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Analytic Smoothing Effect for the Benjamin-Ono Equations

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1. Introduction

We study smoothing effect for the following nonlinear dispersive equation of the Benjamin-Ono type:

$$(1.1)\qquad \begin{cases} \partial_t u + H_x \partial_x^2 u + \partial_x u^2 = 0, & t \in (-T,T), \quad x \in \mathbb{R}, \\ v(0,x) = \phi(x), \end{cases}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a unknown function and H_x denotes the Hilbert transform defined by $H_x v = \mathcal{F}\frac{\xi}{i|\xi|}\hat{v}$. This equation arises in the water wave theory and u describes long internal gravity wave in deep stratified fluid (see [2], [31]). Our problem here is to investigate a sufficient condition of the initial data ϕ on which the solution has regularizing property up to analyticity.

The existence and well-posedness problem of this equation is studied by many authors. We refer to T. Kato [21], Iorio Jr. [14], Ponce [32], Kenig, Ponce and Vega [26] and reference therein. In the recent studies for the nonlinear dispersive equations, large amounts of studies are devoted to the smoothing effect. When we consider the well-posedness of those type of equation, L^2 based (Sobolev) space is usually considered and the same order of the regularity for solutions is derived as the initial data ϕ. Concerning the dispersive equation such as KdV, nonlinear Schrödinger and the Benjamin-Ono type equations, local or somewhat restricted version (in terms of weighted norm) of smoothing effect was observed. As the most well understood example, we would refer to the case of nonlinear Schrödinger equations in [3], [4], [6], [9], [10], [11], [12],[18], [20], [30] and case of linear Schrödinger equations in [16] and [33]. Since the Benjamin-Ono equation has a similar dispersive structure in its linear part $\partial_t u + H_x \partial_x^2 u$ as the Schrödinger equations, we would expect that an analogous result holds for the nonlinear problem (1.1).

Concerning the analytic smoothing effect, we know that a drastic smoothing effect holds for the KdV equation and nonlinear Schrödinger equations. Especially for the KdV equation, it is shown that for a weak initial data including the Dirac delta measure, the corresponding weak solution gains the regularity up to analytic in both space and time variable by virtue of the conormal vector fields (see K.Kato and Ogawa [17] and also for the Schrödinger cases [18] and K.Kato and Taniguchi [20]). In this paper, we would extend these results to the Benjamin-Ono case (1.1). Our method is based on an operator method which is common to the cases of the KdV equation or nonlinear Schrödinger equations: We introduce the generator of space-time dilation $P = 2t\partial_t + x\partial_x$ that plays a compensating role where the main linear operator $L = \partial_t + H_x\partial_x^2$ can not gain the regularity. As a consequence, we observe analytic smoothing effect for the solution to (1.1) with an initial data having a singularity at one point. We state this more specifically as follows. Let $H^s = H^s(\mathbb{R})$ be the Sobolev space of order s defined by

$$\|f\|_{H^s} \equiv \|\langle\xi\rangle^s \hat{f}\|_2,$$

where $\hat{f} = \mathcal{F}f$ denotes the Fourier transform of f and $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$.

Theorem 1.1. *Let $s > 3/2$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

then there exists a unique solution $u \in C(\mathbb{R}, H^s)$ to the nonlinear dispersive equation (1.1) and for any $(t,x) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$, we have for some $A > 0$

$$|\partial_t^j \partial_x^l u(t,x)| \le C\langle t^{-1}\rangle^{j+l} \langle x\rangle^{2l+3j} A^{j+l} (j+l)!$$

for any $j, l \in \mathbb{N}$. Namely $u(t,\cdot)$ is a real analytic function in both space and time variables for $t \neq 0$.

Remark 1. The assumption on the initial data implies that the data have to be analytic except $x = 0$. On this point the data is assumed to have only H^s regularity. Hence the above theorem states that this singularity disappears after time passed. The weakness of this singularity on the data is depending on the space where we may establish the well-posedness of the equation.

The existence and uniqueness result of the Benjamin-Ono equation can be found in the articles by Iorio Jr. [14], Ponce [32]. The global well-posedness in time is also discussed in Kenig, Ponce and Vega [26]. Our result is based on those well-posedness results in the Sobolev space $H^s(\mathbb{R})$ with $s > 3/2$. It seems that the well-posedness in a weaker spaces than $H^{3/2}$ is not well established so far as the authors know. If this is improved into the lower regularity classes like H^s with $s \le 3/2$, we may extend our result into such a weak space even negative exponent Sobolev spaces. See [17] for this direction for the KdV equation case.

Remark 2. It is well-known that the global in time solution has been obtained to Benjamin-Ono type equations by both the inverse scattering and analytical (continuing) methods. Since our result shows that the solution reaches analytic in space time variables, one can show that the result is valid globally in time through the result by T.Kato and Masuda [23].

By the similar argument as in Theorem 1.1, one can also show the following weaker theorem in the analytic and Gevrey regularity.

Theorem 1.2. *Let $s > 3/2$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{(k!)^2} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

then there exists a unique solution $u \in C(\mathbb{R}, H^s)$ to the nonlinear dispersive equation (1.1) and for any $(t,x) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$, $u(t,\cdot)$ is an analytic function in space variable and for $x \in \mathbb{R}$, $u(t,x)$ is of Gevrey 2 as a time variable function for $t \neq 0$.

Remark 3. In both Theorems, the assumption on the initial data implies the analyticity and Gevrey 2 regularity except the origin respectively. In this sense, these results state that the singularity at the origin immediately disappears after $t > 0$ or $t < 0$ up to analyticity.

Some related results are obtained for linear and nonlinear Schrödinger equations. For linear variable coefficient case, see Kajitani and Wakabayashi [16], Robbiano and Zuily [33] and for nonlinear case, Chihara [3]. They give a global weighted uniform estimates of the solution with arbitrary order derivative in space variable.

The essential difference in proving the above type results from the case for the nonlinear Schrödinger or KdV equation is the appearance of the nonlocal operator H_x. Since our method uses some localization technique, it is required to treat the non local term carefully to show the higher regularity. We then introduce a weight function which has an explicit commutation estimate with H_x. This enables us to handle the nonlocal term H_x in the linear part of the equation. We explain this part in the following sections.

Here we summarize some notation that we would use in what follows. $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. H^s is the Sobolev space of order s. Let $L = \partial_t + H_x\partial_x^2$ be the linear part of the Benjamin-Ono equation and $P = 2t\partial_t + x\partial_x$ be the dilation operator associated with L. For operators A and B, $[A,B]$ stands for the commutator $AB - BA$. The free propagator group for the linear Benjamin-Ono type evolution is denoted by $e^{-tH_x\partial_x^2}$ which is a unitary operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

2. METHOD

In this section we give an overview of the proof and present some difference from the proof of the former cases in [17] and [18]. The results are based on the following observation.

Noting the commutation relation between the generator of the dilation $P = 2t\partial_t + x\partial_x$ and the linear dispersive operator $L \equiv \partial_t + H_x\partial_x^2$:

$$[L, P] = 2L,$$

we have

$$\begin{aligned} LP^k &= (P+2)^k L, \\ (P+2)^k\partial_x &= \partial_x(P+1)^k, \qquad k = 1, 2, \cdots. \end{aligned} \tag{2.1}$$

Applying $P = 2t\partial_x + x\partial_x$ to the equation (1.1) iteratively, we have

$$\partial_t(P^k u) + H_x\partial_x^2(P^k u) = (P+2)^k Lu = -(P+2)^k\partial_x(u^2). \tag{2.2}$$

Setting $u_k = P^k u$ and $B_k(u,u) = -(P+2)^k\partial_x u^2$, we have

$$\partial_t u_k + H_x\partial_x^2 u_k = B_k(u,u) = -\partial_x \sum_{k=k_0+k_1+k_2} \frac{k!}{k_0!k_1!k_2!} u_{k_1}u_{k_2}. \tag{2.3}$$

An important point here is that the nonlinear terms $B_k(u,u)$ maintain the bilinear structure similar to the original Benjamin-Ono equation. This is due to the fact that the Leibniz law can be applicable for an operation of P. Thus each of u_k satisfies the following system of equations;

$$\begin{cases} \partial_t u_k + H_x\partial_x^2 u_k = B_k(u,u), \qquad t, x \in \mathbb{R}, \\ u_k(0,x) = (x\partial_x)^k\phi(x). \end{cases} \tag{2.4}$$

Firstly we establish the local well-posedness of the solution to the following infinitely coupled system of dispersive equation in a proper Sobolev space:

$$\begin{cases} \partial_t u_k + H_x\partial_x^2 u_k = B_k(u,u), \qquad t, x \in \mathbb{R}, \\ u_k(0,x) = \phi_k(x). \end{cases} \tag{2.5}$$

Taking $\phi_k = (x\partial_x)^k\phi(x)$, the uniqueness and local well-posedness allow us to say $u_k = P^k u$ for all $k = 0, 1, \cdots$.

Through showing the existence and uniqueness process, we obtain the estimate

$$\|P^k u\|_{H^s} \le CA^k k!.$$

Until this step, there is no effect from the appearance of the non local operator H_x.

Next we would derive the pointwise derivative estimate by using the equation:

$$H_x\partial_x^2 P^k u = -\frac{1}{2t}P^{k+1}u + \frac{1}{2t}x\partial_x P^k u + B_k(u,u). \tag{2.6}$$

To treat the second term of the right hand side of (2.6), we employ localization argument. With a suitable decaying weight function $a = a(x)$, we can show that

$$\|a\partial_x^l P^k u(t)\|_{H^1(\mathbb{R})} \le C\langle t^{-1}\rangle^l A^{k+l}(k+l)!, \quad k,l = 0,1,2,\cdots.$$

and then by iterative argument, we can shift from the estimate with the operator P to the one with $t\partial_t$ and conclude

$$\|(t\partial_t)^{l_1}\partial_x^{l_2}u(t)\|_{L^\infty(x_0-\delta,x_0+\delta)} \le C\langle t^{-1}\rangle^{l_1+l_2}\langle x_0\rangle^{3l_1+2l_2}A^{l_1+l_2}(l_1+l_2)!, \tag{2.7}$$

for $l_1, l_2 = 0,1,2,\cdots$. A crucial step for obtaining the above derivative estimates is to treat the nonlocal operator H_x which is an essential difference from the KdV equation or nonlinear Schrödinger equations. In order to handle this term, it is required to show an explicit dependence of the iteration of the commutator estimate

$$\|[H_x, a^k]\|_{\mathcal{L}(L^2\to L^2)} \le C_k,$$

where $a = a(x)$ is a cut-off function and $a^k = a(x)^k$. We then choose a particular weight function $a(x) = \langle x\rangle^{-2}$, where $\langle x\rangle = (1+|x|^2)^{1/2}$ and derive an explicit commutation estimate with the Hilbert transform and a^k. By this step, we may use the equation (2.6) to gain the regularity and to show the analyticity (2.7). Here we only exhibit the following lemma which treats the commutator of H_x and a^k.

Lemma 2.1. *If $\|a^l\partial_x^l f\|_2 \le CA^l l!\|f\|_2$ for $0 \le l \le N-1$, then we have*

$$\|[H_x, a^N]\partial_x^N f\|_2 \le CA^N N!\|f\|_2.$$

Proof of Lemma 2.1. The result is obtained by using the explicit expression of the commutator $[H_x, a]$. [1] Let $f \in \mathcal{S}$. Since

$$\|[H_x, a^N]\partial_x^N f\|_2 \le \sum_{j=0}^{N-1}\|a^j[H,a]a^{N-1-j}\partial_x^N f\|_2, \tag{2.8}$$

it suffices to show that

$$\|a^j[H_x,a]a^{N-1-j}\partial_x^N f\|_2 \le CA^N(N-1)!\|f\|_2.$$

An elementary computation gives

$$[H_x, a]f = p.v.\int_{\mathbb{R}}\frac{a(y)-a(x)}{x-y}f(y)dy = \int_{\mathbb{R}}\frac{x+y}{\langle x\rangle^2\langle y\rangle^2}f(y)dy.$$

[1] It is also possible to show the N dependence of the operator norm of $[H_x, a^N]$ directly by passing the Fourier transform.

By integration by parts, we have

$$
\begin{aligned}
\|a^j[H_x,a]a^{N-1-j}\partial_x^N f\|_2^2 &= \int_{\mathbb{R}}\left|\langle x\rangle^{-2j}\int_{\mathbb{R}}\frac{x+y}{\langle x\rangle^2\langle y\rangle^2}\langle y\rangle^{-2(N-1-j)}\partial_x^N f(y)dy\right|^2 dx \\
&= \iiint \frac{(x+y)(x+z)}{\langle x\rangle^{4(j+1)}}\langle y\rangle^{-2(N-j)}\langle z\rangle^{-2(N-j)}\partial_y^N f(y)\partial_z^N \bar{f}(z)dydzdx \\
&= \iint \partial_y^{j+1}\partial_z^{j+1}\left\{\left(\int_{\mathbb{R}}\frac{(x+y)(x+z)}{\langle x\rangle^{4(j+1)}}dx\right)\langle y\rangle^{-2(N-j)}\langle z\rangle^{-2(N-j)}\right\} \\
&\qquad \times \partial_y^{N-j-1}f(y)\partial_z^{N-j-1}\bar{f}(z)dydz.
\end{aligned}
\tag{2.9}
$$

If we set $\sigma(y,z) = \displaystyle\int_{\mathbb{R}}\frac{(x+y)(x+z)}{\langle x\rangle^{4(j+1)}}dx = \sigma_1+\sigma_0 yz$ and $\max(\sigma_0,\sigma_1) = \tilde{\sigma}$, where σ_i $(i=0,1)$ are constants of order $j^{1/2}$, then

$$
\begin{aligned}
&\left|\partial_y^{j+1}\partial_z^{j+1}\left(\sigma(y,z)\langle y\rangle^{-2(N-j)}\langle z\rangle^{-2(N-j)}\right)\right| \\
&\le \sigma_0 j^2\partial_y^j\langle y\rangle^{-2(N-j)}\partial_z^j\langle z\rangle^{-2(N-j)} + \sigma_0 jy\partial_y^{j+1}\langle y\rangle^{-2(N-j)}\partial_z^j\langle z\rangle^{-2(N-j)} \\
&\qquad + \sigma_0 jz\partial_y^j\langle y\rangle^{-2(N-j)}\partial_z^{j+1}\langle z\rangle^{-2(N-j)} \\
&\qquad\qquad + (\sigma_1+\sigma_0 yz)\partial_y^{j+1}\langle y\rangle^{-2(N-j)}\partial_z^{j+1}\langle z\rangle^{-2(N-j)} \\
&\le C_0 j^2 A_0^{j+1}\left(\frac{2^j N!}{(N-j-1)!}\right)^2\langle y\rangle^{-2N+j}\langle z\rangle^{-2N+j} \\
&\quad + \sigma_0 jA_0^{j+1}\left(\frac{2^j N!}{(N-j-1)!}\right)\left(\frac{2^j(N+1)!}{(N-j-1)!}\right) \\
&\quad\times\{\langle y\rangle\langle y\rangle^{-2N+j-1}\langle z\rangle^{-2N+j}+\langle z\rangle\langle y\rangle^{-2N+j}\langle z\rangle^{-2N+j-1}\} \\
&\qquad + \tilde{\sigma}\langle y\rangle\langle z\rangle A_0^{j+1}\left(\frac{2^j(N+1)!}{(N-j-1)!}\right)^2\langle y\rangle^{-2N+j-1}\langle z\rangle^{-2N+j-1} \\
&\le C\tilde{\sigma}(j+1)^2 A_1^{j+1}\left(\frac{N!}{(N-j-1)!}\right)^2\langle y\rangle^{-2N+j}\langle z\rangle^{-2N+j}.
\end{aligned}
\tag{2.10}
$$

Hence it follows by the assumption that

(2.11)

$$
\begin{aligned}
\|a^j[H_x,a]a^{N-1-j}\partial_x^N f\|_2^2 &\le C\tilde{\sigma}(j+1)^2 A_1^{j+1}\left(\frac{N!}{(N-j-1)!}\right)^2\|a^{N-j/2}\partial_x^{N-j-1}f\|_1^2 \\
&\le C\tilde{\sigma}(j+1)^2 A_1^{j+1}\left(\frac{N!}{(N-j-1)!}\right)^2\|a^{j/2+1}\|_2^2\|a^{N-j-1}\partial_x^{N-j-1}f\|_2^2 \\
&\le C\tilde{\sigma}(j+1)^2 A_1^{j+1}A_1^{2(N-j-1)}(N!)^2\|f\|_2^2 \\
&\le 4^{-(j+1)}C^2A^{2N}(N!)^2\|f\|_2^2
\end{aligned}
$$

and we conclude

$$\|[H_x, a^N]f\|_2 \le CA^N N! \sum_{j=1}^{N-1} 2^{-(j+1)}\|f\|_2 \le CA^N N!\|f\|_2.$$

□ □

Based upon the above Lemma 2.1, we can show the analyticity.

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