数理解析研究所講究録 1203 巻 2001 年 46-50

ON THE PAINLEVÉ I HIERARCHY

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Every solution of the first Painlevé equation

$$\text{(I)} \qquad Z'' = 6Z^2 + 4t$$

($' = d/dt$) is meromorphic in $\mathbf{C}$, that is to say, equation (I) admits the Painlevé property. It is known that the fourth-order equation

$$(\mathrm{I}_4) \qquad Z^{(4)} = 20ZZ'' + 10(Z')^2 - 40Z^3 + 16t$$

also admits the Painlevé property, which is proved by using Miwa's result concerning the isomonodromic deformation ([1,2,3]). In this note, we show that

(1) there exists a hierarchy of systems of nonlinear equations, from which we can derive (I), (I_4) and

$$(\mathrm{I}_6) \qquad Z^{(6)} = 28ZZ^{(4)} + 56Z'Z^{(3)} + 42(Z'')^2 - 280(Z^2Z'' + Z(Z')^2 - Z^4) + 64t,$$

(2) all the systems in the hierarchy and the nonlinear equations derived from it such as (I), (I_4), (I_6) admit the Painlevé property.

1. Results

Cosider the following formal power series in ξ :

$$Q(\xi) = \sum_{\nu\geq 1} Z_\nu \xi^\nu,$$

$$R(\xi) = \sum_{\nu\geq 1} U_\nu \xi^\nu,$$

$$F(\xi) = 2\xi^{-1}Q(\xi)(1 + Z_1\xi) + \left(\xi^{-1}Q(\xi)^2 - R(\xi)^2\right)(1 - Q(\xi))^{-1} - u_0^2,$$

where u_0, Z_ν, U_ν ($\nu \in \mathrm{N}$) are parameters. Then, $F(\xi)$ is written in the form

$$F(\xi) = \sum_{\nu\geq 0} F_\nu \xi^\nu$$

$$F_0 = 2Z_1 - u_0^2,$$
$$F_\nu = 2Z_{\nu+1} + G_\nu(Z_j, U_k; 1 \le j \le \nu, 1 \le k \le \nu - 1) \quad (\nu \in \mathbf{N}).$$

Here $G_\nu(Z_j, U_k; ...)$ denotes a polynomial in Z_j and U_k $(1 \le j \le \nu, 1 \le k \le \nu - 1)$. Let m be a nonnegative integer and t a variable. Then the relations

$$\frac{d}{dt}(u_0 + R(\xi)) \equiv F(\xi) + 2(t - Z_{m+1})\xi^m \quad (\mathrm{mod}\ \xi^{m+1}),$$
$$\frac{d}{dt}Q(\xi) \equiv 2R(\xi) \quad (\mathrm{mod}\ \xi^{m+1}),$$

define the following systems:
for $m = 0$,

$$(\mathrm{S}_1) \qquad u_0' = 2t - u_0^2;$$

for $m \ge 1$,

$$(\mathrm{S}_m) \qquad \begin{aligned} u_0' &= 2Z_1 - u_0^2, \\ Z_\nu' &= 2U_\nu, \\ U_\nu' &= 2Z_{\nu+1} + G_\nu(Z_j, U_k; 1 \le j \le \nu, 1 \le k \le \nu - 1), \\ Z_m' &= 2U_m, \\ U_m' &= 2t + G_m(Z_j, U_k; 1 \le j \le m, 1 \le k \le m - 1) \end{aligned}$$

$(1 \le \nu \le m - 1)$. Then we have

Theorem 1.1. *Every solution* $(u_0(t), Z_\nu(t), U_\nu(t))$ $(1 \le \nu \le m)$ *of* (S_m) $(m \ge 0)$ *is meromorphic in* $\mathbf{C}$.

As an immediate corollary of this theorem, we have

Corollary 1.2. *Every solution of* (I_4) *or* (I_6) *is meromorphic in* $\mathbf{C}$.

It is known that, for an arbitrary solution $P(t)$ of (I), every solution of

$$y'' - 2P(t)y = 0$$

is meromorphic in $\mathbf{C}$. Futhermore we have

Corollary 1.3. *Let* $P_4(t)$ *(resp.* $P_6(t)$*) be an arbitrary solution of* (I_4) *(resp.* (I_6)*). Then every solution of*

$$y'' - 2P_4(t)y = 0 \quad (\textit{resp. } y'' - 2P_6(t)y = 0)$$

is meromorphic in $\mathbf{C}$.

2. Outline of the proof of Theorem 1.1

Consider the 2 by 2 matrix linear differential equation

$$\frac{d\Xi}{dx} = A(x)\Xi, \quad A(x) = -\sum_{j=0}^{2(m+1)} A_{-j}x^j + A_1 x^{-1}. \tag{2.1}$$

Here $A_{-\nu}$ are given as below:

$$\begin{aligned}
&A_{-2(m+1)} = J, \quad A_{-(2m+1)} = -u_0 L,\\
&A_{-2m} = v_1 K - w_1 J, \quad A_{-(2m-1)} = -u_1 L,\\
&A_{-2(m+1)+2i} = v_i K - w_i J, \quad A_{-(2m+1)+2i} = -u_i L \quad (1 \le i \le m),\\
&A_0 = s(J+K), \quad A_1 = (I-L)/2
\end{aligned}$$

with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 2.1. *Let $t, u_0, u_1, \ldots, u_m, v_1, \ldots, v_m$ be arbitrary parameters. System (2.1) admits a formal matrix solution of the form*

$$\begin{aligned}
&\Xi = \Xi(x) = Y(x)\exp T(x),\\
&T(x) = -\frac{J}{2m+3}x^{2m+3} - tJx + \frac{I}{2}\log(1/x), \quad Y(x) = \sum_{j\ge 1} Y_j x^{-j},
\end{aligned}$$

if and only if

$$\begin{aligned}
&w_1 = u_0^2/2,\\
&w_\nu = \frac{1}{2}\left(\sum_{j=1}^{\nu-1} w_j w_{\nu-j} - \sum_{j=1}^{\nu-1} v_j v_{\nu-j} + \sum_{j=1}^{\nu} u_{j-1}u_{\nu-j}\right),\\
&s = t - \frac{1}{2}\left(\sum_{j=1}^{m} w_j w_{m+1-j} - \sum_{j=1}^{m} v_j v_{m+1-j} + \sum_{j=1}^{m+1} u_{j-1}u_{m+1-j}\right)
\end{aligned} \tag{2.2}$$

$(1 \le \nu \le m)$.

For the deformation parameter t, the deformation equation with respect to (2.1) is written in the form

$$\begin{aligned}
&dA(x) = \frac{\partial}{\partial x}\Omega(x,t) + [\Omega(x,t), A(x)],\\
&\Omega(x,t) = \Phi_{-1}(t)x + \Phi_0(t),
\end{aligned} \tag{2.3}$$

where $\Phi_{-1}(t)$ and $\Phi_0(t)$ are 1-forms of t defined by

$$\sum_{k=-\infty}^{1} \Phi_{-k}(t)x^k = Y(x)(-x\,dt)JY(x)^{-1}.$$

Proposition 2.2. *Equation* (2.3) *is equivalent to*

$$u'_{\nu-1} = 2v_\nu, \quad v'_\nu = 2u_\nu + 2u_0 w_\nu, \quad w'_\nu = 2u_0 v_\nu,$$
$$u'_m = 2s, \quad s' = 1 - 2u_0 s$$

($1 \le \nu \le m$), *where* w_ν, s *are the parameters defined by* (2.2).

System (2.1) possesses an apparent singularity at $x = 0$, and Miwa's theorem [2] is not applicable. To remove it, we employ the Schlesinger transformation

$$W = \Psi(x)\Xi, \quad \Psi(x) = \begin{pmatrix} 1 & 1 \\ u_0/2 & u_0/2 + x \end{pmatrix}.$$

Then system (2.1) is changed into

$$\frac{dW}{dx} = B(x)W, \quad B(x) = -\sum_{j=0}^{2(m+1)} B_{-j}x^j, \tag{2.4}$$

where

$$B_{-2(m+1)} = J,$$
$$B_{-(2\nu+1)} = \begin{pmatrix} -u_{m-\nu} - u_0(v_{m-\nu} + w_{m-\nu}) & 2(v_{m-\nu} + w_{m-\nu}) \\ -u_0^2(v_{m-\nu} + w_{m-\nu})/2 - u_0 u_{m-\nu} - v_{m-(\nu-1)} & u_{m-\nu} + u_0(v_{m-\nu} + w_{m-\nu}) \end{pmatrix}$$
$$B_{-2\nu} = \begin{pmatrix} -(v_{m+1-\nu} + w_{m+1-\nu}) & 0 \\ -u_0(v_{m+1-\nu} + w_{m+1-\nu}) - u_{m+1-\nu} & v_{m+1-\nu} + w_{m+1-\nu} \end{pmatrix},$$
$$B_{-1} = \begin{pmatrix} -u_m - u_0(v_m + w_m) & 2(v_m + w_m) \\ -u_0^2(v_m + w_m)/2 - u_0 u_m - s & u_m + u_0(v_m + w_m) \end{pmatrix},$$
$$B_0 = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix}$$

($1 \le \nu \le m$), $v_0 = w_0 = 0$. Applying Miwa's theorem to (2.4), we can show that u_0, $Z_\nu = v_\nu + w_\nu$ and $U_\nu = u_\nu + u_0 Z_\nu$ are meromorphic in $\mathbf{C}$. Since the isomonodromy property is invariant under the Schlesinger transformation, from (2.2) and Proposition 2.2 we derive the deformation equation with respect to Z_ν, U_ν, which coincides with (S_m). This completes the proof.

3. Derivation of the corollaries

Eliminating the unkown variables other than Z_1, from (S_2) and (S_3) we get equations (I_4) and (I_6), respectively. Thus we have Corollary 1.2.

To show Corollary 1.3, let us consider, for example, system (S_3). By Corollary 1.2, an arbitrary solution $Z = P_6(t)$ of (I_6) is meromorphic in $\mathbf{C}$, and, around each pole $t = t_0$, it is expanded into one of the following Laurent series:

$$(t-t_0)^{-2} + \cdots, \quad 3(t-t_0)^{-2} + \cdots, \quad 6(t-t_0)^{-2}. \tag{3.1}$$

By Theorem 1.1, every solution of

$$u' = 2P_6(t) - u^2, \tag{3.2}$$

is meromorphic in $\mathbf{C}$, which is the first equation of (S_3). The transformation $u = y'/y$ takes (3.2) into

$$y'' - 2P_6(t)y = 0. \tag{3.3}$$

Let $y(t)$ be an arbitrary solution of (3.3). It is sufficient to show that an arbitrary pole $t = t_0$ of $P_6(t)$ is at most a pole of $y(t)$. To do this, we note that $u(t) = y'(t)/y(t)$ is written in the form

$$u(t) = c(t - t_0)^{-1} + \cdots ,$$

around it, where c is an integer equal to one of $-3, -2, -1, 2, 3, 4$. Hence we get an expression of the form

$$y(t) = (t - t_0)^c \sum_{j=0}^{\infty} C_j (t - t_0)^j,$$

from which Corollary 1.3 follows.

References

1. M. Jimbo, T. Miwa and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, I, – General theory and τ-function –*, Phys. **D2** (1981), 306–352.
2. T. Miwa, *Painlevé property of monodromy preserving deformation equations and analyticity of τ-functions*, Publ. Res. Inst. Math. Sci. **17** (1981), 703–721.
3. S. Shimomura, *Painlevé property of a degenerate Garnier system of (9/2)-type and of a certain fourth order non-linear ordinary differential equation*, Ann. Scuola Norm. Sup. Pisa **29** (2000), 1–17.