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Positive solutions for nonhomogeneous elliptic equations

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0. Introduction

In this paper, we study the existence of positive solutions for a nonhomogeneous elliptic problem in $\mathbf{R}^N$:

$$\begin{cases} -\Delta u + u = g(x,u) + f(x) & \text{in } \mathbf{R}^N, \\ \quad u > 0 & \text{in } \mathbf{R}^N, \\ \quad u \in H^1(\mathbf{R}^N), \end{cases} \tag{0.1}$$

where $g(x,s) \in C(\mathbf{R}^N \times \mathbf{R})$ is a function of superlinear growth, i.e.,

$$\lim_{s\to\infty} \frac{g(x,s)}{s} = \infty,$$

and $f(x) \in H^{-1}(\mathbf{R}^N)$, $f(x) \geq 0$, $f(x) \not\equiv 0$. Here $H^1(\mathbf{R}^N)$ denotes the usual Sobolev space over $\mathbf{R}^N$ and $H^{-1}(\mathbf{R}^N)$ is the dual space of $H^1(\mathbf{R}^N)$. We denote the duality product between $H^{-1}(\mathbf{R}^N)$ and $H^1(\mathbf{R}^N)$ by $\langle \cdot\, , \, \cdot \rangle_{H^{-1}(\mathbf{R}^N),H^1(\mathbf{R}^N)}$ and for $f(x) \in H^{-1}(\mathbf{R}^N)$, we say $f(x) \geq 0$ if $\langle f\, , \varphi \rangle_{H^{-1}(\mathbf{R}^N),H^1(\mathbf{R}^N)} \geq 0$ holds for any non-negative function $\varphi \in H^1(\mathbf{R}^N)$.

Our main aim is to study the effects of the shape of $g(x,u)$ and $f(x)$ on the existence and multiplicity of solutions of (0.1). We first consider the existence and multiplicity of solutions of (0.1) for the general nonlinearity $g(x,u)$ and next consider for $g(x,u) = a(x)u^p$ in particular.

1. Existence of two positive solutions for general nonlinearity $g(x,u)$

In this section, we will show the existence of at least two positive solutions of (0.1) under suitable conditions. In some cases we prove the existence of two positive solutions of (0.1), even if the existence of a positive solution of (0.1) with $f(x) \equiv 0$ is not known.

1.1. Assumption on $g(x,u)$

We assume that

(A1) $g(x,s) \in C^1(\mathbf{R}^N \times \mathbf{R},\ \mathbf{R})$.

(A2) There exist constants $\delta_0 \in [0,1)$ and $m_0 > 0$ such that

$$0 < g(x,s) \le \delta_0 s + m_0 s^p \qquad \text{for all } x \in \mathbf{R}^N \text{ and } s > 0,$$

where $1 < p < \dfrac{N+2}{N-2}$ if $N \ge 3$, $1 < p < \infty$ if $N = 1,\ 2$.

(A3) There exists a constant $\theta > 2$ such that

$$0 < \theta G(x,s) \le g(x,s)s \qquad \text{for all } x \in \mathbf{R}^N \text{ and } s > 0,$$

where $G(x,s) = \displaystyle\int_0^s g(x,\tau)\,d\tau$.

(A4) $\dfrac{g(x,s)}{s}$ is strictly increasing in $s > 0$ uniformly in $x \in \mathbf{R}^N$ in the following sense:

$$\inf_{s\in[r_1,r_2],x\in\mathbf{R}^N} \frac{d}{ds}\left(\frac{g(x,s)}{s}\right) > 0 \qquad \text{for all } 0 < r_1 < r_2.$$

Moreover, we consider the situation that $g(x,s)$ approaches to some limit function $g^\infty(s) \in C^1(\mathbf{R},\ \mathbf{R})$ as $|x| \to \infty$:

(A5) $g(x,s) \to g^\infty(s)$ as $|x| \to \infty$ uniformly on any compact subset of $[0,\infty)$.

Moreover we assume

(A6) There exists a constant $\lambda > 2$ such that for any $\varepsilon > 0$ we can find a constant $C_\varepsilon > 0$ which satisfies

$$g(x,s) - g^\infty(s) \ge -e^{-\lambda|x|}(\varepsilon s + C_\varepsilon s^p) \qquad \text{for all } x \in \mathbf{R}^N \text{ and } s \ge 0.$$

Here the constant λ is corresponding to a convergent rate (from below) and the condition $\lambda > 2$ plays an important role in our existence result.

We remark that it follows from (A1)–(A5) that the limit function $g^\infty(s)$ satisfies similar conditions to (A1)–(A4):

(A1') $g^\infty(s) \in C^1(\mathbf{R},\ \mathbf{R})$.

(A2') $0 < g^\infty(s) \le \delta_0 s + m_0 s^p$ for all $s > 0$.

(A3') $0 < \theta G^\infty(s) \le g^\infty(s)s$ for all $s > 0$, where $G^\infty(s) = \displaystyle\int_0^s g^\infty(\tau)\,d\tau$.

(A4') $\dfrac{d}{ds}\left(\dfrac{g^\infty(s)}{s}\right) > 0$ for all $s > 0$.

1.2. Known results

Cao-Zhou [**CZ**], Jeanjean [**J**] (c.f. Hirano [**H**], Zhu [**Z**]) studied the problem (0.1) as a perturbation from the following homogeneous equation:

$$\begin{cases} -\Delta u + u = g(x,u) & \text{in } \mathbf{R}^N, \\ \quad u > 0 & \text{in } \mathbf{R}^N, \\ \quad u \in H^1(\mathbf{R}^N). \end{cases} \tag{1.1}$$

In addition to similar assumptions to (A1)–(A5), they needed

$$g(x,s) \geq g^\infty(s) \quad \text{for all } x \in \mathbf{R}^N \text{ and } s > 0 \tag{1.2}$$

and they succeeded to show that there exists a constant $M > 0$ such that if $f \geq 0$, $f \not\equiv 0$, $\|f\|_{H^{-1}(\mathbf{R}^N)} \leq M$, then (0.1) has at least two positive solutions. Here the constant $M > 0$ was chosen so that the corresponding functional:

$$I(u) = \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbf{R}^N} G(x,u)\,dx - \int_{\mathbf{R}^N} fu\,dx \,:\, H^1(\mathbf{R}^N) \to \mathbf{R},$$

where

$$\|u\|_{H^1} = \left(\int_{\mathbf{R}^N} |\nabla u|^2 + |u|^2\,dx\right)^{\frac{1}{2}},$$

possesses the mountain pass geometry. That is, if $\|f\|_{H^{-1}(\mathbf{R}^N)} \leq M$, then $I(u)$ satisfies

(i) there exists a constant $\rho_0 > 0$ such that

$$I(u) \geq 0 \quad \text{for all } u \in H^1(\mathbf{R}^N) \text{ with } \|u\|_{H^1(\mathbf{R}^N)} = \rho_0,$$

(ii) $\{u \in H^1(\mathbf{R}^N)\,;\, \|u\|_{H^1(\mathbf{R}^N)} > \rho_0 \text{ and } I(u) < 0\} \neq \emptyset$,

(iii) $\inf_{\|u\|_{H^1(\mathbf{R}^N)} < \rho_0} I(u) < 0.$

To see the role of the condition (1.2), we consider here the homogeneous problem (1.1). The corresponding functional is

$$J(u) = \frac{1}{2}\|u\|_{H^1(\mathbf{R}^N)}^2 - \int_{\mathbf{R}^N} G(x,u)\,dx.$$

It is well-known that the mountain pass critical value for $J(u)$ is attained at some critical point $u \in H^1(\mathbf{R}^N)$ under condition (1.2). However, without the condition (1.2), the mountain pass value is not attained in general. For example, it is not under condition: $g(s,x) < g^\infty(s)$ for all $a \in \mathbf{R}^N$, $a > 0$. See Lions [**PLL1**], [**PLL2**] for similar arguments.

We also remark that is seems that the existence of positive solution for (1.1) is not known without (1.2) in general. As far as we know, it is obtained just for the case $g(x,s) = a(x)s^p$. See Bahri-Li [**BaYL**], Bahri-Lions [**BaPLL**] for details. Thus the aim of our paper is to show the existence of positive solutions of (0.1) without (1.2). Even the existence of a positive solution for homogeneous problem (1.1) is not known, we can show the existence of at least two positive solutions for nonhomongeneous problem (0.1).

1.3. Main results

Our main result is as follows.

Theorem 1.1. *Assume that (A1)–(A6). Then there exists a constant $M > 0$ such that if $f \geq 0$, $f \not\equiv 0$, $\|f\|_{H^{-1}(\mathbf{R}^N)} \leq M$, then* (0.1) *has at least two positive solutions.*

We will prove Theorem 1.1 via variational methods. We find positive solutions of (0.1) as critical points of $I(u)$. First we find one positive solution $u_0(x)$ as a local minimum of $I(u)$ near 0. We remark that if $f \not\equiv 0$, then 0 is not a solution of our problem and the first positive solution is obtained as a perturbation of 0. Next we find a positive solution of (0.1) different from $u_0(x)$ by using the Mountain Pass Theorem. When we seek critical points of $I(u)$, we need to pay attention to the breaking down of Palais-Smale condition for $I(u)$.

2. Existence of four positive solutions in the case $g(x,u) = a(x)u^p$

In this section, we consider the equation (0.1) with $g(x,u) = a(x)u^p$, that is:

$$\begin{cases} -\Delta u + u = a(x)u^p + f(x) & \text{in } \mathbf{R}^N, \\ \quad u > 0 & \text{in } \mathbf{R}^N, \\ \quad u \in H^1(\mathbf{R}^N), \end{cases} \tag{2.1}$$

where $1 < p < \dfrac{N+2}{N-2}$ $(N \geq 3)$, $1 < p < \infty$ $(N = 1, 2)$. We also assume that for $a(x) \in C(\mathbf{R}^N)$

(H1) $a(x) > 0$ for all $x \in \mathbf{R}^N$,
(H2) $a(x) \to 1$ as $|x| \to \infty$,
(H3) there exist $\delta > 0$ and $C > 0$ such that

$$a(x) - 1 \geq -Ce^{-(2+\delta)|x|} \qquad \text{for all } x \in \mathbf{R}^N.$$

By Theorem 1.1, we see that if $\|f\|_{H^{-1}(\mathbf{R}^N)}$ is not so large, then (2.1) has at least two positive solutions without order relation between $a(x)$ and 1. We remark that the equation (2.1) with $f(x) \equiv 0$:

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{in } \mathbf{R}^N, \\ u > 0 & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), \end{cases} \tag{2.2}$$

possesses at least one positive solution only under condition (H1)–(H3). See Bahri-Li [**BaYL**]. (c.f. Bahri-Lions [**BaPLL**]). We also remark that Kwong [**K**] showed that the limit equation:

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbf{R}^N, \\ u > 0 & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), \end{cases} \tag{2.3}$$

possesses a unique positive radial solution $\omega(x) = \omega(|x|) > 0$ and any positive solution $u(x)$ of (2.3) can be written as

$$u(x) = \omega(x - x_0) \quad \text{for some } x_0 \in \mathbf{R}^N.$$

(c.f. Kabeya-Tanaka [**KT**]).

In this section, we consider (2.1) under

(H4) $a(x) \in (0,1]$ for all $x \in \mathbf{R}^N$, $a(x) \not\equiv 1$.

in addition to (H1)–(H3). We will show the existence of more positive solutions under (H1)–(H4). The uniqueness of positive solution of the limit equation (2.3) plays an important role in our existence results. Our main results are the following

Theorem 2.1 ([AT1]). *We assume (H1)–(H4). Then there exists a $\delta_0 > 0$ such that for non-negative function $f(x)$ satisfying $0 < \|f\|_{H^{-1}(\mathbf{R}^N)} \le \delta_0$, (2.1) possesses at least four positive solutions.*

As to an asymptotic behavior of solutions obtained in Theorem 2.1 as $\|f\|_{H^{-1}(\mathbf{R}^N)} \to 0$, we have

Theorem 2.2 ([AT1]). *Assume that a sequence of non-negative functions $(f_j(x))_{j=1}^\infty \subset H^{-1}(\mathbf{R}^N)$ satisfies $f_j(x) \not\equiv 0$ and*

$$\|f_j\|_{H^{-1}(\mathbf{R}^N)} \to 0 \qquad \text{as } j \to \infty.$$

Then there exist a subsequence of $(f_j(x))_{j=1}^\infty$ — still denoted by $(f_j(x))_{j=1}^\infty$ — and four sequences $(u_j^{(k)}(x))_{j\in\mathbf{N}}$ $(k = 1,2,3,4)$ of positive solutions of (2.1) with $f(x) = f_j(x)$ such

(i) $\|u_j^{(1)}\|_{H^1(\mathbf{R}^N)} \to 0$ *as* $j \to \infty$.
(ii) *There exist sequences* $(y_j^{(2)})_{j=1}^{\infty}$, $(y_j^{(3)})_{j=1}^{\infty} \subset \mathbf{R}^N$ *such that*

$$|y_j^{(k)}| \to \infty, \quad \|u_j^{(k)}(x) - \omega(x - y_j^{(k)})\|_{H^1(\mathbf{R}^N)} \to 0$$

as $j \to \infty$ *for* $k = 2, 3$.
(iii) *There exists a positive solution* $v_0(x)$ *of* (2.2) *such that*

$$\|u_j^{(4)}(x) - v_0(x)\|_{H^1(\mathbf{R}^N)} \to 0 \qquad \text{as } j \to \infty.$$

We use variational methods to find positive solutions of (2.1). We define for given $a(x)$ and $f(x)$

$$I_{a,f}(u) = \frac{1}{2}\|u\|_{H^1(\mathbf{R}^N)}^2 - \frac{1}{p+1}\int_{\mathbf{R}^N} a(x)u_+^{p+1}\,dx - \int_{\mathbf{R}^N} fu\,dx : H^1(\mathbf{R}^N) \to \mathbf{R},$$
$$J_{a,f}(v) = \max_{t>0} I_{a,f}(tv) : \Sigma_+ \to \mathbf{R},$$

where

$$\Sigma = \{v \in H^1(\mathbf{R}^N)\,;\, \|v\|_{H^1(\mathbf{R}^N)} = 1\},$$
$$\Sigma_+ = \{v \in \Sigma\,;\, v_+ \not\equiv 0\}.$$

We will see that critical points of $I_{a,f}(u) : H^1(\mathbf{R}^N) \to \mathbf{R}$ or $J_{a,f}(v) : \Sigma_+ \to \mathbf{R}$ are corresponding to positive solutions of (2.1).

We will find critical point of $I_{a,f}(u)$, $J_{a,f}(v)$ in the following way. First we find one positive solution $u^{(1)}(a, f; x) = u_{loc\ min}(a, f; x)$ as a local minimum of $I_{a,f}(u)$ near 0. Next we see that the Palais-Smale compactness condition for $I_{a,f}(u)$ and $J_{a,f}(v)$ breaks down only at levels

$$I_{a,f}(u_0(x)) + \ell I_{1,0}(\omega) \quad \ell = 1, 2, \ldots$$

where $I_{1,0}(u)$ is a functional corresponding to the limit equation (2.3), $\omega(x)$ is a unique positive radial solution of (2.3) and $u_0(x)$ is a critical point of $I_{a,f}(u)$. In particular, we will see that the Palais-Smale condition holds under the level $I_{a,f}(u_{loc\ min}(a, f; x)) + I_{1,0}(\omega)$.

Next we find two critical points different from $u_{loc\ min}$ under the first level of breaking down of Palais-Smale condition, that is, under the level $I_{a,f}(u_{loc\ min}(a, f; x)) + I_{1,0}(\omega)$. We use notation:

$$[J_{a,f} \le c] = \{u \in \Sigma_+\,;\, J_{a,f}(u) \le c\}$$

for $c \in \mathbf{R}$. We will observe that for sufficiently small $\varepsilon > 0$

$$[J_{a,f} \le I_{a,f}(u_{loc\ min}(a, f; x)) + I_{1,0}(\omega) - \varepsilon]$$

is not empty and

$$\mathrm{cat}([J_{a,f} \leq I_{a,f}(u_{loc\ min}(a,f;x)) + I_{1,0}(\omega) - \varepsilon]) \geq 2 \tag{2.4}$$

provided $f(x) \geq 0$, $f(x) \not\equiv 0$ and $\|f\|_{H^{-1}(\mathbf{R}^N)}$ is sufficiently small. Here $\mathrm{cat}(\cdot)$ stands for the Lusternik-Schnirelman category. We find two positive solutions $u^{(2)}(a,f;x)$ and $u^{(3)}(a,f;x)$ satisfying

$$I_{a,f}(u^{(k)}(a,f;x)) < I_{a,f}(u_{loc\ min}(a,f:x)) + I_{1,0}(\omega) \quad \text{for } k = 2,3. \tag{2.5}$$

We remark that for $f \equiv 0$, we see that

$$u_{loc\ min}(a,0;x) \equiv 0$$

and

$$[J_{a,0} \leq I_{a,0}(u_{loc\ min}(a,0;x) + I_{1,0}(\omega)] = \emptyset \tag{2.6}$$

and (2.4) is the key of our proof. To get (2.4), we use the following interaction phenomenon as in [**AT2**] (c.f. Bahri-Coron [**BaC**], Bahri-Li [**BaYL**], Bahri-Loins [**BaPLL**], Taubes [**T**]):

$$I_{a,f}(u_{loc\ min}(a,f;x) + \omega(x-y)) < I_{a,f}(u_{loc\ min}(a,f;x)) + I_{1,0}(\omega)$$

for sufficiently large $|y| \geq 1$.

To find the fourth positive solution, we adapt the minimax method of Bahri-Li [**BaYL**] to our functional $J_{a,f}(v)$. More precisely, we define

$$b_{a,f} = \inf_{\gamma\in\Gamma} \sup_{y\in\mathbf{R}^N} J_{a,f}(\gamma(y)),$$

where

$$\Gamma = \{\gamma \in C(\mathbf{R}^N, \Sigma_+)\,;\, \gamma(y) = \frac{\omega(\cdot - y)}{\|\omega\|_{H^1(\mathbf{R}^N)}} \quad \text{for large } |y|\}.$$

Then we will find a positive solution $u^{(4)}(a,f;x)$ corresponding to the minimax value $b_{a,f}$ which satisfies

$$I_{a,f}(u^{(4)}(a,f;x)) > I_{a,f}(u_{loc\ min}(a,f;x)) + I_{1,0}(\omega)$$

for sufficiently small $\|f\|_{H^{-1}(\mathbf{R}^N)}$. To show Theorem 2.2, we also use (2.5) and (2.6) in an essential way.

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