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A polynomial time approximation scheme for the minimum maximal matching problem in planar graphs

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Abstract: Given an undirected graph G, the minimum maximal matching problem asks to find a minimum matching that is inclusionwise maximal. The problem is known to be NP-hard even if the graph is planar. We consider the problem for planar graphs, and show that a polynomial time approximation scheme (PTAS) can be obtained by a divide-and-conquer method based on the planar separator theorem. For a given $\epsilon > 0$, our scheme delivers in $O(n \log n + \alpha^{\frac{1}{\epsilon^2}} \epsilon^{-1} n)$ time a solution with size at most $(1+\epsilon)$ times the optimal value, where n is the number of vertices in G and α is a constant number.

Keywords: graph algorithm, approximation algorithm, matching, planar graph, separator.

1 Introduction

Given an undirected graph $G = (V, E)$, a matching is a subset M of E containing no two adjacent edges. A matching M is said to be *maximal* if there is no matching M' which strictly contains M. The *minimum maximal matching problem* asks to find a maximal matching containing the minimum number of edges. The problem is one of the NP-hard problems included in the list of NP-complete problems [3, p.192], and the problem remains NP-hard for planar graphs and for bipartite graphs, in both cases even if no vertex degree exceeds 3 [10]. As to approximability, the problem is shown to be APX-hard for general graphs [1, p.374]. In this paper, we consider the complexity status of the minimum maximal matching problem for planar graphs.

An algorithm is called an α-approximation algorithm to a minimization problem if it outputs a solution whose weight is at most α times of the weight of an optimal solution. A polynomial time approximation scheme (PTAS) to a minimization problem A is an algorithm that, given an instance of A and a precision $\epsilon > 0$, finds a $(1+\epsilon)$-approximate solution in time that is polynomial for each fixed ϵ. For planar graphs, several NP-hard problems admit PTASs. For example, the *maximum independent set problem* and the *minimum vertex cover problem* are known to have PTASs in planar graphs [2, 6, 8]. The algorithms in [6, 8] are based on a divide-and-conquer approach based on the planar separator theorem [5]. In this method the problem of interest is divided into two or more smaller problems. The subproblem are solved by applying the method recursively, and then solutions to the subproblems are recursively combined into a solution to the original problem. In the scheme a planar separator is used as a method to divide a given planar graph.

Based on a decomposition of planar graphs different from planar separators, Baker [2] presented a general method for providing PTASs for a variety of the optimization problems on planar graphs, which includes the minimum vertex cover problem and the maximum independent set problem. The paper also pointed out some NP-hard problems to which her method cannot be applied in a straightforward way. For example, it says that the minimum maximal matching problem is one of such problems because the restriction of an optimal solution M in G on a vertex subset X (i.e., the set of edges in M whose endvertices belong to X) may not be a maximal matching in the graph induced by X. (We remark that the similar difficulty necessarily arises for the minimum edge dominating set problem, which is claimed to admit a PTAS in [2].)

In this paper we use the divide-and-conquer approach to solve the minimum maximal matching problem for planar graphs. However, a naive application of this approach does not yield a PTAS to the problem. One of the reasons is that the size of a minimum maximal matching can be arbitrarily small, compared with the size $|V|$ of a graph $G = (V, E)$. For this, we reduce an arbitrary planar graph $G = (V, E)$ to a particular planar graph, called an *irreducible planar graph*, so that the size of a minimum maximal matching is $\Omega(|V|)$ (this property is important to obtain a PTAS by the divide-and-conquer approach). Another difficulty of the problem is that the restriction of an optimal solution M in G on a vertex subset X may not be feasible. We overcome this by a careful analysis of the performance of our divide-and-conquer method. As a result, for a given $\epsilon > 0$, our scheme delivers a $(1+\epsilon)$-approximate solution in $O(n \log n + \alpha^{\frac{1}{\epsilon^2}} \epsilon^{-1} n)$ time, where n is the number of vertices in a given planar graph and α is a constant number.

2 Preliminaries

Let $G = (V, E)$ stand for a simple undirected graph with a vertex set V and an edge set E. The vertex set (resp., edge set) of a graph G may be denoted by $V(G)$ (resp., $E(G)$). For a subset $X \subseteq V(G)$, $G-X$ denotes the graph obtained from G by removing the vertices in X together with edges incident to them. Let $V[e]$ be the set of endpoints of an edge e. Let $d_G(v)$ denote the degree of a vertex $v \in V$. Let $\delta(G) = \min_{v \in V} d_G(v)$. A vertex v with $d_G(v) = 1$ is called a *leaf vertex*. An edge incident to a leaf vertex is called a *leaf edge*. An non-leaf edge one of whose endpoints is incident to only leaf edges is called a *fringe edge*. A *maximum matching* is a matching of the maximum size. Let $\mu(G)$ denote the size of a maximum matching of G and $\rho(G)$ denote the size of a minimum maximal matching.

We introduce lower bounds on the size of a minimum maximal matching in a graph G. It is easy to see that the size of any maximal matching is at least half of the size of a maximum matching. That is,

$$\rho(G) \geq \frac{1}{2}\mu(G). \tag{1}$$

Thus we can use any lower bound on the size of a maximum matching as that on the size of a minimum maximal matching within a constant factor. We always assume that a given planar graph is equipped with a fixed plane embedding. Namely G is a *plane* graph. The following fact about planarity is known.

Theorem 2.1 [4] *Every planar graph with $n \geq 3$ vertices contains no more than $3n - 6$ edges.* □

A following lower bound on $\mu(G)$ for planar graphs is known.

Theorem 2.2 [7] *If $G = (V, E)$ is a connected planar graph with $\delta(G) \geq 3$ and $n = |V|$, then $\mu(G) \geq \min\{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n+2}{3} \rceil\}$.* □

Our algorithm uses the following partition of a vertex set of a planar graph.

Theorem 2.3 [5] *Let G be a planar graph with n vertices. Then the vertices of G can be partitioned into three sets A, B, C such that no edge joins a vertex in A with a vertex in B, neither A nor B contains more than $\frac{2}{3}n$ vertices, and C contains no more than $2(2n)^{1/2}$ vertices. Such a partition can be found in linear time.* □

A vertex set C in the theorem is called a *planar separator*. In the following sections, we prove the next theorem.

Theorem 2.4 *Given a connected planar graph $G = (V, E)$ and $\epsilon > 0$, the minimum maximal matching problem is $(1 + \epsilon)$-approximable in $O(n \log n + \alpha^{\frac{1}{\epsilon^2}} \epsilon^{-1} n)$ time, where $n = |V|$ and α is a constant number.* □

A subset D of edges in $G = (V, E)$ is called an *edge dominating set* if every edge in $E - D$ is adjacent to an edge in D. The *edge dominating set problem* asks to find an edge dominating set of the minimum size. As pointed out in [10], the size of a minimum maximal matching of a graph G is equal to that of a minimum edge dominating set in G and, from any maximal matching M, an edge dominating set D with $|D| = |M|$ can be constructed in linear time. Then the above theorem implies the next result.

Corollary 1 *The edge dominating set problem in a planar graph G admits a PTAS with the same performance in Theorem 2.4.* □

3 Algorithm

3.1 Preprocess

For an arbitrary planar graph G, $\rho(G)$ cannot be bounded from below by $c|V(G)|$ for any constant c. In this subsection, we present how to process a given graph to obtain a graph G' with $\rho(G') = \Omega(|V(G')|)$ without losing the optimality of the problem.

Definition 1 *A graph G (not necessarily planar) is called irreducible if*
(i) *G is simple and connected,*
(ii) *G has no fringe edges,*
(iii) *each vertex $v \in V(G)$ has at most one leaf vertex adjacent to it,*
(iv) *any two vertices $u, v \in V(G)$ have at most two common neighbors of degree 2.* □

The following procedure converts a given graph G into an irreducible one without changing the optimality of the minimum maximal matching problem.

Algorithm REDUCE
Input: A connected graph G.
Output: An irreducible graph G' and a matching M' of G such that $\rho(G) = \rho(G') + |M'|$.
Let $M' := \emptyset$.

while there is a fringe edge e **do**
  Choose a fringe edge e and let $M' := M' \cup \{e\}$, discarding all edges adjacent to e.
end while

while there is a vertex u to which at least two leaf vertices are adjacent **do**
  Choose such a vertex u.
  Choose one leaf vertex adjacent to u and discard the rest of all leaf vertices adjacent to u.
end while

while there is a pair of vertices u and v which have at least three common neighbors of degree 2 **do**
  Choose such a pair of vertices u and v.
  Choose two vertices of degree 2 adjacent to u and v and discard the rest of all vertices of degree 2 adjacent to u and v.
end while

Let G' be the resulting graph. □

Then we have the following result (the proof is omitted). We denote by $E^{opt}(G)$ a minimum maximal matching in G.

Lemma 1 *For a given graph $G = (V, E)$, let G' and M' be the graph and the matching obtained from G by Algorithm REDUCE. Then for any $E^{opt}(G')$, $E^{opt}(G') \cup M'$ is a minimum maximal matching in G, and G' is irreducible. REDUCE can be implemented to run in $O(n+m)$ time, where $n = |V|$ and $m = |E|$.* □

For a planar graph G with n vertices, Algorithm REDUCE runs in $O(n)$ time by Theorem 2.1. In what follows, we consider how to find an approximation solution to an irreducible planar graph G. If $|V(G)| \leq 36$, then we find a minimum maximal matching $E^{opt}(G)$ by checking every subset of E. Otherwise (i.e., $|V(G)| \geq 37$), we use the property that $\mu(G) = \Omega(n)$ in an irreducible planar graph G.

Lemma 2 *Let $G = (V, E)$ be an irreducible planar graph with $n = |V| \geq 37$. Then, $\mu(G) \geq \frac{1}{42}n + \frac{13}{21}$.*

Proof: See Appendix. □

3.2 Approximation algorithm

For a graph G with a sufficiently small number of vertices, we find a minimum maximal matching by using the next lemma.

Lemma 3 *For a graph G with n vertices and m edges, a minimum maximal matching can be found in $O(2^n\sqrt{n}m)$ time*

Proof: Omitted. □

Now we are ready to describe our approximation algorithm.

Algorithm DIVIDE

Input: An irreducible planar graph G and a real number $\epsilon > 0$.

Output: A maximal matching $E^{apx}(G)$ of G such that $|E^{apx}(G)| \leq (1+\epsilon)\rho(G)$.

1. Let $L := (\frac{1943}{\epsilon})^2$, and $C^* := \emptyset$.

2. **while** $G - C^*$ has a connected component with more than L vertices **do**
   Choose such a connected component G'.
   Find a planar separator $C \subseteq V(G')$ by applying Theorem 2.3 to G'.
   $C^* := C^* \cup C$.
   end while

3. For each of connected components $G_i = (V_i, E_i)$ $i = 1, 2, \ldots, p$ of $G - C^*$ (where $|V_i| \leq L$ for all i), find a minimum maximal matching $E^{opt}(G_i)$ by using Lemma 3.

4. Extend a matching $\bigcup_{1\leq i\leq p} E^{opt}(G_i)$ to a maximal one in G by adding a set M^* of independent edges. Let $E^{apx}(G) = \bigcup_{1\leq i\leq p} E^{opt}(G_i) \cup M^*$. □

Notice that each edge in M^* must be incident to a vertex in the final C^* by the maximality of each $E^{opt}(G_i)$. Thus $|M^*| \leq |C^*|$.

3.3 Analysis

We first analyze the approximation ratio of algorithm DIVIDE.

Lemma 4 *Let $E^{apx}(G)$ be a maximal matching obtained from an irreducible planar graph G with $|V(G)| \geq 37$ by Algorithm DIVIDE. Then $|E^{apx}(G)| \leq (1+\epsilon)\rho(G)$ holds.*

Proof: Let $n = |V(G)|$. By the inequality (1) and Lemma 2, it holds

$$|E^{opt}(G)| = \rho(G) \geq \frac{1}{2}\mu(G) \geq \frac{1}{84}n. \quad (2)$$

By $|M^*| \leq |C^*|$,

$$\begin{aligned} |E^{apx}(G)| &= \sum_i |E^{apx}(G_i)| + |M^*| \\ &\leq \sum_i |E^{opt}(G_i)| + |C^*|. \quad (3) \end{aligned}$$

We now compare $\sum_i |E^{opt}(G_i)|$ with $|E^{opt}(G)|$. It should be noted that $E^{opt}(G) \cap E_i$ is not necessarily a maximal matching in G_i in Step 3, and we may need to add some edges from $E(G_i)$ to $E^{opt}(G) \cap E_i$ in order to make it maximal in G_i. Then, each of these edges joins two vertices u and v that are adjacent to vertices in C^* via some edges $e_u, e_v \in E^{opt}(G)$. Thus the number of such edges $e_u, e_v \in E^{opt}(G)$ is at most $|C^*|$. Hence the number of edges to be added to $E^{opt}(G) \cap E_i$ over all G_i is at most $\frac{1}{2}|C^*|$. Therefore we have

$$\sum_i |E^{opt}(G_i)| \leq |E^{opt}(G)| + \frac{1}{2}|C^*|. \quad (4)$$

By (3) and (4), we get

$$|E^{apx}(G)| \leq |E^{opt}(G)| + \frac{3}{2}|C^*|. \quad (5)$$

Now, we claim that $|C^*| \leq d\epsilon n$ holds for a constant number d. Consider all the connected components which appeared during an execution of the above procedure. Assign a *level* to each component as follows: the final components (with at most L vertices) have level 0; and each of the components has a level one greater than the maximum level of the components produced from it. Obviously any two components of the same level are disjoint.

Since a component of level i has at least $(\frac{3}{2})^i$ vertices, the maximum level ℓ must satisfy $(\frac{3}{2})^i \leq n$ or $\ell \leq \log_{\frac{3}{2}} n$. Since every component of level 1 has at least L vertices, every components of level i has at least $(\frac{3}{2})^{i-1}L$ vertices. Therefore the number c_i of components of level i is at most $(\frac{2}{3})^{i-1}\frac{n}{L}$ because of $c_i(\frac{3}{2})^{i-1}L \leq n$.

Now we can bound the size of C^* as follows. Let $n^i_j, 1 \leq j \leq c_i$, be the number of vertices in the jth component of level i. Then we have

$$\begin{aligned} |C^*| &\leq \sum_{1\leq i\leq \ell}\sum_{1\leq j\leq c_i} 2(2n^i_j)^{1/2} \\ &\leq 2(2)^{1/2}\sum_{1\leq i\leq \ell}(c_i\sum_{1\leq j\leq c_i} n^i_j)^{1/2} \\ &\leq 2(2)^{1/2}\sum_{1\leq i\leq \ell} c_i^{1/2} n^{1/2} \\ &\leq 2(2)^{1/2}(\frac{n}{\sqrt{L}})\sum_{1\leq i\leq \ell}(\sqrt{\frac{2}{3}})^{(i-1)} \\ &\leq 2\sqrt{6}\frac{1-(\sqrt{\frac{2}{3}})^{\ell}}{\sqrt{3}-\sqrt{2}}\cdot\frac{n}{\sqrt{L}} \\ &\leq \frac{6\sqrt{2}+4\sqrt{3}}{1943}\epsilon n. \quad (6) \end{aligned}$$

The approximate ratio is evaluated by (2), (5) and (6) as follows.

$$\frac{|E^{apx}(G)|}{|E^{opt}(G)|} \leq 1 + \frac{\frac{3}{2}|C^*|}{|E^{opt}(G)|}$$
$$\leq 1 + (\frac{n}{84})^{-1}\cdot\frac{3}{2}\cdot\frac{6\sqrt{2}+4\sqrt{3}}{1943}\epsilon n \leq 1+\epsilon. \ \square$$

We finally evaluate the running time of Algorithm DIVIDE for a planar graph $G = (V, E)$, where $n = |V|$ and $m = |E|$. First, we consider the running time by Step 2. A planar separator C is found in linear time by Theorem 2.3 and the number of recurrences during Step 2 is $O(\log n)$ since the separator partitions a graph into two graphs so that the size of these graphs decrease by a constant factor. Therefore it takes $O(n \log n)$ time to decompose a given graph into subgraphs with at most L vertices. Next, we consider the running time to find optimal solutions for all subgraphs. By applying Lemma 3 to G_i, where $|E(G_i)| = O(|V(G_i)|)$, an $E^{opt}(G_i)$ can be found in $O(2^L L^{3/2})$ time. Thus, the time to compute all $E^{opt}(G_i)$ is $O(2^L L^{3/2} \frac{n}{L}) = O(2^L \sqrt{L} n)$. Thus Algorithm DIVIDE can be implemented to run in $O(n \log n + 2^L \sqrt{L} n) = O(n \log n + \alpha^{\frac{1}{\epsilon^2}} \epsilon^{-1} n)$ time for $\alpha = 2^{(1943)^2}$.

This completes the proof of Theorem 2.4.

4 Conclusion

In this paper, we proved that the minimum maximal matching problem in planar graphs admits a PTAS. However, the current trade-off of the PTAS between the running time and the approximation ratio is not effective. Thus it is a future work to design a PTAS with a better trade-off.

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Appendix

Proof of Lemma 2: Let G be a given irreducible planar graph. To prove the lemma via Lemma 2.2, we convert G into graphs G_1, G_2, G_3, G_4 in this order to obtain a planar graph with the minimum degree at least 3. We first construct a graph G_1 from G by applying the following procedures 1 and 2.

1. If there is a pair of leaf edges (u, u') and (v, v') which are adjacent to the same edge, say (u', v'), then add three new edges $(u, v), (u, v'), (v, u')$ to the graph (where the resulting graph remains simple and planar, and each of u and v has degree 3). We repeat this until there is no such pair of leaf edges.

2. If there is a leaf edge (u, v) with a leaf vertex u, then add new edges $(u, w), (u, w')$ with two neighbors $w, w' (\neq u)$ of v by choosing w, w' so that the augmented graph remains planar (such a pair w, w' exists since the current graph has no fringe edge and no two leaf edges adjacent to the same edge). The resulting graph remains simple and the degree of u becomes 3. We repeatedly apply this until there is no leaf edge.

Claim 1 *G_1 remains irreducible and planar, and satisfies $V(G_1) = V(G)$, $\delta(G_1) \geq 2$, $\mu(G_1) \leq 2\mu(G)$.*

Proof: Omitted. □

We next augment G_1 to a graph G_2 by adding a maximal set of new edges such that the resulting graph remains simple and planar and has the same size of a maximum matching of G_1.

Claim 2 *G_2 remains irreducible and planar, and satisfies $V(G_2) = V(G_1)$, $\delta(G_2) \geq 2$, and $\mu(G_2) = \mu(G_1)$. In G_2,*

(i) *the two neighbors of a vertex of degree 2 are joined by an edge,*

(ii) *no two vertices of degree 2 are adjacent,*

(iii) *each vertex of degree 2 is adjacent to two vertices of degree at least 4, and*

(iv) *if two vertices of degree 2 are adjacent to the same two vertices u and v, then both u and v have degree at least 5.*

Proof: Omitted. □

For a graph H, let $V_2(H)$ denote the set of vertices of degree 2 in H. Let us call an edge e *covered* if there is a vertex of degree 2 adjacent to both end-vertices of e, and *uncovered* otherwise.

The graph $G_2 - V_2(G_2)$ may have a vertex of degree at most 2. Let u be such a vertex. By the irreducibility and (i) of Claim 2, the degree of u in $G_2 - V_2(G_2)$ is exactly 2. Let v, w be the neighbors of u in $G_2 - V_2(G_2)$. Let t_i, $1 \leq i \leq p$ denote the all vertices in $V_2(G_2)$ that are adjacent to u in G_2; $t_i \neq v, w$ by $t_i \in V_2(G_2)$. By (i) and (ii) of Claim 2, each t_i is adjacent to v or w. By the irreducibility and (iii) and (iv) of Claim 2, we see that $2 \leq p \leq 4$, and there exist t_1 and t_2 which are adjacent to v and w, respectively, and we can assume that t_3 and t_4 (if any) are adjacent to v and w, respectively (note that if $p = 2$ and both t_1 and t_2 are adjacent to u and v then u would be of degree 4, contradicting (iv) of Claim 2). Notice that in G_2 such a set of vertices u, v, w and t_i, $1 \leq i \leq p$ induces a connected subgraph S_u in which only vertices v and w are adjacent to the rest of the vertices. We then apply the following procedure to each of such induced subgraphs. For the above vertices u, v, w and t_i, $1 \leq i \leq p$, we remove t_3 and t_4 (if any), and add two new edges $(v, t_2), (w, t_2)$. Observe that the resulting graph remains irreducible and planar, and the degrees of v and w never decrease (hence Claim 2 remains valid). Also it is easy to check that the size of a maximum matching never increases. In particular, each vertex of t_1, t_2 has degree 3, and each of the eight edges that are incident to t_1, t_2 or u is uncovered. We repeat applying this procedure to a subgraph S_u as long as $G' - V_2(G')$ has no vertex u of degree 2 in the current graph G'. Let G_3 be the resulting graph. We then obtain the next property.

Claim 3 *G_3 remains irreducible and planar, and satisfies $\mu(G_3) \leq \mu(G_2)$. For $n^* = |V(G_2)| - |V(G_3)|$, G_3 has at least $4n^*$ uncovered edges.* □

Thus, the graph $G_4 = G_3 - V_2(G_3)$ satisfies $\delta(G_4) \geq 3$ and $|V_2(G_3)| \leq 2(|E(G_4)| - 4n^*)$.

We are now ready to prove Lemma 2. Let $n_i = |V(G_i)|$ for $i = 1, 2, 3, 4$. By Theorem 2.1, $|E(G_4)| \leq 3n_4 - 6$. From this and $|V_2(G_3)| \leq 2(|E(G_4)| - 4n^*)$, we have $|V_2(G_3)| \leq 6n_4 - 12 - 8n^*$. By $n_4 = n_3 - |V_2(G_3)|$, we have

$$\begin{aligned} n_4 &\geq n_3 - 6n_4 + 12 + 8n^* \\ &\geq (n_2 - n^*) - 6n_4 + 12 + 8n^*. \end{aligned}$$

Therefore, we get $n_4 \geq \frac{n_2+12}{7}$. By Theorem 2.2, $\mu(G_4) \geq \min\{\frac{n_4-1}{2}, \frac{n_4+2}{3}\} = \frac{n_4+2}{3}$ (since $n_4 \geq 7$ by $n_2 = n_1 = n \geq 37$, $n_4 \geq 7$). Then, we obtain

$$\begin{aligned} \mu(G_1) &\geq \mu(G_2) \geq \mu(G_3) \geq \mu(G_4) \\ &\geq \frac{n_2}{21} + \frac{26}{21} = \frac{n}{21} + \frac{26}{21}. \end{aligned}$$

Finally, by $\mu(G) \geq \frac{1}{2}\mu(G_1)$, we obtain

$$\mu(G) \geq \frac{n}{42} + \frac{13}{21}.$$