

Fig. 8 Exact u, q and approximate u^h, q^h on Γ

6. Conclusions

A boundary inverse problem is considered for the Laplace equation in two dimensions. By introducing a convex functional to be minimized, the solution of the inverse problem is understood as the minimizer of the functional. The necessary condition for the functional to attain the minimum is paraphrased by the primary and adjoint boundary value problems of the Laplace equation. The boundary element method is applied to obtain numerical solution of the problems, yielding an augumented system of linear algebraic equations. The linear system of equations can be solved directly. Four test examples suggest the validity of this direct method for the inverse boundary value problem.

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