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Cubic spline approximation of offset curves of planar cubic splines

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Abstract We derive an easier to calculate algorithm for a cubic spline approximation of an offset curve of a given planar cubic spline and a sufficient condition on an offset length for its existence. We show that the cubic offset approximation is C- (or S-) shaped if the original cubic is C- (or S-) shaped.

1 Introduction and description of the method

The construction of offset-curves and offset surfaces plays an important role in the automobile industry. For example, if we are given the inner surface of a part of a car body we can reach the outer surface by an offset with material thickness. Klass has proposed an algorithm for a cubic spline approximation of an offset curve of a planar cubic spline ([2]).

The object of this paper is to obtain an easier to calculate algorithm for the cubic approximation method and a sufficient condition on an offset length for its existence. A planar cubic spline segment $\boldsymbol{z}(t), 0 \leq t \leq 1$ is defined by the equation:

$$\boldsymbol{z}(t)=\boldsymbol{P}_0(1-t^2(3-2t))+\boldsymbol{T}_0(1-t)^2t-\boldsymbol{T}_1(1-t)t^2+\boldsymbol{P}_1t^2(3-2t) \tag{1.1}$$

where $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ are the tangent vectors at endpoints $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$. For simplicity of analysis, we assume that the tangent directions (the counterclockwise angles from the x-axis to the tangent vectors) $\pi-\theta, \pi+\psi$ at the end points $(1,0),(-1,0)$, i.e.,

$$\boldsymbol{P}_0=(1,0), \boldsymbol{P}_1=(-1,0), \boldsymbol{T}_0=r_0(-\cos\theta, \sin\theta), \boldsymbol{T}_1=r_1(-\cos\psi,-\sin\psi) \tag{1.2}$$

with $r_i>0, i=0,1; 0<\theta, |\psi|<\pi/2, \theta+\psi>0$. Its signed curvature $\kappa(t)$ is given by

$$\kappa(t)=(\boldsymbol{z}'\times\boldsymbol{z}'')(t)/\left\|\boldsymbol{z}'(t)\right\|^3, \quad 0\leq t\leq 1 \tag{1.3}$$

where "$\times$" means the cross product of two vectors. Now we want to construct the offset cubic spline approximation $\tilde{\boldsymbol{z}}$ with an offset length d to the original cubic spline $\boldsymbol{z}$. Since offset curves of splines are themselves not splines, Klass ([2]) has considered an offset cubic spline approximation $\tilde{\boldsymbol{z}}$ of the form (1.1) with an offset of d to $\boldsymbol{z}$ as

$$\tilde{\boldsymbol{P}}_j = \boldsymbol{P}_j + d\boldsymbol{N}_j, \quad \tilde{\boldsymbol{T}}_j = c_j\boldsymbol{T}_j, \quad c_j > 0, j = 0,1 \tag{1.4}$$

where the unit normal vectors $\boldsymbol{N}_j$ to the corresponding to the tangent vectors $\boldsymbol{T}_j$ at $\boldsymbol{P}_j, j = 0,1$ are given by

$$N_0 = (\sin\theta, \cos\theta), \quad N_1 = (-\sin\psi, \cos\psi) \tag{1.5}$$

Letting the curvature of $\tilde{\boldsymbol{z}}$ be $\tilde{\kappa}$, then the unknown positive parameters $c_j, j = 0,1$ are determined for the given offset length d as

$$\frac{1}{\tilde{\kappa}(t)} = \frac{1}{\kappa(t)} + d, t = 0,1 \tag{1.6}$$

For later use, we define the following four quantities $D_i, i = 0,1$ and p, q:

$$D_0 = \frac{r_0 \sin(\theta+\psi)}{6\sin\psi}, \quad D_1 = \frac{r_1 \sin(\theta+\psi)}{6\sin\theta}; \quad p = \frac{D_0^2}{1-D_1}, \quad q = \frac{D_1^2}{1-D_0} \tag{1.7}$$

By a simple but long calculation (if necessary, with help of *Mathematica*), eqs (1.6) give a quadratic system of equations in $C_0(= r_0c_0), C_1(= r_1c_1)$:

$$C_0^2 = b_0 - a_0C_1, \quad C_1^2 = b_1 - a_1C_0 \tag{1.8}$$

where

$$\begin{aligned}
&w_0 = 2d + r_0^2/\{6(1-D_1)\sin\theta\}, \quad w_1 = 2d + r_1^2/\{6(1-D_0)\sin\psi\}\\
&a_0 = w_0 \sin(\theta+\psi), \quad a_1 = w_1\sin(\theta+\psi)\\
&b_0 = 6w_0\{\sin\theta + d\sin^2\{(\theta+\psi)/2\}\}, \quad b_1 = 6w_1\{\sin\psi + d\sin^2\{(\theta+\psi)/2\}\}
\end{aligned}$$

The above system of equations (1.8) in (C_0, C_1) is easier to treat than the one given in [2]. The solubility of the above system has been studied as follows ([1]). Letting $(\rho_0, \rho_1) = (a_1C_0/b_1, a_0C_1/b_0)$, then (1.8) is reduced to the system:

$$\rho_0 = 1 - R_1\rho_1^2, \quad \rho_1 = 1 - R_0\rho_0^2 \tag{1.9}$$

with

$$(R_0, R_1) = \left(b_1^2/(b_0a_1^2), b_0^2/(b_1a_0^2)\right)$$

Fig. 1 gives the number of the solutions of the system of (1.9) with respect to the positive $R_i, i = 0,1$ where the curve through $(3/4, 3/4)$ is given by $256R_0R_1\{R_0R_1 - (R_0+R_1)+288\} = 27$. Here, we consider the case when $(R_0-1)(R_1-1) > 0, R_0, R_1 > 0$. Then, note that the system (1.8) has at least one positive solution (C_0, C_1), as is to be proved.

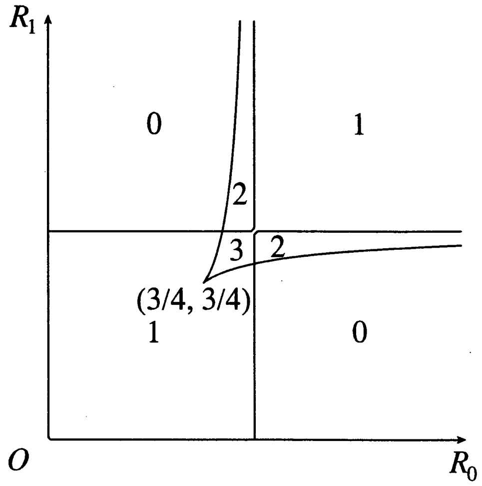

Fig. 1. Number of solutions of quadratic system (1.9) with respect to (R_0, R_1) ([1]).

Positive offset length d: First we derive a sufficient condition for the system (1.8) to have a positive solution (C_0, C_1) for on the positive offset length d. Depending on the sign of ψ, we consider the following two cases.

C**-shaped data** $(\theta > 0, \psi > 0)$: Delete C_1 from the system (1.8) to obtain a quartic equation $\phi(C_0) = 0$ as

$$\phi(C_0) = C_0^4 - 2b_0C_0^2 + a_0^2a_1C_0 + b_0^2 - a_0^2b_1 \tag{1.10}$$

Since $C_1 > 0$ requires $0 < C_0 < \sqrt{b_0}$, we seek a sufficient condition that $\phi(C_0) = 0$ has a root in $(0, \sqrt{b_0})$. Since $\phi(0) = b_0^2 - b_1a_0^2$ and $\phi(\sqrt{b_0}) = a_0^2(a_1\sqrt{b_0} - b_1)$, we have sufficient ones on $z(= d\sin^2\{(\theta+\psi)/2\})$.

Case 1 $(0 < p, q < 1; 0 < \theta + \psi < \pi/3)$: Note a sufficient condition for (1.8) to have positive root(s) as follows

$$\begin{aligned}
&(i) \quad \frac{b_0^2 - b_1a_0^2}{12w_0^2} = \{2\cos(\theta+\psi) - 1\}z^2 + \{\eta(\theta,\psi,p)\sin\psi\}z + 3(p-1)\sin^2\psi < 0 \\
&(ii) \quad \frac{b_1^2 - b_0a_1^2}{12w_1^2} = \{2\cos(\theta+\psi) - 1\}z^2 + \{\eta(\psi,\theta,q)\sin\theta\}z + 3(q-1)\sin^2\theta < 0
\end{aligned} \tag{1.11}$$

with

$$\eta(\theta,\psi,p) = \frac{3p\sin\psi}{\sin\theta} + \frac{2\sin\theta + \sin(2\theta+\psi)}{\sin\psi} - 7$$

Letting $u(\theta,\psi,p)$ be the positive root of 1.11(i) (where "inequality" is to be replaced by "equality") to obtain :

$$u(\theta,\psi,p) = \frac{6(1-p)\sin\psi}{\eta(\theta,\psi,p) + \sqrt{12(1-p)\{2\cos(\theta+\psi) - 1\} + \eta^2(\theta,\psi,p)}}$$

Then, we obtain a sufficient condition from (1.11) as

$$0 < d < d_0 = \mathrm{Min}\left[\frac{u(\theta,\psi,p)}{\sin^2\{(\theta+\psi)/2\}}, \frac{u(\psi,\theta,q)}{\sin^2\{(\theta+\psi)/2\}}\right] \tag{1.12}$$

Case 2 ($p, q > 1; \pi/3 < \theta + \psi < \pi$): Note a sufficient condition for (1.8) to have positive root(s) as follows

$$(i) \quad \frac{b_0^2 - b_1 a_0^2}{12 w_0^2} = \{2\cos(\theta+\psi) - 1\}z^2 + \{\eta(\theta,\psi,p)\sin\psi\}z + 3(p-1)\sin^2\psi > 0$$

$$(ii) \quad \frac{b_1^2 - b_0 a_1^2}{12 w_1^2} = \{2\cos(\theta+\psi) - 1\}z^2 + \{\eta(\psi,\theta,q)\sin\theta\}z + 3(q-1)\sin^2\theta > 0 \tag{1.13}$$

Letting $v(\theta,\psi,p)$ be the positive root of 13(i) in z (where "inequality" is to be replaced by "equality") to obtain :

$$v(\theta,\psi,p) = \frac{6(p-1)\sin\psi}{-\eta(\theta,\psi,p) + \sqrt{12(1-p)\{2\cos(\theta+\psi)-1\} + \eta^2(\theta,\psi,p)}}$$

Then, two inequalities (1.13) require

$$0 < d < \bar{d}_0 = \mathrm{Min}\left[\frac{v(\theta,\psi,p)}{\sin^2\{(\theta+\psi)/2\}}, \frac{v(\psi,\theta,q)}{\sin^2\{(\theta+\psi)/2\}}\right] \tag{1.14}$$

Note that two inequalities in (1.11) (or (1.13)) are equivalent to $R_0, R_1 > 1$ (or $R_0, R_1 < 1$); refer to Fig. 1. To study the shape of the cubic spline curve $\boldsymbol{z}$, we consider the linear system of equations in λ, μ:

$$\Delta\boldsymbol{z} (= \boldsymbol{z}(1) - \boldsymbol{z}(0)) = \lambda\boldsymbol{z}'(0) + \mu\boldsymbol{z}'(1) \tag{1.15}$$

Then we obtain the distribution of inflections and singularities with respect to (λ,μ) where regions $N_i, 0 \le i \le 2$ mean i-inflection points, and the curve C (or L) means the cusp (or loop) where A (or B) is $\mu^2 = \lambda(3\mu - 1)$ (or $\lambda^2 = \mu(3\lambda - 1)$) and C is denoted by $(\lambda - 1/3)(\mu - 1/3) = 1/36, \lambda, \mu < 1/3$ ([3]).

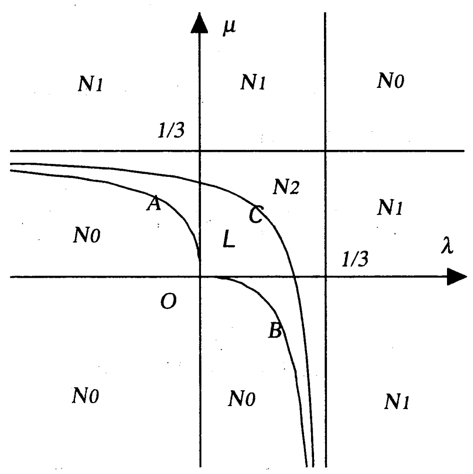

Fig. 2. Inflections and singularities with respect to (λ, μ) ([3]).

Theorem 1.1 *(C-shaped data): If $0 < p, q < 1$, then there exists a fair (without inflection points and singularities) spline approximation $\tilde{z}$ to a fair cubic z with an offset length $d \in (0, d_0)$ for $0 < \theta + \psi < \pi/3$ where for $\pi/3 < \theta + \psi < \pi$, $p, q > 1$ and $d \in (0, \bar{d}_0)$.*

Proof. First, note that the original C-shaped cubic segment z of the form (1.1) has neither inflection points nor singularities (loop, cusp) since the linear system (1.15) has the solutions $\lambda(= 1/(3D_0)), \mu(= 1/(3D_1))$ $(\geq 1/3)$ since $0 < D_0, D_1 < 1$ from $p, q > 0$ and (1.7); refer to Figure 2. Next, to show that the approximation $\tilde{z}$ has neither inflection points nor singularities, we have only to check that the solutions of the following linear system satisfies the inequalities: $\bar{\lambda}, \bar{\mu} \geq 1/3$:

$$\Delta \tilde{z} = \bar{\lambda} \tilde{z}'(0) + \bar{\mu} \tilde{z}'(1) \tag{1.16}$$

from which we have

$$\bar{\lambda} - \frac{1}{3} = \frac{C_1^2}{3C_0 w_1 \sin(\theta + \psi)}, \quad \bar{\mu} - \frac{1}{3} = \frac{C_0^2}{3C_1 w_0 \sin(\theta + \psi)} \tag{1.17}$$

Since $w_0, w_1 > 0$ from $D_0, D_1 < 1$, $\bar{\lambda}, \bar{\mu} > 1/3$.

This completes the proof of this theorem.

Here we remark the remaining cases for the C-shaped original cubic curve which are not included in the above theorem since otherwise the theorem is a little too complicated; only a numerical example (Example 2) is treated in the next section:

(i) $(0 < p, q < 1; \pi/3 \leq \theta + \psi < \pi)$: Only if $\eta(\theta, \psi, p) > 0$, then 1.11(i) requires $z < u(\theta, \psi, p)$ when the real $u(\theta, \psi, p)$ does exist or is always valid when it does not exist. If $\eta(\theta, \psi, p) < 0$, no restriction on z is required from 1.11(i). The same analysis can be applicable to 1.11(ii).

(ii) $(p, q > 1; 0 < \theta + \psi \leq \pi/3)$: Only if $\eta(\theta, \psi, p) < 0$, then 1.13(i) requires $z < v(\theta, \psi, p)$ when the real $v(\theta, \psi, p)$ does exist or is always valid when it does not exist. The same analysis can be applicable to 1.13(ii).

S-shaped data $(\theta > 0, \psi < 0)$: Then note that the original S-shaped cubic segment of the form (1.1) has just one inflection point without a singularity since $D_0 < 0 < D_1 < 1$. Then, we require $b_1 > 0$ (a necessary condition for the solubility of the system (1.8)) to give $d \in (0, d_1)$ where

$$d_1 = \mathrm{Min}\left[\frac{-\sin\psi}{\sin^2\{(\theta+\psi)/2\}}, \frac{-r_1^2}{12(1-D_0)\sin\psi}\right] \tag{1.18}$$

Let

$$d_2 = \mathrm{Min}\left[\frac{u(\theta, \psi, p)}{\sin^2\{(\theta+\psi)/2\}}, \frac{v(\psi, \theta, q)}{\sin^2\{(\theta+\psi)/2\}}\right]$$

$$\bar{d}_2 = \mathrm{Min}\left[\frac{v(\theta, \psi, p)}{\sin^2\{(\theta+\psi)/2\}}, \frac{u(\psi, \theta, q)}{\sin^2\{(\theta+\psi)/2\}}\right] \tag{1.19}$$

to obtain

Theorem 1.2 *(S-shaped data): If $0 < p, q < 1$ and $d < d_1$, there exists a cubic spline approximation $\tilde{\boldsymbol{z}}$ to the S−shaped cubic spline $\boldsymbol{z}$ with an offset of $d \in (0, d_2)$ for $\pi/3 < \theta + \psi < \pi$ where for $\pi/3 < \theta + \psi < \pi$, $p, q > 1$ and $d \in (0, \bar{d}_2)$. Then, the cubic segments $\boldsymbol{z}$ and $\tilde{\boldsymbol{z}}$ have just one infection point, respectively.*

Proof. Since $w_0 > 0, w_1 < 0$, note $\bar{\lambda} < 1/3, \bar{\mu} > 1/3$ from (1.18) to prove that the cubic segment $\tilde{\boldsymbol{z}}$ have just one infection points. For the original cubic $\boldsymbol{z}$, the linear system (1.15) has the solutions $\lambda(= 1/(3D_0) < 0), \mu(= 1/(3D_1))$ $(> 1/3)$ where $q > 0$ and (1.7) give $0 < D_1 < 1$.

Here we remark the remaining cases for the S-shaped original cubic curve as for the above C-shaped cubic one where note $\sin\theta > 0, \sin\psi < 0$ in derivation of the condition on 1.13(ii):

(i) $(0 < p, q < 1; \pi/3 \leq \theta + \psi < \pi)$: Only if $\eta(\theta, \psi, p) < 0$, then 1.11(i) requires $z < v(\theta, \psi, p)$ when the real v does exist or is always valid when it does not exist. If $\eta(\theta, \psi, p) < 0$, no restriction on z is required from 1.11(i). The same analysis can be applicable to 1.11(ii).

(ii) $(p, q > 1; 0 < \theta + \psi \leq \pi/3)$: Only if $\eta(\theta, \psi, p) > 0$, then 1.13(i) requires $z < u(\theta, \psi, p)$ when the real u does exist or is always valid when it does not exist. The same analysis can be applicable to 1.13(ii).

Negative offset length d: Next we derive a sufficient condition for the system (1.8) to have a positive solution (C_0, C_1) for the negative offset length d. First, we define the following quantities d_3 (or d_4) for the C (or S)-shaped original cubic curve:

$$d_3 = \mathrm{Max}\left[\frac{-\sin\psi}{\sin^2\{(\theta+\psi)/2\}}, \frac{-\sin\theta}{\sin^2\{(\theta+\psi)/2\}}, \frac{-r_0^2}{12(1-D_1)\sin\theta}, \frac{-r_1^2}{12(1-D_0)\sin\psi}\right] \tag{1.20}$$

$$d_4 = \mathrm{Max}\left[\frac{-\sin\theta}{\sin^2\{(\theta+\psi)/2\}}, \frac{-r_0^2}{12(1-D_1)\sin\theta}\right] \tag{1.21}$$

Then, the same analysis for the case positive offset d gives the following theorems where in the definition of $d_i, \bar{d}_i, i = 0, 2$, "Min" should be be replaced by "Max".

Theorem 1.3 *(C-shaped data): If $0 < p, q < 1$ and $d > d_3$, then there exists a fair (without inflection points and singularities) spline approximation $\tilde{\boldsymbol{z}}$ to a fair cubic $\boldsymbol{z}$ with an offset length $d \in (\bar{d}_0, 0)$ for $0 < \theta + \psi < \pi/3$ where for $\pi/3 < \theta + \psi < \pi$, $p, q > 1$ and $d \in (d_0, 0)$.*

Here we remark the remaining cases for the C-shaped original cubic curve:

(i) $(0 < p, q < 1; \pi/3 \leq \theta + \psi < \pi)$: if $\eta(\theta, \psi, p) < 0$, then 1.11(i) requires $z < v(\theta, \psi, p)$ when the real v does exist or is always valid when it does not exist. If $\eta(\theta, \psi, p) > 0$, no restriction on z is required from 1.11(i). The same analysis can be applicable to 1.11(ii) where note that $\sin\theta > 0, \sin\psi < 0$.

(ii) $(p, q > 1; 0 < \theta + \psi \leq \pi/3)$: if $\eta(\theta, \psi, p) > 0$, then 1.13(i) requires $z < u(\theta, \psi, p)$ when the real u does exist or is always valid when it does not exist.

Theorem 1.4 *(S-shaped data): If $0 < p, q < 1$ and $d > d_4$, then there exists a spline approximation $\tilde{z}$ to the s-shaped cubic z with an offset length $d \in (\bar{d}_2, 0)$ for $0 < \theta + \psi < \pi/3$ where for $\pi/3 < \theta + \psi < \pi$, $p, q > 1$ and $d \in (d_2, 0)$. Then, the cubic segments z and $\tilde{z}$ have just one infection points, respectively.*

Here we remark the remaining cases for the S-shaped original cubic curve:

(i) $(0 < p, q < 1; \pi/3 \leq \theta + \psi < \pi)$: Only if $\eta(\theta, \psi, p) > 0$, then 1.11(i) requires $z < u(\theta, \psi, p)$ when the real $u(\theta, \psi, p)$ does exist or is always valid when it does not exist. If $\eta(\theta, \psi, p) < 0$, no restriction on z is required from 1.11(i). The same analysis can be applicable to 1.11(ii) note the different signs of $\sin\theta, \sin\psi$.

(ii) $(p, q > 1; 0 < \theta + \psi \leq \pi/3)$: Only if $\eta(\theta, \psi, p) < 0$, then 1.13(i) requires $z < v(\theta, \psi, p)$ when the real $v(\theta, \psi, p)$ does exist or is always valid when it does not exist. The same analysis can be applicable to 1.13(ii).

2 Numerical examples

Note that all the theorems given in Section 1 are sufficient (a little too restrictive) ones. Hence, in practical calculation of the approximate offset curves, it would be better and easier to obtain the region on $z(= d\sin^2\{(\theta+\psi)/2\})$ containing 0 and satisfying

$$\begin{aligned} & b_0 > 0, \quad b_1 > 0 \\ & [\{2\cos(\theta+\psi) - 1\}z^2 + \{\mu(\theta, \psi, p)\sin\psi\}z + 3(p-1)\sin^2\psi] \\ & \times[\{2\cos(\theta+\psi) - 1\}z^2 + \{\mu(\psi, \theta, q)\sin\theta\}z + 3(q-1)\sin^2\theta] > 0 \end{aligned} \tag{2.1}$$

Given r_0, r_1, θ, ψ, first determine R_0, R_1, p, q from (1.7), and then solve the above inequalities (2.1).

Example 1: For $(r_0, r_1, \theta, \psi) = (1, 2, \pi/8, \pi/6) : \theta + \psi < \pi/3$, we obtain $(p, q) \approx (0.226357, 0.649232)$ and

$$\begin{aligned} & (2d + 0.539451)(0.391239d2d + 0.765367) > 0, (2d + 0.906353)(0.391239d + 1) > 0 \\ & (0.217523z^2 - 1.32508z - 0.580232)(0.217523z^2 - 0.116872z - 0.154106) > 0 \end{aligned}$$

from which we get $-2.09721 < d < 5.88991$.

Example 2. For $(r_0, r_1, \theta, \psi) = (1, 2; \pi/8, \pi/4) : \theta + \psi > \pi/3$, we obtain $(p, q) \approx$

(0.242851, 0.827883) and in addition

$$(2d+0.853553)(0.617317d+0.765367)>0,(2d+0.852254)(0.617317d+1.41421)>0$$
$$(-0.234633z^2-2.3248z-1.13572)(-0.234633z^2+0.17369z-0.0756177)>0$$

from which $d>-0.426127$ since the second quadratic equation is negative for any z.
Example 3: Next we choose an original S-shaped cubic spline with $(r_0, r_1, \theta, \psi) = (2, 4, \pi/3, -\pi/8)$; note that $\theta+\psi<\pi/3$. For $d=0.2$, $(c_0, c_1) \approx (1.17097, 0.96644)$. Figs 5-1 and 5-3 give the graphs of the original cubic curve, the exact offset and its cubic approximate curves (which should be improved by use of the following Klass's algorithm) with $d=\pm 0.2$ where Klass has proposed the following numerical algorithm([2]):

1. Calculate the offset segment.
2. Determine the distance between the original cubic spline and the offset one.
3. If the result is good enough, divide the original segment into two parts and start again.

Fig. 5-2 gives the original segment and its improved cubic offset composed of three segments for $t\in[0,0.3],[0.3,0.7],[0.7,1]$ where $(c_0,c_1)\approx(0.48809,0.43367),(0.45913,0.36613),(0.25386,0.27013)$, respectively. Figs 3-3 and 3-4 give the graphs for $d=-0.2$ where $(c_0,c_1)\approx(0.13277,0.104168)$.

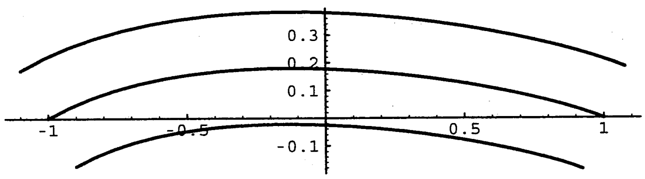

Fig. 3. Cubic spline and its spline offset curves with $d=\pm 0.2$ (Ex.1).

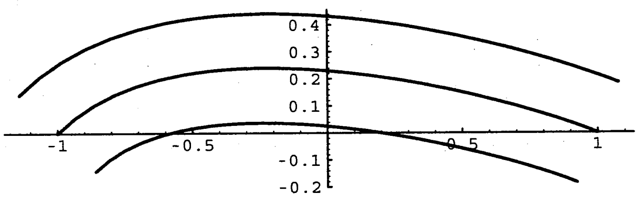

Fig. 4. Cubic spline and its spline offset curves with $d=\pm 0.2$ (Ex.2).

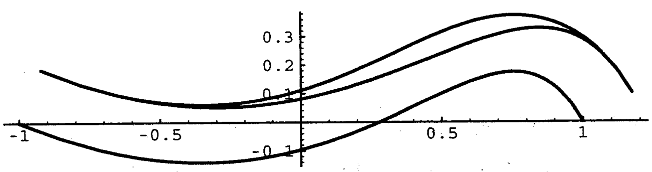

Fig. 5-1. Cubic spline, its offset curve and spline offset curve with $d=0.2$ (Ex.3).

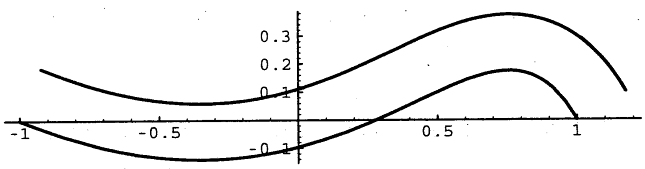

Fig. 5-2. Cubic spline and its spline offset curve of three parts with $d=0.2$ (Ex.3).

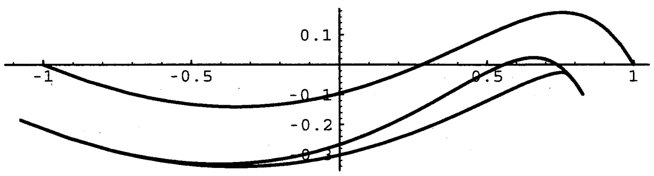

Fig. 5-3. Cubic spline, its offset curve and spline offset curve with $d = -0.2$ (Ex.3).

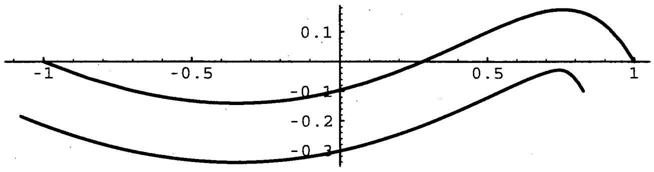

Fig. 5-4. Cubic spline and its spline offset curve composed of three parts with $d = -0.2$ (Ex.3).

Even when $0 < \theta, |\psi| < \pi$, we note that our analysis could be applicable even to the case, i.e., the nice cubic offset curve is obtained as Fig. 6-2.
Example 4. Consider the data: $(r_0, r_1, \theta, \psi) = (2, 4, \pi/3, 3\pi/4)$; note that $\theta + \psi = 13\pi/12 > \pi$. This case requires $d > -0.075$ and so the offset curve with the negative d is of no practical use.

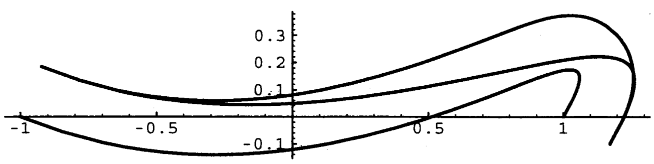

Fig. 6-1. Cubic spline and its spline offset curve composed of three parts with $d = 0.2$ (Ex.4).

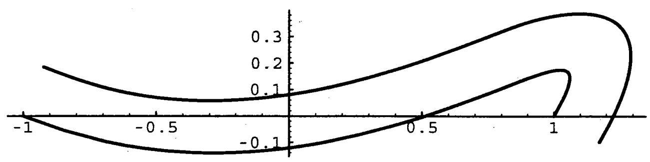

Fig. 6-2. Cubic spline and its spline offset curve composed of three parts with $d = 0.2$ (Ex.4).

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