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Euler systems for Drinfeld modular varieties

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1 Introduction

A conjecture of Beilinson relates elements in the K-group of schemes with special values of L-functions. Beilinson gave such elements in the K-group of modular curves. Kato ([3]) constructs Euler systems, i.e., elements in K-groups that satisfy certain property under norm maps, and show that they give rise to special values, using Beilinson's result. We follow the analogy between function fields (resp. Drinfeld module of rank 2) and number fields (resp. elliptic curve) to construct elements in the K-groups of Drinfeld modular curve, and show that they are related, under a regulator map, to special values of L-functions attached to automorphic forms in positive characteristic. For Drinfeld modular varieties of higher dimensions, where the analogy is no longer applicable, we still have a series of elements in K-groups, which is proved to be an Euler system.

2 Euler system

We give the construction of elements in higher K-groups of Drinfeld modular varieties, and show that they form Euler systems. For more detail on the result in this section, see [5]. Let p be a prime, $q=p^f$, $f\in\mathbb{N}$, $A=\mathbb{F}_q[T]$, $K=\mathbb{F}_q(T)$, $O_\infty=\mathbb{F}_q[[1/T]]$, and $K_\infty=\mathbb{F}_q((1/T))$. A (resp. K, K_∞) is the analog of $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$). We refer the reader to [1] for the definition and properties of Drinfeld modules. For an ideal I of A and $d\in\mathbb{N}$, we write M_I^d for the moduli space of rank d, level I Drinfeld modules, and E_I^d for the universal Drinfeld module. The construction is in three steps. First we construct theta function $\theta\in O(E_I^d\setminus\{0\})^*$, which is determined by the location of zeros and norm invariance. Then we construct Siegel units $g_{a_1,\dots,a_d}\in O(M_I^d)^*$ where a_k are elements of $I^{-1}A/A$, as the specialization of the theta function at division points of the universal Drinfeld module. Lastly, we let

$$\kappa_I:=\{g_{\frac{1}{i},0,\dots,0},g_{0,\frac{1}{i},0,\dots,0},\dots,g_{0,\dots,0,\frac{1}{i}}\}\in K_d^M(K(M_I^d))$$

where i is a generator of I, $K(M_I^d)$ is the function field of M_I^d, and K_d^M is the Milnor K-group.

Theorem 2.1 (Norm property of Euler system). *Let $I\subset J\subset A$ be ideals, $d\in\mathbb{N}$.*

$$\mathrm{Norm}(\kappa_I)=\prod_{\substack{\wp:\mathrm{prime}\\ \wp|I,\ \wp\nmid J}}\sum_{k=0}^{d}[(-1)^k(\mathrm{N}\wp)^{\frac{k(k-1)}{2}}T_{k,\wp}^{(d)}](\kappa_J)$$

where $T_k, \wp^{(d)} : K_d^M(K(M_J^d)) \to K_d^M(K(M_J^d))$ *is induced by the Hecke correspondence associated to*

$$g_k^{(d)} = \begin{pmatrix} \pi & & & & & \\ & \ddots & & & & \\ & & \pi & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in GL_d(\hat{A} \otimes_A K).$$

Here, π *appears* k *times and* 1 *appears* $(d-k)$ *times.* Norm *is the homomorphism induced by the quotient map.*

This theorem shows that the elements constructed above form an Euler system.

Remark 2.2. When $d = 2$ and $I = \wp J$, the above theorem reads

$$\mathrm{Norm}(\kappa_I) = [T_{0,\wp}^{(2)} - T_{1,\wp}^{(2)} + (\mathrm{N}\wp)T_{2,\wp}^{(2)}](\kappa_J).$$

One can see that it resembles the Euler factor at $\wp$ of L-function of modular forms.

3 Regulator

We define a homomorphism, which we call regulator map, from K-group to the space of automorphic forms. The construction has been done in §7 of [4]; we merely translate it into our context.

3.1 the source

We fix an ideal I. The moduli space M_I^2 is a 2-dimensional scheme over $\mathbb{F}_q$. We denote by $\mathscr{M}_I^2$ the compact model over $\mathbb{F}_q$. We let

$$H^0(M_I^2, \mathscr{K}_2) := \mathrm{Ker}\left[\bigoplus_{z \in (M_I^2)_2} K_2^M(\kappa(z)) \to \bigoplus_{z \in (M_I^2)_1} K_1^M(\kappa(z))\right]$$

where $(M_I^2)_2$ (resp. $(M_I^2)_1$) denotes the set of points of codimension 2 (resp. 1), and $\kappa(z)$ the residue field at z. This group is the source of our regulator map.

3.2 the target

The closed complement $Y := \mathscr{M}_I^2 \setminus M_I^2$ is a 1-dimensional scheme over $\mathbb{F}_p$ whose dual graph is the quotient graph of the tree associated to Drinfeld upper half plane by the action of the congruence subgroup

$$\Gamma(I) = \left\{ X \in \mathrm{GL}_2(A) \,\middle|\, X \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{I} \right\}.$$

It is, in general, an infinite graph ([7]) but we simply ignore the half lines and consider its finite subgraph. Let $H^1(M_I^2)$ be the set of functions $f : E \to \mathbb{C}$ satisfying the following conditions:

- $f(\gamma e) = f(e)$ $(\gamma \in \Gamma(I),\ e \in E)$.
- harmonic, i.e., $\displaystyle\sum_{\beta \in \mathrm{GL}_2(O_\infty)/\mathfrak{J}} f(X\beta) = 0$
- alternating, i.e., $f\left(X\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right) = -f(X)$.
- f has compact support modulo $\Gamma(I)$, i.e., there are only finitely many elements X in $\Gamma(I)\backslash \mathrm{GL}_2(K_\infty)/\Gamma_\infty K_\infty^*$ with $f(X) = 0$.

Here we let $\mathfrak{J} = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(O_\infty) \,\middle|\, z \equiv 0 \pmod{1/T} \right\}$. This is the target group of our regulator map. These functions are studied in [6] as automorphic forms of Drinfeld type.

3.3 the map

We define the map:

$$\mathrm{reg} : H^0(M_I^2, \mathcal{K}_2) \to H^1(M_I^2).$$

Let $\{f_1, f_2\} \in H^0(M_I^2, \mathcal{K}_2)$, and choose lifts $f_1', f_2' \in K(M_I^2)$. An oriented edge $e \in E$ corresponds to a singular point where two divisors with support in Y meet. Let the curve corresponding to the source (resp. target) be C_1 (resp. C_2). Then we let

$$\mathrm{reg}(\{f_1, f_2\})(e) := \det \begin{pmatrix} \mathrm{ord}_{C_1} f_1' & \mathrm{ord}_{C_2} f_1' \\ \mathrm{ord}_{C_1} f_2' & \mathrm{ord}_{C_2} f_2' \end{pmatrix}.$$

It can be verified that this map is well defined.

4 Values of L-functions

4.1 Definition

We use Fourier coefficients to define L-function. For details on Fourier analysis over function fields, see [2], [8]. Take an element g of $H^1(M_I^2)$. It has Fourier expansion of the form

$$g\left(\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix}\right) = \sum_{\mathfrak{m}} c(\mathfrak{m}, g)\psi(\mathfrak{m}u).$$

We let

$$L_{\equiv\xi(I)}(g, s) = \sum_{\mathfrak{m}\equiv\xi(I)}{}' \frac{c(\mathfrak{m}, g)}{(\mathrm{N}\mathfrak{m})^s},$$

$$L(g, s) = \sum_{\mathfrak{m}} \frac{c(\mathfrak{m}, g)}{(\mathrm{N}\mathfrak{m})^s}.$$

4.2 Eisenstein series

We compute the image under the regulator map of the special elements in K-group. It turns out to be the product of two types of Eisenstein series, whose definition will be given in this section. We define a function on $\mathrm{GL}_2(K_\infty)/\mathrm{GL}_2(O_\infty)$. Let

$$\varphi^s_{c,d}\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right)=\begin{cases} q^{(k-\deg c)s} & \omega\geq k-\deg c\\ q^{\omega s} & \omega\leq k-\deg c.\end{cases}$$

where $\omega=\mathrm{ord}_\infty(cu+d)$.

Definition 4.1. For $s\in\mathbb{C}$, we let

$$E^s_{1/i,0}=\sum_{c\equiv 1(I),\,d\equiv 0(I)}\varphi_{c,d},$$
$$E^s_{0,1/i}=\sum_{c\equiv 0(I),\,d\equiv 1(I)}\varphi_{c,d}.$$

They converge absolutely for $\mathrm{Re}\, s\gg 0$. They may be analytically continued to the whole complex plane. We are interested in the functions at $s=0$. We let

$$E_{1/i,0}=\frac{\partial}{\partial s}E^s_{1/i,0}\Big|_{s=0},\quad E_{0,1/i}=\frac{\partial}{\partial s}E^s_{0,1/i}\Big|_{s=0}.$$

Let

$$\tilde{\varphi}_{c,d}\left(\begin{pmatrix}\pi^k & u\\ 0 & 1\end{pmatrix}\right)=\begin{cases} -q^{k-2\deg c-1} & \omega\geq k-\deg c\\ q^{2\omega-k} & \omega\leq k-\deg c.\end{cases}$$

Definition 4.2. For more properties of the Eisenstein series below, see [2]. Let

$$\tilde{E}_{1/i,0}=\sum_{c\equiv 1(I),\,d\equiv 0(I)}\tilde{\varphi}_{c,d}$$
$$\tilde{E}_{0,1/i}=\sum_{c\equiv 0(I),\,d\equiv 1(I)}\tilde{\varphi}_{c,d}.$$

Theorem 4.3.

$$\mathrm{reg}(\kappa_I)=C'[E_{1/i,0}\tilde{E}_{0,1/i}-\tilde{E}_{1/i,0}E_{0,1/i}]$$

where $C'\in\mathbb{R}$ is a constant.

Remark 4.4. For the proof, we need an analog of Kronecker limit formula.

4.3 Special values

We introduce a pairing on the space of functions defined in §3.2 (see [6])

$$\langle g_1,g_2\rangle=\int_{X(I)^{(1)}}g_1\cdot\overline{g_2}$$

where $X(I)^{(1)}=\Gamma^{(1)}(I)\backslash\mathrm{GL}_2(K_\infty)/\Gamma_\infty K^*_\infty$, $\Gamma^{(1)}(I)=\mathrm{SL}_2(A)\cap\Gamma(I)$. Our Beilinson type result is the following theorem.

Theorem 4.5. *There exists a computable constant C, which is independent of $g \in H^1(M_I^2)$, such that*

$$\langle g, \mathrm{reg}(\kappa_I) \rangle = C \cdot L(g,1) \left. \frac{\partial}{\partial s} \sum_{\xi \in \mathbf{F}_q^*} L_{\equiv \xi(I)}(g,s) \right|_{s=0}$$

holds.

Remark 4.6. The left hand side, upon substitution of Theorem 4.3, is

$$\int g E_{1/i,0} \tilde{E}_{0,1/i} - \int g \tilde{E}_{1/i,0} E_{0,1/i}$$

up to a computable constant. Each term may be calculated since it is a Rankin-Selberg integral.

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