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Reach Set Computations Using Real Quantifier Elimination

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Abstract

Reach set computations are of fundamental importance in control theory. We consider the reach set problem for open-loop systems described by parametric inhomogeneous linear differential systems and use real quantifier elimination methods to get exact and approximate solutions. The method employs a reduction of the forward and backward reach set and control parameter set problems to the transcendental implicitization problems for the components of special solutions of simpler non-parametric systems. For simple elementary functions we give an exact calculation of the cases where exact semialgebraic transcendental implicitization is possible. For the negative cases we provide approximate alternating using discrete point checking or safe estimations of reach sets and control parameter sets. Examples are computed using the REDLOG and QEPCAD packages.

1 Introduction

Today integrated systems which combine physical processes with information systems (*i.e.* digital programs) are in great demand. In fact complex systems which have been designed recently incorporate both differential equations to model the continuous behavior and discrete event systems to model instantaneous state changes in response to events. Systems that are finite state machines with differential equations at each discrete state are called *Hybrid Systems*.

A lot of research effort has been devoted to develop mathematical models, specification formalisms, analysis/design/control methods and tools to help control engineers in building such systems (see [18, 30, 26]). Most of the applications of hybrid systems are safety critical. Safety is usually encoded as avoidance of an undesirable region of the state space. Consequently, the most important problems for analyzing hybrid systems are verification problems; these are essentially reachability problems, that ask whether trajectories of the hybrid systems reach certain undesirable (unsafe) regions from an initial region.

Computing the reach set of hybrid systems is difficult because hybrid systems have an infinite state space. Due to the difficulty of computing the reach set for systems of differential equations, formal verification methods and tools for hybrid systems have been developed [2, 17]. These methods and tools, however, can deal with only very simple continuous models as, *e.g.* $\dot{x} = 1$,

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$A\dot{x} = b$. What is actually required is to handle hybrid systems with more complicated continuous parts.

Decidability of reachability problem for hybrid systems with linear differential equation of the form $\dot{y} = Ay + Bu$ is discussed in [23, 24]. This is a significant class of linear differential equations that is widely used in linear control theory. The results are based on the notion of "o-minimality"[16] from model theory and "quantifier elimination"[11]. O-minimality is used to define a class of hybrid systems "o-minimal hybrid systems" and it is shown that all o-minimal hybrid systems admit finite bisimulations in [22]. To make the bisimulation algorithm computationally feasible, they utilize mathematical logic, in particular, real quantifier elimination, as main tool to represent and manipulate sets symbolically. Since quantifier elimination, in general, is possible for the polynomial theory of reals [11], they have found subclasses of o-minimal hybrid systems that are definable in the theory.

Remark: There are many results that apply quantifier elimination to control theory [6, 15, 19, 4]. In [28, 3] quantifier elimination is used for verification problems (reachability and observability problems) of discrete-time polynomial systems.

In this paper we study in particular reach set problems for continuous open-loop systems described by parametric systems of linear differential equations [21]. Roughly speaking reach set problems are concerned with the relations between possible values of the state variables at some initial time t_0 and the corresponding values at later points in time. The specific problems studied in this paper are the following:

1. Fix a set M of values of the state variables at t_0; what are the possible corresponding values at later points t in time (up to some bound t_1 or ∞). (*Forward reach set*)

2. Fix a set N of "safe" values of the state variables. Find a set M as large as possible of initial values of the state variables at time t_0 that guarantees that the values of the state variables will for all later time points t (up to some bound t_1 or ∞) remain inside N. (*Backward reach set*)

3. Fix a set M of values of the state variables at t_0 and a set N of "safe" values of the state variables. Find a set P as large as possible of the control parameters such that all state variables with initial values at t_0 in M will have values in N for all later time points t (up to some bound t_1 or ∞.) (*Control parameter set*)

Our main tool is the method of real quantifier elimination in computer algebra. This approach was introduced into reach set computations in [29]. In a series of papers they showed how to get exact solutions of the forward reach set problem for certain homogeneous linear differential systems of special type with constant coefficients [23] and for associated inhomogeneous systems with very special right hand side [24]. The exact solutions are always obtained as semialgebraic sets described by a boolean combination of polynomial inequalities.

Here we extend this ad hoc approach for special types of differential systems to a systematic study of the type of results obtainable by an approach via real quantifier elimination. By reducing the approach to its bare essentials, we obtain a much wider systematic framework applicable to a considerably larger class of systems. The main observation is that all the problems mentioned above can be reduced by exact symbolic algorithms to an implicitization problem for certain basic transcendental functions associated with the given system. Exact solutions for implicitization problems with rational parametrizations are well-known [8, 27]. Here we deal with the corresponding problem for transcendental parametrizations that has been studied only for special cases *e.g.* in [13, 20].

Our main results are as follows: We associate with every parametric linear system of differential equations $\dot{y} = A(t)y + b(t, \underline{r})$ a finite system F of basic functions. Then for semialgebraic sets M, N all three problems can be solved exactly by real quantifier elimination relative to the implicitization problem for the components of the functions in F. Moreover the discrete point version of these problems require only finitely many evaluations of functions in F. We prove a theorem that determines the exact classes of vector-valued functions of the kind arising in linear differential systems with constant coefficients, where exact semialgebraic implicitization is possible. As a corollary we obtain the exact limitations of the approach of [23, 24] for linear differential systems with constant coefficients and special right hand sides.

We propose several ways to overcome these limitations by approximate computations: One way is to compute exact reach sets at a finite selection of discrete time points. This is always possible and practically quite efficient, but may lead to underestimation of the true forward reach set, depending on the selection of time points. Another approach separates the common time variable into different time variables. This leads to an overestimation in the implicitization problem resulting in an overestimation of the forward reach set and an underestimation of the backward reach set and the control parameter set: So all three approximations are on the safe side.

We illustrate some problems and solution methods by examples computed in the REDLOG-package of REDUCE [14] and QEPCAD [12]. We expect that our results can be extended to the hybrid systems with linear continuous parts.

2 Reach sets & transcendental implicitization problem

2.1 Problem statement

We consider parametric inhomogeneous systems S of linear differential systems of the form $\dot{y} = A(t)y + b(t, \underline{r})$ with an $n \times n$ matrix $A(t)$ of real continuous functions $a_{ij}(t)$ and a vector-valued real continuous function $b(t, \underline{r})$ defined on some interval I. The inhomogeneous part is assumed to be a linear combination $b(t, \underline{r}) = \sum_{i=1}^{k} r_i g_i(t)$ with continuous functions $g_i : I \longrightarrow \mathbf{R}^n$, and real parameters r_i. Such a system can be viewed as an continuous open-loop control system with control parameters $\underline{r} = (r_1, \ldots, r_k)$. Let M be some subset of $\mathbf{R}^n$ and fix an initial time point $t_0 \in I$: Then we denote the set of all solution functions $f : I \longrightarrow \mathbf{R}^n$ of the given system with parameters $\underline{r} = (r_1, \ldots, r_k)$, by F_r, and the set of all solution functions $f \in F_r$ with initial value $f(t_0) \in M$ by $F = F_{M,r}$.

We consider the following *forward reach set problems*:

discrete reach sets Compute for finitely many time points $t_1 < \ldots < t_m$ in I the union of the sets $\{f(t_i) \mid f \in F_{M,r}\}$.

bounded reach set Compute for a given time $t_1 > t_0$ in I the set $\{f(t) \mid f \in F_{M,r}, t_0 \leq t \leq t_1\}$.

unbounded reach set Suppose $I \supseteq [t_0, \infty)$, and compute the set $\{f(t) \mid f \in F_{M,r}, t_0 \leq t\}$.

All computations should be performed in explicit dependence on the control parameters $\underline{r}$. Any solution of the discrete reach sets problem yields an lower estimate for the sets to be computed in the bounded and unbounded reach set problems.

Of equal interest are the corresponding "backward" reach set problems that are a kind of "dual" to the corresponding "forward" problems.

Some *backward reach set problems* are as follows: Let N be a subset of $\mathbf{R}^n$.

backward discrete reach sets Compute for finitely many time points $t_1 < \ldots < t_m$ in I the sets $\{f(t_0) \mid f(t_1), \ldots, f(t_m) \in N\}$.

backward bounded reach set Compute for a given time $t_1 > t_0$ in I the set $\{f(t_0) \mid f(t) \in N \text{ for all } t_0 \leq t \leq t_1\}$.

backward unbounded reach set Suppose $I \supseteq [t_0, \infty)$, and compute the set $\{f(t_0) \mid f(t) \in N \text{ for all } t_0 \leq t\}$.

From the viewpoint of control theory these problems have still other variants concerning the determination of suitable control parameter values $\underline{r} = (r_1, \ldots, r_k)$. Let M as before be a subset of $\mathbf{R}^n$, and let N be another subset of $\mathbf{R}^n$. Then we have the following natural *control parameter set problems*:

discrete point control Compute for finitely many time points $t_1 < \ldots < t_m$ in I the set $\{\underline{r} \in \mathbf{R}^k \mid f(t_i) \in N \text{ for all } f \in F_{M,r}, 1 \leq i \leq m\}$.

bounded interval control Compute for a given time $t_1 > t_0$ in I the set $\{\underline{r} \in \mathbf{R}^k \mid f(t) \in N \text{ for all } f \in F_{M,r}, t_0 \leq t \leq t_1\}$.

unbounded interval control Suppose $I \supseteq [t_0, \infty)$, and compute the set $\{\underline{r} \in \mathbf{R}^k \mid f(t) \in N \text{ for all } f \in F_{M,r}, t_0 \leq t\}$.

In order to make these problems mathematically precise, we need to specify the way in which the input sets M and N, and the output sets should be described. For an approach using symbolic computations it is natural to consider *semialgebraic* sets as possible inputs. These are subsets of $\mathbf{R}^n$ described by a boolean combination $\varphi(x_1, \ldots, x_n)$ of real polynomial inequalities. If in addition all the polynomials involved in $\varphi(x_1, \ldots, x_n)$ are linear, then the set described by φ is called *semilinear* [16, 32].

Our goal is to solve the forward and backward reach set and control parameter set problems for semialgebraic input sets as far as possible with descriptions of semialgebraic sets as outputs. This, however, is not always possible. Hence we consider also the computation of overestimating the forward reach sets and underestimating the backward reach set and the control parameter sets by suitable semialgebraic sets.

Our main tool will be a reduction of reach set and control parameter set computations to corresponding implicitization problems for a fixed finite system of functions associated with S, namely a fundamental system $f_1, \ldots, f_n$ for the homogeneous system S_0 associated with S, and special solutions h_i of the parameter-free inhomogeneous system S_i given by $\dot{y} = A(t)y + g_i(t)$ for $1 \leq i \leq k$. We refer to $\{f_1, \ldots, f_n, h_1, \ldots, h_k\}$ as a system of *basic functions* for S.

Implicitization problems for rational parametrizations of algebraic varieties have been widely considered in computer algebra [8, 27]. Here we have to study the corresponding problem for the vector-valued functions $f_1, \ldots, f_n, h_1, \ldots, h_k$, arising from the system S. As these functions will in general be transcendental, we refer to these problems as *transcendental implicitization problems*.

More precisely, we consider the following transcendental implicitization problems for given functions $f_i : I \longrightarrow \mathbf{R}^n$ for $1 \le i \le k$:

discrete points implicitization Compute for finitely many time points $t_1 < \ldots < t_m$ in I the values $(f_1(t_i), \ldots f_k(t_i))$, regarded as points in $\mathbf{R}^{nk}$.

bounded implicitization Compute for a given time $t_1 > t_0$ in I the set $\{(f_1(t), \ldots f_k(t)) \in \mathbf{R}^{nk} \mid t_0 \le t \le t_1\}$.

unbounded implicitization Suppose $I \supseteq [t_0, \infty)$, and compute the set $\{(f_1(t), \ldots f_k(t)) \in \mathbf{R}^{nk} \mid t_0 \le t\}$.

The first problem amounts to simple evaluations of the given functions. Notice that the unbounded and bounded implicitization problem for a single solution of the differential system S is in fact a special case of the unbounded and bounded forward reach set problem for S, respectively, namely for the case of a singleton set M.

2.2 Reduction to implicitization problems

Next we show that all reach set computations and control parameter set computations listed above can for semialgebraic input sets M, N be reduced in an exact symbolic way to one of these implicitization problems. All these reductions require real quantifier elimination as fundamental tool. For the case of discrete points forward and backward reach set and control parameter set and semilinear input sets M, N we find moreover that the output sets are also semilinear.

Let $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ be quantifier-free formulas describing the semialgebraic input sets M and N, respectively. Let $\dot{y} = Ay + b(t, \underline{r})$ with $b(t, \underline{r}) = \sum r_i g_i(t)$ be a parametric linear system S with control parameter r_i. Let f_i be a fundamental system of solutions of $\dot{y} = Ay$. Let h_i be a special solution of the system $\dot{y} = Ay + g_i(t)$. Then by the superposition principle, a special solution of the system S is given by $\sum_{i=1}^{k} r_i h_i$. Note that here r_i's may be regarded as constants or as free parameters. Then it is straightforward to write down first-order formulas describing the respective forward and backward reach sets and control parameter sets in terms of evaluations of the basic functions $f_1, \ldots, f_n, h_1, \ldots, h_k$, the given formulas $\varphi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n)$ and a quantifier-free formula $\mu(y_{11}, \ldots, y_{1n}, \ldots, y_{n1}, \ldots, y_{nn})$ describing the combined range of $(f_1, \ldots, f_n, h_1, \ldots, h_k)$, as a semialgebraic set. All these formulas will involve several quantifiers over real numbers. By real quantifier elimination one can construct equivalent quantifier-free formulas, and thus get the desired semialgebraic descriptions.

We will exhibit concrete first-order formulas for some reach set problems and control parameter set problem. The remaining cases are handled similarly in [5]. The forward discrete reach set problem can be described by the following formula and hence be solved by real quantifier elimination and evaluation of the basic functions at finitely many points.

$$\exists x_1 \ldots \exists x_n (\varphi(\textstyle\sum_i x_i f_i + \sum_i r_i h_i)(t_0) \wedge [\bigwedge_{j=1}^{n} y_j = (\sum_i x_i f_{ij} + \sum_i r_i h_{ij}(t_1)) \\ \vee \ldots \vee \bigwedge_{j=1}^{n} y_j = (\sum_i x_i f_{ij} + \sum_i r_i h_{ij})(t_m)]).$$

Next suppose we have a quantifier-free formula $\mu(y_{11},\ldots,y_{1n},\ldots,y_{n1},\ldots,y_{nn},z_{11},\ldots,z_{1n},\ldots,z_{k1},\ldots,z_{kn})$ describing the combined range of $(f_1,\ldots,f_n,h_1,\ldots,h_k)$ on the interval $[t_0,\infty)$ or $[t_0,t_1]$. So $\mu(y_{11},\ldots,z_{kn})$ holds for $n(k+n)$-tuple in $\mathbf{R}^{n(k+n)}$ if and only if this tuple is in the combined range of $(f_1,\ldots,f_n,h_1,\ldots,h_k)$ on the given interval. Then the forward bounded and unbounded reach set problem, respectively, can be described by the following formula and hence solved by real quantifier elimination:

$$\exists x_1\ldots\exists x_n[\varphi(\textstyle\sum_i x_i f_i+\sum_i r_i h_i)(t_0)\wedge\exists y_{11}\ldots\exists y_{nn}\exists z_{11}\ldots\exists z_{kn}(\mu(y_{11},\ldots,z_{kn})\\ \wedge\bigwedge_{j=1}^{n} y_j=(\sum_i x_i y_{ij}+\sum_i r_i z_{ij}))].$$

With the same formula μ, the backward bounded and unbounded reach set problem, respectively, can be described by the following formula and hence solved by real quantifier elimination:

$$\exists x_1\ldots\exists x_n[\textstyle\bigwedge_{j=1}^{n} y_j=(\sum_i x_i f_{ij}+\sum_i r_i h_{ij})(t_0))\wedge\forall y_{11}\ldots\forall y_{nn}\forall z_{11}\ldots\forall z_{kn}\\ (\mu(y_{11},\ldots,z_{kn})\to\psi(\sum_i x_i\underline{y_i}+\sum_i r_i\underline{z_i})(t))].$$

Finally, the bounded interval control problem and the unbounded interval control problem, respectively, can be described by the following formula and hence solved by real quantifier elimination:

$$\exists x_1\ldots\exists x_n[\varphi(\textstyle\sum_i x_i f_i+\sum_i r_i h_i)(t_0)\wedge\forall y_{11}\ldots\forall y_{nn}\forall z_{11}\ldots\forall z_{kn}\\ (\mu(y_{11},\ldots,z_{kn})\to\psi(\sum_i x_i\underline{y_i}+\sum_i r_i\underline{z_i})(t))].$$

3 Exact transcendental implicitization

Here we consider cases, where the unbounded and bounded transcendental implicitization problem for given functions $f_i: I\longrightarrow\mathbf{R}^n$ $(1\le i\le k)$ has an exact solution. Notice that the transcendental implicitization problem refers only to the component functions $f_{ij}(t)$ of $f_i(t)$; the grouping of these component functions into vector-valued functions is irrelevant here. So we may assume w.l.o.g. that $k=1$ and that we deal with a single vector-valued function $f(t):=(f_1(t),\ldots,f_n(t))$. Then the exact transcendental implicitization problem is to determine the range of $f(t)$ on an unbounded interval $[t_0,\infty)$, or a compact interval $[t_0,t_1]$ contained in I. Since the f is continuous, this range is always a connected subset of $\mathbf{R}^n$.

In particular for $n=1$ the range is a real interval J; moreover J is compact for the bounded implicitization case. In the unbounded implicitization case J is compact iff f is bounded on $[t_0,\infty)$, otherwise it is a closed semiinfinite interval or all of $\mathbf{R}$. In particular J is always a semialgebraic set that can computed explicitly from upper and lower bounds for f. In other words the unbounded and the bounded transcendental implicitization problem always has a positive solution for $n=1$.

For $n=2$ there are two well-known cases, where exact unbounded and bounded implicitization is possible, namely the sin-cos-pair and the sinh-cosh-pair: If f has components $f_1:=\cos(p(x))$, $f_2:=\sin(p(x))$, where $p(x)$ is a real polynomial of positive degree, then the range of $p(x)$ on $[t_0,\infty)$ includes an unbounded interval; consequently the range of f on $[t_0,\infty)$ is exactly the unit circle $\{(x_1,x_2)\mid x_1^2+x_2^2=1\}$. On a bounded interval $[t_0,t_1]$, the range of $p(x)$ is again a compact interval, and so the range of f is a connected subset of the circle that can be easily computed as semialgebraic set from the range of $p(x)$. For the hyperbolic case, where

$f_1 := \cosh(p(x)), f_2 := \sinh(p(x))$, the situation is analogous, except that the role of the circle is replaces by the hyperbola $\{(x_1, x_2) \mid x_1^2 - x_2^2 = 1\}$.

The next theorem shows that exact transcendental implicitization is preserved under composition of functions in a very general sense:

Theorem 1 *Let $f(t) := (f_1(t), \ldots, f_k(t))$ be a vector valued function such that the range of f on every compact or unbounded closed interval I is a semialgebraic set described by a quantifier-free formula $\varphi_I(x_1, \ldots, x_k)$. Let g be a continuous real function defined on some compact or upper semiinfinite closed interval I'. Let h_i $(1 \leq i \leq n)$ be semialgebraic real functions defined on some subset of $\mathbf{R}^n$ extending the range of f. Let $\rho_i(x_1, \ldots, x_n, y)$ be quantifier-free formulas defining the graph $\{(x_1, \ldots, x_n, y) \mid y = h_i(x_1, \ldots, x_n)\}$ of h_i. Then the vector-valued function $f^*(t) := (f_1^*(t), \ldots, f_n^*(t))$ with components $f_i^*(t) := h_i(f_1(g(t)), \ldots, f_n(g(t)))$ for $1 \leq i \leq n$ has a semialgebraic range described by the formula*

$$\psi(x_1, \ldots, x_n) := \exists y_1 \ldots \exists y_n (\varphi_J(y_1, \ldots, y_n) \wedge \bigwedge_{i=1}^{n} \rho_i(y_1, \ldots, y_n, x_i)),$$

where J is the range of $g(t)$ on I'.

The proof is obvious. Notice that the algorithmic quantifier elimination for the ordered field of real numbers this formula is required in order to transform the formula ψ into an equivalent quantifier-free formula that describes the range of f^* as a semialgebraic set. Typical instances of g and h_i are real polynomials or real rational functions. The method can in particular be applied to the situation, where f consists of a sin-cos-pair or a sinh-cosh-pair as described above. Other interesting examples are pairs $(\wp, \wp')$, where $\wp(t)$ is a Weierstrass $\wp$-function [1]. Then the range of $(\wp, \wp')$, on a large enough interval is a real elliptic curve $\{(x, y) \mid y^2 = 4x^3 - g_2 x - g_3\}$. See [5] for the more examples.

4 Semialgebraic implicitization for simple elementary functions

In this section we characterize those cases of linear differential systems S with constant coefficients and "simple right hand side", where an exact implicitization of the system of basic functions for S is possible. The condition on the right hand side $b(t)$ of the system is as follows: All components $b_i(t)$ of $b(t)$ are $\mathbf{R}$-linear combinations of functions of the form $t^{d_i} e^{a_i t} \cos(\omega_i t)$, $t^{d_i} e^{a_i t} \sin(\omega_i t)$, where d_i are non-negative integers and a_i, ω_i, α_i are real numbers. Then it is well known that a special solution of the inhomogeneous system and the fundamental solutions of the homogeneous system are again real linear combinations of functions of this kind. We call linear systems of this form *regular* and functions of type $t^d e^{at} \cos(\omega t)$, $t^d e^{at} \sin(\omega t)$, with a, ω, α real numbers *simple elementary functions.* In some special cases of regular systems, it has been shown how to solve the reach set problem by an implicit semialgebraic implicitization of functions of the following type in [23, 24, 22] : (i) real polynomials $p_i(t)$, (ii) exponential functions $e^{a_i t}$ with rational values of a_i. (iii) trigonometric functions $\cos(\omega_i t)$, $\sin(\omega_i t)$, for rational ω_i.

In the following we show that for simple elementary functions there are only few more cases which allow unbounded exact semialgebraic implicitization; all these cases are covered by Theorem 1 of the last section. In most of the remaining cases the exact semialgebraic implicitization problem is unsolvable. In fact we provide a complete characterization of those cases, where unbounded semialgebraic implicitization is possible.

Let $f(t) := (f_1(t), \ldots, f_n(t))$ with non-constant, pairwise different component functions $f_i(t) := t^{d_i} e^{a_i t} \cos(\omega_i t)$, or $f_i(t) := t^{d_i} e^{a_i t} \sin(\omega_i t)$, where d_i are non-negative integers and a_i, ω_i are real numbers. Moreover we assume that the functions f_i appear in cos-sin-pairs, whenever $\omega_i \neq 0$.

Theorem 2 *Let $f : [t_0, \infty) \longrightarrow \mathbf{R}^n$ be as above and let $n \geq 2$. Then the range of f is a semialgebraic set iff one of the following holds:*

1. *For all $1 \leq i \leq n$, $f_i(t) := t^{d_i}$.*
2. *For all $1 \leq i \leq n$, $d_i = 0$, $f_i(t) := e^{a_i t}$ and $dim_{\mathbf{Q}}(span(a_1, \ldots, a_n)) \leq 1$.*
3. *For all $1 \leq i \leq n$, $d_i \neq 0$, $a_i \neq 0$, $f_i(t) := t^{d_i} e^{a_i t}$, and $dim_{\mathbf{Q}}(span(a_1, \ldots, a_n)) \leq 1$, and $\frac{d_i}{d_1} = \frac{a_i}{a_1}$.*
4. *For all $1 \leq i \leq n$, $f_i(t) := \cos(\omega_i t)$, or $f_i(t) := \sin(\omega_i t)$, and $dim_{\mathbf{Q}}(span(\omega_1, \ldots, \omega_n)) \leq 1$.*

Moreover in these positive cases a quantifier-free formula describing the range of f can be computed algorithmically over the reals.

Idea of the Proof. In the cases mentioned above the unbounded semialgebraic implicitization is always achieved by the methods of the previous section, in particular Theorem 1. It remains to show that in all other cases the range of f is not a semialgebraic set. This requires a case distinction. In each case we show that the assumption that the range of f is semialgebraic leads to a contradiction. Based on the assumption that the range of f is semialgebraic we construct new semialgebraic sets with impossible properties. Either this set is one dimensional such that neither the set nor its complement is a finite union of intervals or it describes the graph of a semialgebraic function with an impossible rate of growth (compare [9]). See [5] for details of the proof.

This theorem clearly shows the limitations of the approach presented in [23, 24]. In fact we have the following immediate corollary:

Corollary 3 *Let $\dot{y} = Ay$ with constant $n \times n$-matrix A be a homogeneous system of linear differential equations. Then exact semialgebraic implicitization is possible for a fundamental system of solutions of the system iff one of the following cases holds:*

1. *All eigenvalues of A are zero,* i.e. *A is a nilpotent matrix.*
2. *All eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are non-zero, pairwise distinct reals, and $dim_{\mathbf{Q}}(span(\lambda_1, \ldots, \lambda_n)) \leq 1$.*
3. *All eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are purely imaginary, say of the form $\lambda_i = \mu_i \sqrt{-1}$ with non-zero pairwise distinct reals μ_i, and $dim_{\mathbf{Q}}(span(\mu_1, \ldots, \mu_n)) \leq 1$.*

5 Approximate solutions

In this section we study the cases, where an exact semialgebraic unbounded or bounded implicitization is definitely not possible. In these cases we want to find a semialgebraic superset of the true forward reach set and a semialgebraic subset of the true backward reach set or the true control parameter set, both if possible such that the set difference to the true reach set or control parameter set is in some sense "small enough." Then an inspection of the reduction formulas shows that an overestimation of the implicitization problem leads to an overestimation of the forward reach set and an underestimation of the backward reach set and of the control parameter set *i.e.* for "safe" estimations. Hence we are reduced to the problem of finding a semialgebraic superset of the true range of a transcendental vector valued function on a compact or upper semiinfinite closed interval.

One strategy to find overestimations of the range is *separation of variables:* It comes in two flavours: Separation of variables in different components, and separation of variables in products.

Separation of variables in different components : Let $f(t) = (f_1(t), \ldots, f_n(t))$ be defined on an interval I. Then separation of variables in different components yields the function $g(t) = (f_1(t_1), \ldots, f_n(t_n))$ defined on the cube I^n with range$(g) \supseteq$ range(f). The range of g is easily computed as a box $J_1 \times \cdots \times J_n$, where J_i is the range of f_i. Notice that this box is in fact the smallest box containing the range of f.

Separation of variables in products : Suppose the component functions of the given functions are products $f_i(t) := f_{i,1}(t) \cdots f_{i,m}(t)$, where each $f_{i,j}(t)$ is defined on the interval I. Put $g_j(t) := (f_{1,j}, \ldots, f_{n,j})^T$. Then each g_j is also defined on the interval I. Let B_j be the range g_j, and put $C := B_1 \cdots B_m$, where the multiplication is performed on the elements componentwise. Then C is obviously a superset of the range of f.

Example. 1 Let I be the upper semiinfinite interval $[0, \infty)$.

1. Let $f_1 := \cos(t)$, $f_2 := \sin(t)$. Then the true range of f is the unit circle. Separation of variables in different components yields as overestimation the closed unit square.

2. Let $f_1 := \cosh(t)$, $f_2 := \sinh(t)$. Then the true range of f is the hyperbola $\{(x, y) \mid x^2 - y^2 = 1\}$. Separation of variables in different components yields as overestimation the "quadrant" $\{(x, y) \mid x, y \geq 1\}$.

3. Let $f_1 := e^t \cos(t)$, $f_2 := e^t \sin(t)$. Then the true range of f is an expanding exponential spiral. Separation of variables in different components yields as overestimation the full plane $\mathbf{R}^2$. Separation of variables in products yields as better overestimation the annulus $\{(x, y) \mid x^2 + y^2 \geq 1\}$.

4. Let $f_1 := e^{-t} \cos(t)$, $f_2 := e^{-t} \sin(t)$. Then the true range of f is a contracting exponential spiral. Separation of variables in different components yields as overestimation a closed box $\{(x, y) \mid -e^{\pi} \leq x \leq 1, -e^{3\pi/2} \leq y \leq e^{\pi/2}\}$. Separation of variables in products yields as overestimation the closed disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$. These approximations are incomparable. So their intersection is a common improvement of both.

6 Complexity

In this section we briefly discuss the complexity of our algorithms. From the results on complexity of quantifier elimination in [7] we can give upper bounds for the asymptotic complexity of our approach:

Discrete point reach set problems are described by purely existential formulas. Hence the complexity of quantifier elimination is at most simply exponential in the dimension of the differential system. For fixed dimension it the computation runs in a polynomial time. The complexity of bounded and unbounded reach set problems is the same as for the discrete reach set problem for a fixed number m of points. The backward discrete reach set problems can be solved in singly exponential time. The complexity of backward bounded and unbounded reach set computation is of type $e^{n^{O(1)}}$ (generalized singly exponential). The upper complexity bounds for the control parameter set problems are same as for the corresponding backward reach set problems.

7 Computational example in REDLOG and QEPCAD

In this section we report on experimental results in reach set and control parameter set computation. In [5] we have presented experimental results for numerous examples that illustrate the different problem types and solution methods. Here we display only one of these examples with non-constant coefficients to show the generality of the approach. All computations are performed in the REDLOG package [14] of REDUCE 3.7 and QEPCAD [12] [1]. The main algorithm employed is the linear and quadratic quantifier elimination [25, 31] of REDLOG and quantifier elimination based on cylindrical algebraic decomposition [12] of QEPCAD.

Example. 2 Consider the inhomogeneous system $\dot{y} = Ay + b$ with

$$A := \begin{pmatrix} 0 & 2t \\ -2t & 0 \end{pmatrix}, \quad b := r_1 \begin{pmatrix} 2t\cos(t^2) \\ 2t\sin(t^2) \end{pmatrix}.$$

Then basic functions are $\begin{pmatrix} \sin(t^2) \\ \cos(t^2) \end{pmatrix}, \begin{pmatrix} \cos(t^2) \\ -\sin(t^2) \end{pmatrix}, \begin{pmatrix} \sin(t^2) \\ 0 \end{pmatrix}$. For this system we illustrate the computations in the forward/backward unbounded reach set and the control parameter set problems below (Note that we set $t_0 = 0$):

• **Forward unbounded reach set:** A quantifier-free formula $\mu(y_{11}, y_{12}, y_{21}, y_{22}, z_{11}, z_{12})$ is obtained from the following first-order formula μ_o

$$\mu_o = \exists u \exists v (u^2 + v^2 = 1 \wedge y_{11} = v \wedge y_{12} = u \wedge y_{21} = u \wedge y_{22} = -v \wedge z_{11} = v \wedge z_{12} = 0))$$

by using quantifier elimination. By using REDLOG we have

$$\mu := y_{11}^2 + y_{12}^2 - 1 = 0 \wedge y_{11} + y_{22} = 0 \wedge y_{11} - z_{11} = 0 \wedge y_{12} - y_{21} = 0 \wedge z_{12} = 0$$

in 10 ms. Then we set $r_1 = 1$ and moreover $\varphi = (0 \leq x_1 \leq 1 \ \wedge \ x_2 = 0)$. Then forward unbounded reach set problem is solved by using real quantifier elimination for the following first-order formula $freach$;

[1] All the computations are executed on a SUN SPARC station Ultra I (140MHz).

$$freach = \exists x_1(\varphi \wedge \ freachaux)$$

where

$$freachaux = \exists y_{11}\exists y_{12}\exists y_{21}\exists y_{22}\exists z_{11}\exists z_{12}(\mu \wedge y_1 = x_1y_{11} + x_2y_{21} + r_1z_{11} \\ \wedge y_2 = x_1y_{12} + x_2y_{22} + r_1z_{12})$$

By using QEPCAD for $freach$ we obtain as an answer for the forward unbounded reach set; $y_1^2 + 4y_2^2 - 4 <= 0$ in 10 ms.

• **Backward unbounded reach set:** μ is the same formula as in forward unbounded reach set. We also set $r_1 = 1$ and $\psi(x_1, x_2) = (-\frac{1}{2} \leq x_1 \leq \frac{1}{2} \wedge -\frac{1}{2} \leq x_1 \leq \frac{1}{2})$. Then the backward unbounded reach set problem is solved by using real quantifier elimination for the following first-order formula $breach$;

$$breach = \exists x_1\exists x_2(y_1 = x_2 \wedge y_2 = x_1 \wedge breachaux)$$

where

$$breachaux = \forall y_{11}\forall y_{12}\forall y_{21}\forall y_{22}\forall z_{11}\forall z_{12}(\mu \rightarrow (-\tfrac{1}{2} < x_1y_{11} + x_2y_{21} + r_1z_{11} < \tfrac{1}{2} \\ \wedge -\tfrac{1}{2} < x_1y_{12} + x_2y_{22} + r_1z_{12} < \tfrac{1}{2})$$

By using REDLOG for $breach$ we obtain in 420 ms a semialgebraic description of the backward unbounded reach set consisting of 21 atomic formulas.

• **Control parameter set:** The formula μ is the same as in the reach set cases. We also set $\varphi = (0 \leq x_1 \leq 1 \wedge x_2 = 0)$ and $\psi(x_1, x_2) = (-\frac{1}{2} < x_1 < \frac{1}{2} \wedge -\frac{1}{2} < x_1 < \frac{1}{2})$. Then control parameter set problem is solved by using real quantifier elimination for the following first-order formula p$control$;

$$control = \exists x_1(\varphi \wedge controlaux)$$

where

$$controlaux = \forall y_{11}\forall y_{12}\forall y_{21}\forall y_{22}\forall z_{11}\forall z_{12}(\mu \rightarrow (-\tfrac{1}{2} < x_1y_{11} + x_2y_{21} + r_1z_{11} < \tfrac{1}{2} \\ \wedge -\tfrac{1}{2} < x_1y_{12} + x_2y_{22} + r_1z_{12} < \tfrac{1}{2})$$

By using REDLOG for $control$ we obtain in 70 ms a semialgebraic description of control parameter set consisting of 12 atomic formulas. It can be simplified to the result $-1 \leq r_1 < \frac{1}{2}$ by hand calculation.

8 Conclusions

In this paper we have studied forward and backward reach set and control parameter set problems for continuous parametric open-loop systems described by a system of parametric linear differential equations with arbitrary coefficients.

The approach using quantifier elimination was introduced into reach set computations in [29]. We extend their ad hoc approach for special types of differential systems to a systematic study of the type of results obtainable by an approach via real quantifier elimination. Thus we obtain a much wider systematic framework applicable to a considerably larger class of systems. The main observation is that all the problems can be reduced by exact symbolic algorithms to an implicitization problem for certain basic transcendental functions associated with the given system.

We have proved a theorem that determines the exact classes of vector-valued functions of the kind arising in linear differential systems with constant coefficients, where exact semialgebraic implicitization is possible. As a corollary we have obtained the exact limitations of the approach of [23, 24] for linear differential systems with constant coefficients and simple elementary inhomogeneous part. We have also proposed several ways to overcome these limitations by approximate computations. The problems have been illustrated by examples computed in the REDLOG-package of REDUCE and QEPCAD.

Further research will be concerned with an extension of these results to hybrid systems.

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