数理解析研究所講究録 1203 巻 2001 年 71-88

Painlevé-Calogero correspondence

Kanehisa Takasaki *

Department of Fundamental Science

Faculty of Integrate Human Studies, Kyoto University

Abstract

The so called "Painlevé-Calogero correspondence" relates the sixth Painlevé equation with an integrable system of the Calogero type. This relation was recently generaized to the other Painlevé equations and a "multi-component" analogue. This paper reviews these results.

1 Historical background

It was at the beginning of the twentieth century that Painlevé discovered what are nowadays called the "Painlevé equations" [1]. Painlevé obtained those equations in the course of classification of second order nonlinear algebraic ordinary differential equations "without movable critical point". The classification was eventually completed by his student Gambier[2], who supplemented several cases (in particular, the sixth equation) that Painlevé overlooked.

The property that the differential equation be free of movable critical point, which lies in the heart of Painlevé's work, is now called the "Painlevé property". This kind of analysis is generally referred to as "Painlevé analysis". Actually, a prototype of Painlevé's method can be found in Kowalevskaya's work on integrability of the motion of a rigid body [3]. In this respect, this method should be rather called "Kowalevskaya-Painlevé analysis".

At the time when Painlevé's classification was being completed, R. Fuchs (son of L. Ruchs, who's name is coined in the notion of "Fuchsian differential equations", "Fuchsian groups", etc.) proposed two new approaches to the sixth Painlevé equations [4]:

*高崎金久，京都大学総合人間学部基礎科学科

1. isomonodromic deformations

2. elliptic integrals

The first approach was soon generalized by Schlesinger[5] and Garnier[6] to a large extent, and (after a long break) revived in the seventies along with an unexpected application in mathematical physics, such as the Ising model, soliton theory, etc.

The second approach, meanwhile, had been almost forgotten after Painlevé improved it slightly [7]. It should be mentioned that Okamoto [8] touched upon this work of Painlevé in his study on symmetries of the sixth Painlevé equation. Manin [9] reexamined the work of Fuchs and Painlevé after ninety years, and presented a very remarkable result.

Manin discovered that the sixth Painlevé equation can be converted to a non-autonomous Hamiltonian system with an elliptic potential. This is achieved by change of variables in two steps. The first step was done by Painlevé, who interpreted Fuchs' elliptic integral as a new dependent variable. Manin proposed to use the modulus of the elliptic curve as a new independent variable, The outcome is the aforementioned non-autonomous Hamiltonian system. Furthermore, Manin noticed that this Hamiltonian system is reminiscent of a problem of integrable systems studied by Treibich and Verdier [10].

Levin and Olshanetsky [11] pointed out that Manin's Hamiltonian coincides with the Hamiltonian of an integrable system called the "Inozemtsev system" [13, 14]. Since the Inozemtsev system is an integrable generalization of the Calogero system [12], Levin and Olshanetsky called this connection between the Painlevé equations and the integrable systems of the Calogero type the "Painlevé-Calogero correspondence".

We shall briefly review this correspondence in the next section, then turn to recent results on generalizations of the Painlevé-Calogero correspondenc to the other Painlevé equations [15].

2 Painlevé-Calogero correspondence for PVI

The sixth Painlevé equation (PVI) is a second order nonlinear differential equation of the following form:

$$\begin{aligned} \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &\quad +\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{\lambda^2}+\frac{\gamma(t-1)}{(\lambda-1)^2}+\frac{\delta t(t-1)}{(\lambda-t)^2}\right). \end{aligned} \tag{1}$$

We now follow the work of Fuchs, Painlevé and Manin in the historical order.

2.1 What Fuchs did

Fuchs presents the following expression for this equation:

$$
\begin{aligned}
& t(1-t)\mathcal{L}_t \int_\infty^\lambda \frac{dz}{\sqrt{z(z-1)(z-t)}} \\
= \; & \sqrt{\lambda(\lambda-1)(\lambda-t)}\left[\alpha + \frac{\beta t}{\lambda^2} + \frac{\gamma(t-1)}{(\lambda-1)^2} + \left(\delta - \frac{1}{2}\right)\frac{t(t-1)}{(\lambda-t)^2}\right].
\end{aligned} \tag{2}
$$

Here $\mathcal{L}_t$ is the second order linear differential operator

$$
\mathcal{L}_t = t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4} \tag{3}
$$

called the "Picard-Fuchs operator", whichs originates in the differential equation (the Picard-Fuchs equation[1])

$$
\mathcal{L}_t \oint_\gamma \frac{dz}{\sqrt{z(z-1)(z-t)}} = 0 \tag{4}
$$

for complete elliptic integrals. Fuchs thus applied it to an incomplete elliptic integral, and discovered the miraculous phenomena that all derivative terms in the sixth Painlevé equation are absorbed therein. We shall see that the same phenomena takes place for the other Painlevé equations, too.

2.2 What Painlevé did

Painlevé uses the Weierstrass $\wp$ function to rewrite Fuchs' equation in terms of a new dependent variable. Following Painlevé, we now change the dependent variable from λ to

$$
q = \frac{1}{2(e_2-e_1)^{1/2}} \int_\infty^\lambda \frac{dz}{\sqrt{z(z-1)(z-\lambda)}}. \tag{5}
$$

(Actually, this is already modified à la Manin.) This equation can be solved for λ as

$$
\lambda = \frac{\wp(q) - e_1}{e_2 - e_1}, \tag{6}
$$

where $\wp(u)$ is the $\wp$ function

$$
\wp(u) = \wp(u \mid 1, \tau) = \frac{1}{u^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(u+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right) \tag{7}
$$

[1]named after L. Fuchs

with fundamental periods $1, \tau$, and e_1, e_2, e_3 denotes the values

$$e_n = \wp(\omega_n) \tag{8}$$

of $\wp$ at the three half-periods

$$\omega_1 = \frac{1}{2}, \quad \omega_2 = -\frac{1}{2} - \frac{\tau}{2}, \quad \omega_3 = \frac{\tau}{2}.$$

This change of dependent variables stems from the parametrization

$$z = \frac{\wp(u) - e_1}{e_2 - e_1}, \quad y = \frac{\wp'(u)}{2(e_3 - e_1)^{3/2}}, \tag{9}$$

of the elliptic curve

$$y^2 = z(z-1)(z-t) \tag{10}$$

that lies behind Fuchs' elliptic integral. The three half-periods corresponds to the three branch points $z = 0, 1, t$ of the covering map from the elliptic curve onto the z-plane:

$$\begin{aligned} u = \omega_1 &\longleftrightarrow z = 0, \\ u = \omega_2 &\longleftrightarrow z = 1, \\ u = \omega_3 &\longleftrightarrow z = t. \end{aligned}$$

$u = 0$ correspond to the fourth branch point at $z = \infty$.

This change of dependent variable $\lambda \to q$ transforms (2) as follows. The left hand side turns into

$$t(1-t)\mathcal{L}_t\big(2(e_2 - e_1)^{12} q\big).$$

Let us examine the right hand side term by term by expanding the parentheses. Recalling the differential equation

$$\wp'(u)^2 = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3), \tag{11}$$

one can rewrite the first term as

$$\begin{aligned} \sqrt{\lambda(\lambda-1)(\lambda-t)} &= (e_2 - e_1)^{-3/2}\sqrt{(\wp(q) - e_1)(\wp(q) - e_2)(\wp(q) - e_3)} \\ &= \frac{1}{2}(e_2 - e_1)^{-3/2}\wp'(q). \end{aligned}$$

Furthermore, by the functional identity

$$\wp(u + \omega_j) = e_j + \frac{(e_j - e_k)(e_j - e_\ell)}{\wp(u) - e_j)} \tag{12}$$

(j, k, ℓ being a cyclic permutation of $1,2,3$), the other terms give a linear combination of

$$\frac{(e_2-e_1)^{3/2}}{2}\wp'(q+\omega_n) \quad (n=1,2,3).$$

Thus Fuchs' equation can be eventually rewritten

$$t(1-t)\mathcal{L}_t(2(e_2-e_1)^{1/2}q) = \frac{1}{2}(e_2-e_1)^{-3/2}\sum_{n=0}^{3}\alpha_n\wp(q+\omega_n), \tag{13}$$

where

$$\alpha_0=\alpha,\ \alpha_1=-\beta,\ \alpha_2=\gamma,\ \alpha_3=-\delta+\frac{1}{2}. \tag{14}$$

This is essentially what Painlevé did.

2.3 What Manin did

Manin further changes the dependent variable from t to τ. t and τ are connected by the functional relation

$$t=\frac{e_3-e_1}{e_2-e_1}. \tag{15}$$

(Note that e_n are special values of $\wp$ at half-periods, thereby depends on τ.) Geometrically, they are both a modulus of a family of elliptic curves — t is the modulus in Jacobi's elliptic function theory, and τ is its counterpart in the modular upper half plane. Manin shows the beautiful formula

$$\frac{d\tau}{dt}=\frac{\pi i}{t(t-1)(e_2-e_1)}, \tag{16}$$

for the Jacobian of this change of variable, and, with the aid of this formula, derives the relation

$$(e_2-e_1)^{3/2}t(1-t)\circ\mathcal{L}_t\circ(e_2-e_1)^{1/2}=(\pi i)^2\frac{d^2}{d\tau^2} \tag{17}$$

that connects the Picard-Fuchs operator with a differential operator in the new dependent variable τ. ($\circ$ stands for composition of operators.) Equation (13) can be thereby rewritten

$$(2\pi i)^2\frac{d^2q}{d\tau^2}=\sum_{n=0}^{3}\alpha_n\wp'(q+\omega_n). \tag{18}$$

This is the equation that Manin discovered.

This equation has another expression as a non-autonomous Hamiltonian system

$$2\pi i \frac{dq}{d\tau} = \frac{\partial \mathcal{H}}{\partial p}, \quad 2\pi i \frac{dp}{d\tau} = -\frac{\partial \mathcal{H}}{\partial q} \tag{19}$$

with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - \sum_{n=0}^{3} \alpha_n \wp(q + \omega_n). \tag{20}$$

Note that the Hamiltonian depends on τ explicitly via the τ dependence of $\wp$, so that the Hamiltonian system is non-autonomous.

2.4 Relation to elliptic Inozemtsev system

The elliptic Inozemtsev system is a many-body particle system on a line. The equations of motion take the Hamiltonian form

$$\frac{dq_j}{dt} = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial \mathcal{H}}{\partial q_j} \tag{21}$$

with the Hamiltonian (ℓ being the number of particles)

$$\mathcal{H} = \sum_{j=1}^{\ell} \left(\frac{p_j^2}{2} + \sum_{n=0}^{3} g_n^2 \wp(q_j + \omega_n) \right) + g_4^2 \sum_{j \neq k} \left(\wp(q_j - q_k) + \wp(q_j + q_k) \right), \tag{22}$$

where $g_0, \dots, g_4$ are coupling constants. The modulus τ of the $\wp$ functions, too, is treated as an independent constant. Naturally, this is an autonomous system. Inozemtsev [13] discovered this system (along with analogues with hyperbolic and rational potentials) as a generalization of the Calogero system.

Levin and Olshanetsky noticed that Manin's Hamiltonian is nothing but a special case ($\ell = 1$) of Inozemtsev's Hamiltonian. (The two-body potential is absent therein.) An essential difference is that τ plays the role of time variable in Manin's equation. One may similarly consider the non-autonomous system

$$2\pi i \frac{dq_j}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_j}, \quad 2\pi i \frac{dp_j}{d\tau} = -\frac{\partial \mathcal{H}}{\partial q_j} \tag{23}$$

for any value of ℓ. This generalization of Manin's equation turns out to give a system of isomonodromic deformations on the torus (Takasaki[16]).

3 Other Painlevé equations

We now show how this correspondence was extended to the other Painlevé equations [15]. A crucial idea can be obtained from the degeneration relation among the six Painlevé equations (Okamoto[17]). This relation can be schematically displayed by the following diagram:

$$\begin{array}{ccccc} P_{\mathrm{VI}} & \longrightarrow & P_{\mathrm{V}} & \longrightarrow & P_{\mathrm{IV}} & & \\ & & \downarrow & & \downarrow & & \\ & & P_{\mathrm{III}} & \longrightarrow & P_{\mathrm{II}} & \longrightarrow & P_{\mathrm{I}} \end{array}$$

The arrows stand for a degeneration process between two equations. One will notice, in particular, that the five equations other than PVI can be reached from PVI. Remarkably, a similar diagram of degeneration was known (without any connection with the Painlevé equations) for the Inozemtsev systems (van Diejen [18]). After all, the "Painlevé-Calogero correspondence" for PVI turns out to be inherited by the other Painlevé equations in accordance with the degeneration relations on both the Painlevé and Calogero sides.

The correspondence can be formulated in both the second order formalism and Hamiltonian formalism. We here present the second order formalism, which is a rather straightforward generalization of the work of Fuchs, Painlevé and Manin.

3.1 Analogue of Fuchs' equation for PV

The fifth Painlevé V equation (PV) is a second order nonlinear differential equation of the following form:

$$\begin{aligned} \frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt} \\ &\quad +\frac{\lambda(\lambda-1)^2}{t^2}\left(\alpha+\frac{\beta}{t^2}+\frac{\gamma t}{(\lambda-1)^2}+\frac{\delta t^2(\lambda+1)}{(\lambda-1)^3}\right). \end{aligned} \tag{24}$$

This equation can be derived from PVI by degeneration. This process is achieved by setting

$$t=1+\epsilon\tilde{t},\quad \alpha=\tilde{\alpha},\quad \beta=\tilde{\beta},\quad \gamma=\frac{\tilde{\gamma}}{\epsilon}-\frac{\tilde{\delta}}{\epsilon^2},\quad \delta=\frac{\tilde{\delta}}{\epsilon^2} \tag{25}$$

and taking the limit as $\epsilon\to 0$ while leaving $\tilde{\alpha},\ldots,\tilde{\delta}$ finite. This is a kind of "scaling limit" that is frequently used in physics.

In this scaling limit, the building blocks of Fuchs' equation (2) behave as follows:

1. Picard-Fuchs operator:

$$t(1-t)\mathcal{L}_t \longrightarrow \tilde{t}^2\frac{d^2}{d\tilde{t}^2} + \tilde{t}\frac{d}{d\tilde{t}} = \left(\tilde{t}\frac{d}{d\tilde{t}}\right)^2.$$

2. The part $\alpha + \cdots$ on the right hand side:

$$\alpha + \frac{\beta t}{\lambda^2} + \frac{\gamma(t-1)}{(\lambda-1)^2} + \left(\delta - \frac{1}{2}\right)\frac{t(t-1)}{(\lambda-t)^2} \longrightarrow \tilde{\alpha} + \frac{\tilde{\beta}}{\lambda^2} + \frac{\tilde{\gamma}\tilde{t}}{(\lambda-1)^2} + \frac{\tilde{\delta}\tilde{t}^2(\lambda+1)}{(\lambda-1)^3}.$$

3. The square root on the right hand side:

$$\sqrt{\lambda(\lambda-1)(\lambda-t)} \longrightarrow \sqrt{\lambda}(\lambda-1).$$

4. The incomplete elliptic integral:

$$\int_\infty^\lambda \frac{dz}{\sqrt{z(z-1)(z-t)}} \longrightarrow \int_\infty^\lambda \frac{dz}{\sqrt{z}(z-1)}.$$

Renaming $\tilde{\alpha}, \ldots, \tilde{\delta}$ and $\tilde{t}$ as $\alpha, \ldots, \delta$ と t, we eventually obtains the equation

$$\left(t\frac{d}{dt}\right)^2 \int_\infty^\lambda \frac{dz}{\sqrt{z}(z-1)} = \sqrt{\lambda}(\lambda-1)\left(\alpha + \frac{\beta}{\lambda^2} + \frac{\gamma t}{(\lambda-1)^2} + \frac{\delta t^2(\lambda+1)}{(\lambda-1)^3}\right) \tag{26}$$

as an analogue of Fuchs' equation.

3.2 Analogue of Manin's equation for PV

We now change the dependent variable from q to

$$q = \int_\infty^\lambda \frac{dz}{\sqrt{z}(z-1)}. \tag{27}$$

in (26). A remark is in order: To be faithful to the construction for PVI, one should rather define

$$q = \frac{1}{2\pi i}\int_\infty^\lambda \frac{dz}{\sqrt{z}(z-1)}.$$

In fact, the degeneration process is associated with the degeneration

$$y^2 = z(z-1)(z-t) \to y^2 = z(z-1)^2$$

of the elliptic curve to a singular rational curve as $t \to 1$, and the constant factor $2(e_2 - e_1)^{1/2}$ on the right hand side of the definition of q behaves as

$$2(e_2 - e_1)^{1/2} \to 2\pi i$$

in this limit. Since omitting the numerical factor $1/2\pi i$ causes no substantial difference, we adopt the simpler definition. If $1/2\pi i$ is inserted, one will obtain a trigonometric function rather than the hyperbolic function that arises in the following calculation.

The integral defining q can be calculated by elementary calculus:

$$q = \log\left(\frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1}\right) \tag{28}$$

Solving this relation for λ yields

$$\sqrt{\lambda} = -\coth(q/2). \tag{29}$$

Thus the $\wp$ function is now replaced by the hyperbolic function $\coth^2$. Geometrically, this means that the torus turns into a cylinder as $\operatorname{Im}\tau \to \infty$. It should be mentioned that a similar change of variable for PV is already known in the literature (Iwasaki et al. [19]).

Let us rewrite (26) in terms of the new dependent variable q. Each term on the right hand side of the equation can be written

$$\begin{aligned}
\sqrt{\lambda}(\lambda-1) &= -\frac{\cosh(q/2)}{\sinh^3(q/2)}, \\
\sqrt{\lambda}(\lambda-1)\frac{1}{\lambda^2} &= -\frac{\sinh(q/2)}{\cosh^3(q/2)}, \\
\sqrt{\lambda}(\lambda-1)\frac{1}{(\lambda-1)^2} &= -\frac{1}{2}\sinh(q), \\
\sqrt{\lambda}(\lambda-1)\frac{(\lambda+1)}{(\lambda-1)^3} &= -\frac{\lambda^{3/2}+\lambda^{1/2}}{(\lambda-1)^2} = -\frac{1}{4}\sinh(2q),
\end{aligned}$$

so that (26) turns into the equation

$$\left(t\frac{d}{dt}\right)^2 q = -\frac{\partial V(q)}{\partial q}, \tag{30}$$

for q. $V(q)$ is the "potential"

$$V(q) = -\frac{\alpha}{\sinh^2(q/2)} - \frac{\beta}{\cosh^2(q/2)} + \frac{\gamma t}{2}\cosh(q) + \frac{\delta t^2}{8}\cosh(2q). \tag{31}$$

This gives an analogue of Manin's equation for PV.

This second order equation can be readily converted to a Hamiltonian system of the form

$$t\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad t\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \tag{32}$$

with the Hamiltonian $\mathcal{H} = p^2/2 + V(q)$ and the time variable $\log t$. Note that this is a non-autonomous system.

Remarkably, this Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - \frac{\alpha}{\sinh^2(q/2)} - \frac{\beta}{\cosh^2(q/2)} + \frac{\gamma t}{2}\cosh(q) + \frac{\delta t^2}{8}\cosh(2q), \tag{33}$$

too, is a special case of the Hamiltonian

$$\begin{aligned}\mathcal{H} &= \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2} + \frac{g_0^2}{\sinh^2(q_j/2)} + \frac{g_1^2}{\cosh^2(q_j/2)} + g_2^2\cosh(q_j) + g_3^2\cosh(2q_j)\right) \\ &\quad + g_4^2\sum_{j\neq k}\left(\frac{1}{\sinh^2((q_j-q_k)/2)} + \frac{1}{\sinh^2((q_j+q_k)/2)}\right)\end{aligned} \tag{34}$$

that Inozemtsev [13] considered as a generalization of the Calogero system. More precisely, this Hamiltonian was first discovered by Levi and Wojciechowski [20], and Inozemtsev rediscovered it in the course of his classification of generalized Calogero systems. The foregoing Hamiltonian amounts to the case with $\ell = 1$, for which the two-body potential is absent. Furthermore, as opposed to the autonomous case, some of the coupling constants now depends on t.

3.3 Direct method

Although the same method, in principle, works for the other Painlevé equations, complexity of calculations soon increases as one proceeds deep into the lower equations. Fortunately, a simpler and direct method is available. This method, which works for the four Painlevé equations other than PII and PI, tells us how to find an appropriate definition of the q variable by simply inspecting the second order nonlinear differential equations.

To illustrate the idea, let us return to PVI and PV, for which we already know the explicit form of the q variable. A clue lies in the coefficient of the $(d\lambda/dt)^2$ term of the equation. This coefficient is connected with the integrand of the definition of q by the following simple relation:

$$\begin{aligned}\frac{1}{\sqrt{z(z-1)(z-t)}} &= \exp\left[-\int\frac{1}{2}\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-t}\right)dz\right], \\ \frac{1}{\sqrt{z}(z-1)} &= \exp\left[-\int\frac{1}{2}\left(\frac{1}{2z} + \frac{1}{z-1}\right)dz\right].\end{aligned} \tag{35}$$

One will be naturally expect the same relation for PIV

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2-\alpha)\lambda + \frac{\beta}{\lambda} \tag{36}$$

and PIII

$$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\lambda^2}{4t^2}\left(\alpha + \frac{\beta t}{\lambda^2} + \gamma\lambda + \frac{\delta t^2}{4\lambda^3}\right). \tag{37}$$

This is indeed the case, as we shall show below.

Let us note that this idea obviously does not apply to PII

$$\frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \alpha \tag{38}$$

and PI

$$\frac{d^2\lambda}{dt^2} = 6\lambda^2 + t \tag{39}$$

(simply because they have no $(d\lambda/dt)^2$ term). These equations have to be treated in a different way.

3.4 Case of PIV

Applying the foregoing idea to PIV suggests to use the function

$$\exp\left(-\int\frac{dz}{2z}\right) = \frac{1}{\sqrt{z}} \tag{40}$$

for defining q. We thus define

$$q = \int^{\lambda}\frac{dz}{\sqrt{z}} = 2\sqrt{\lambda}. \tag{41}$$

This can be solved for λ as

$$\lambda = \left(\frac{q}{2}\right)^2. \tag{42}$$

By direct calculation, one finds that PIV turns into the equation

$$\frac{d^2q}{dt^2} = -\frac{\partial V(q)}{\partial q} \tag{43}$$

with the potential

$$V(q) = -\frac{1}{2}\left(\frac{q}{2}\right)^6 - 2t\left(\frac{q}{2}\right)^4 - 2(t^2-\alpha)\left(\frac{q}{2}\right)^2 + \beta\left(\frac{q}{2}\right)^{-2}. \tag{44}$$

This equation, too, can be converted to a Hamiltonian system:

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \tag{45}$$

The Hamiltonian takes the form

$$\mathcal{H} = \frac{p^2}{2} - \frac{1}{2}\left(\frac{q}{2}\right)^6 - 2t\left(\frac{q}{2}\right)^4 - 2(t^2 - \alpha)\left(\frac{q}{2}\right)^2 + \beta\left(\frac{q}{2}\right)^{-2}. \tag{46}$$

A similar Hamiltonian can be found in the work of Levi, Wojciechowski [20] and Inozemtsev [13]:

$$\mathcal{H} = \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2} + g_0^2 q_j^6 + g_1^2 q_j^4 + g_2^2 q_j^2 + g_3^2 q_j^{-2}\right) + g_4^2 \sum_{j \neq k}\left(\frac{1}{(q_j - q_k)^2} - \frac{1}{(q_j + q_k)^2}\right). \tag{47}$$

The foregoing Hamiltonian amounts to the case where $\ell = 1$. Some of the coupling constants, too, are time-dependent.

3.5 Case of PIII

The integrand for PIII reads

$$\exp\left(-\int \frac{dz}{z}\right) = \frac{1}{z}. \tag{48}$$

The q variable is given by

$$q = \int^{\lambda} \frac{dz}{z} = \log \lambda, \tag{49}$$

and solved for λ as

$$\lambda = e^q. \tag{50}$$

Now PIII is converted to the equation

$$\left(t\frac{d}{dt}\right)^2 q = -\frac{\partial V(q)}{\partial q} \tag{51}$$

with potential

$$V(q) = -\frac{\alpha}{4}e^q + \frac{\beta t}{4}e^{-q} - \frac{\gamma}{8}e^{2q} + \frac{\delta t^2}{8}e^{-2q}. \tag{52}$$

The associated Hamiltonian system is somewhat similar to the case of PV:

$$t\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad t\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \tag{53}$$

The Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - \frac{\alpha}{4}e^q + \frac{\beta t}{4}e^{-q} - \frac{\gamma}{8}e^{2q} + \frac{\delta t^2}{8}e^{-2q} \tag{54}$$

has no analogue in Inozemtsev's classification. One can, however, derive this Hamiltonian (and its generalization with ℓ degrees of freedom) from the hyperbolic Inozemtsev Hamiltonian by a degeneration process; this is exactly what van Diejen considered [18].

4 Correspondence in Hamiltonian formalism

The foregoing correspondence in the second order formalism can be reformulated as a time-dependent canonical transformation between Hamiltonian systems. The status of PII and PI is also clarified in this Hamiltonian formalism.

4.1 Hamiltonian structure of Painlevé equations

As first pointed out by Malmquist[21], the six Painlevé equations can be expressed as a Hamiltonian system of the form

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\lambda}{dt} = -\frac{\partial H}{\partial \lambda}.$$

This expression is not unique; we now consider the one with the following "polynomial Hamiltonians" (Okamoto[17]):

$$\text{PVI} \quad H = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[\mu^2 - \left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta-1}{\lambda-t}\right)\mu + \frac{\kappa}{\lambda(\lambda-1)}\right].$$

$$\text{PV} \quad H = \frac{\lambda(\lambda-1)^2}{t}\left[\mu^2 - \left(\frac{\kappa_0}{\lambda} + \frac{\theta_1}{\lambda-1} - \frac{\eta_1 t}{(\lambda-1)^2}\right)\mu + \frac{\kappa}{\lambda(\lambda-1)}\right].$$

$$\text{PIV} \quad H = 2\lambda\left[\mu^2 - \left(\frac{\lambda}{2} + t + \frac{\kappa_0}{\lambda}\right)\mu + \frac{\theta_\infty}{2}\right].$$

$$\text{PIII} \quad H = \frac{\lambda^2}{t}\left[\mu^2 - \left(\eta_\infty + \frac{\theta_0}{\lambda} - \frac{\eta_0 t}{\lambda^2}\right)\mu + \frac{\eta_\infty(\theta_0+\theta_\infty)}{2\lambda}\right].$$

$$\text{PII} \quad H = \frac{\mu^2}{2} - \left(\lambda^2 + \frac{t}{2}\right)\mu - \left(\alpha + \frac{1}{2}\right)\lambda.$$

$$\text{PI} \quad H = \frac{\mu^2}{2} - 2\lambda^3 - t\lambda.$$

$\kappa_0, \kappa_1, \theta$ are constants that are connected with the parameters of the Painlevé equations by simple algebraic relations.

4.2 How to find canonical transformation

We now show the outline of the construction of a canonical transformation that connects the Hamiltonian form of the Painlevé equations with the Hamiltonian system of the Inozemtsev type.

A clue lies in the equation for λ in the foregoing Hamiltonian system. For illustration, let us consider PVI. The equation for λ reads:

$$\frac{d\lambda}{dt} = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \left(2\mu - \frac{\kappa_0}{\lambda} - \frac{\kappa_1}{\lambda-1} - \frac{\theta-1}{\lambda-t} \right). \tag{55}$$

This equation can be solvled for μ as follows:

$$\mu = \frac{t(t-1)}{2\lambda(\lambda-1)(\lambda-t)} \frac{d\lambda}{dt} + \frac{1}{2} \left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta-1}{\lambda-t} \right). \tag{56}$$

Our goal is to rewrite this expression of μ to a function of p, q (and τ). To this end, let us recall (29). Differentiating (29) against t results in the equation

$$\frac{d\lambda}{dt} = \left(\frac{\wp'(q)}{e_2 - e_1} \frac{dq}{d\tau} + f_\tau(q) \right) \frac{d\tau}{dt}, \tag{57}$$

where

$$f(u) = \frac{\wp(u) - e_1}{e_2 - e_1}, \quad f_\tau(u) = \frac{\partial f(u)}{\partial \tau}.$$

The equation of motion for q gives

$$\frac{dq}{d\tau} = \frac{1}{2\pi i} \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{2\pi i} \tag{58}$$

Furthermore, one can use (16) to rewrite $d\tau/dt$. Thus one eventually obtain the expression

$$\begin{aligned} \mu &= \frac{e_2 - e_1}{\wp'(q)} p + \frac{2\pi i (e_2 - e_1)^2}{\wp'(q)^2} f_\tau(q) \\ &\quad + \frac{e_2 - e_1}{2} \left(\frac{\kappa_0}{\wp(q) - e_1} + \frac{\kappa_1}{\wp(q) - e_2} + \frac{\theta - 1}{\wp(q) - e_3} \right) \end{aligned} \tag{59}$$

for μ in terms of p, q and τ.

Having derived (59), we now change the point of view: We now interpret (59) and (6) as defining a time-dependent map $(q, p) \to (\lambda, \mu)$. By somewhat lengthy calculations, one can prove that this map satisfies the equation

$$\mu d\lambda - H dt = p dq - \mathcal{H} \frac{d\tau}{2\pi i} + \text{exact form}, \tag{60}$$

which means that this map is a time-dependent canonical transformation between the two Hamiltonian systems on the Painlevé and Calogero sides.

In much the same way, the correspondence for PV, PIV and PIII can be reformulated as a time-dependent canonical transformation.

4.3 PII and PI

PII and PI have to be treated separately. In view of the result for PIV – PIII, one should seek for a transformation to a Hamiltonian system with a Hamiltonian of the "standard form "

$$\mathcal{H} = \frac{p^2}{2} + V(q).$$

Since PI is already of that form, what is left is PII only.

One can readily find such a canonical transformation:

$$\lambda = q, \quad \mu = p + \lambda^2 + \frac{t}{2} \tag{61}$$

This converts PII to a Hamiltonian system with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2 - \alpha q. \tag{62}$$

Besides this rather ad hoc way, one can show that this Hamiltonian can be derived from the others (for PIV and PIII) by a degeneration process. Thus the six Hamiltonians on the Calogero side, like those on the Painlevé side, are connected by degeneration relation.

5 Multi-component analogues

All the Hamiltonian systems in the Painlevé-Calogero correspondence have just one degree of freedom. Since the Inozemtsev systems themselves are generalized to a many-body system, one will naturally ask if the Painlevé equations have a many-body analogue.

This question, too, is answered afirmatively [15]. Namely, an "ℓ-body" generalization of the Painlevé equations can be constructed. This generalization has ℓ pairs (λ_j, μ_j) ($j = 1, \ldots, \ell$) of canonically conjugate variables, each of which is connected with a canonical pair (q_j, p_j) on the Calogero side by the same functional relation as in the $\ell = 1$ case. The Hamiltonian H is a rational function of the canonical variables and the time variable, and take the form

$$H = \sum_{j=1}^{\ell} H_j + \text{two-body interaction.} \tag{63}$$

H_j is the "one-body" Hamiltonian in (λ_j, μ_j) with the same functional form as the corresponding Painlevé equation. The two-body interaction tersm have singularities of the Calogero-type (i.e., $(\lambda_j - \lambda_k)^{-2}$) along $\lambda_j = \lambda_k$. In other words, this is a "perturbation" of ℓ independent Painlevé equations by two-body interaction terms. For instance, the ℓ-body Hamiltonian for the PVI type reads:

$$
\begin{aligned}
H &= \sum_{j=1}^{\ell} \frac{\lambda_j(\lambda_j-1)(\lambda_j-t)}{t(t-1)} \left[\mu_j^2 - \left(\frac{\kappa_0}{\lambda_j} + \frac{\kappa_1}{\lambda_j - 1} + \frac{\theta - 1}{\lambda_j - t}\right)\mu_j + \frac{\kappa}{\lambda_j(\lambda_j - 1)}\right] \\
&\quad + \frac{g_4^2}{2t(t-1)} \sum_{j \neq k} \left[\frac{\lambda_j(\lambda_j-1)(\lambda_j-t) + \lambda_k(\lambda_k-1)(\lambda_k-t)}{8(\lambda_j-\lambda_k)^2} - 2(\lambda_j + \lambda_k)\right]. \quad (64)
\end{aligned}
$$

This is a kind of "multi-component" version of the Painlevé equations, similar but obviously different from another family of multi-dimensional generalizations called the "Garnier systems" (Garnier[6], Okamoto[17]). Presumably, these multi-component generalizations, too, will describe isomonodromic deformations on the Riemann sphere.

References

[1] P. Painlevé, Memoire sur les équations différentielles dont l'intégrale générale est uniforme, Bull. Soc. Math. Phys. France **28** (1900), 201-261; Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, Acta Math. **21** (1902), 1-85.

[2] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critique fixés, C.R. Acad. Sci. (Paris) **142** (1906), 266-269; ditto, Acta. Math. **33** (1910), 1-55.

[3] S. Kowalevski, Sur le problème de la rotation d'un corps solide autour d'un point fixe, Acta Math. **12** (1889), 177-232.

[4] R. Fuchs, Sur quelques équations différentielles linéaires du second ordre, C.R. Acad. Sci. (Paris) **141** (1905), 555-588; Über lineare homogene Differentialgleichungern zweiter Ordnung mit im endlich gelegene wesentlich singulären Stellen, Math. Ann. **63** (1907), 301-321.

[5] L. Schlesinger, Über eine Klasse von Differentialsystemen beliebliger Ordnung mit festen kritischer Punkten, J. für Math. **141** (1912), 96-145.

[6] R. Garnier, Sur des quations différentielles du troisième ordre dont l'intégrale est uniform et sur une classe d'équations nouvelles d'ordre supérieur dont l'intǵrale génŕale a ses point critiques fixés, Ann. Sci. de l'ENS **29** (1912), 1-126; Etudes de l'intégrale

générale de l'équation VI de M. Painlevé dans le voisinage de ses singularité transcendentes, Ann. Sci. Ecole Norm. Sup. **(3)** **34** (1917), 239-353.

[7] P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixés, C.R. Acad. Sci. (Paris) **143** (1906), 1111-1117.

[8] K. Okamoto, Studies on the Painlevé equations, I: Sixth Painlevé equation, Annali Mat. Pura Appl. **146** (1987), 337-381.

[9] Yu. I. Manin, Sixth Painlevé equation, universal elliptic curve, and mirror of $\mathbb{P}^2$, AMS Transl. (2) **186** (1998), 131-151.

[10] A. Treibich and J.-L. Verdier, Revêtements tangentiels et sommes de 4 nombre triangulaires, C.R. Acad. Sci. Paris, Sér. I Math., **311** (1990), 51-54.

[11] A.M. Levin and M.A. Olshanetsky, Painlevé-Calogero correspondence, e-print `alg-geom/9706012`; Classical limit of the Knizhnik-Zamolodchikov-Bernard equations as hierarchy of isomonodromic deformations, e-print `hep-th/9709207`.

[12] F. Calogero, Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. **12** (1971) 419-436; Exactly solvable one-dimensional many body problems, Lett. Nuovo Cim. **13** (1975) 411-416.

[13] V.I. Inozemtsev and D.V. Meshcheryakov, Extension of the class of integrable dynamical systems connected with semisimple Lie algebras, Lett. Math. Phys. **9** (1985), 13-18.

[14] V.I. Inozemtsev, Lax representation with spectral parameter on a torus for integrable particle systems, Lett. Math. Phys. **17** (1989), 11-17.

[15] K. Takasaki, Painlevé-Calogero correspondence revisited, e-print `math.QA/0004118`, J. Math. Phys. (to appear).

[16] K. Takasaki, Elliptic Calogero-Moser systems and isomonodromic deformations, J. Math. Phys. 40 (1999), 5787-5821.

[17] K. Okamoto, Isomonodromic deformations and Painlevé equations, and the Garnier system, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **33** (1986), 575-618.

[18] J.F. van Diejen, Difference Calogero-Moser systems and finite Toda chains, J. Math. Phys. **36** (1995), 1299-1323.

[19] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, From Gauss to Painlevé (Vieweg, Braunschweig, 1991).

[20] D. Levi and S. Wojciechowski, On the Olshanetsky-Perelomov many-body system in an external field, Phys. Lett. **A103** (1984), 11-14.

[21] J. Malmquist, Sur les équations différentielles du second ordre dont l'intégrale générale a ses points critique fixes, Ark. Mat. Astr. Fys. **17** (1922/23), 1-89.