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The Competitiveness of Diversified Investment on Portfolio Selection Problem

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Abstract

On Portfolio Selection Problem, it is one of central issues to study a variety of investment policies that decide asset allocations to several securities. In this paper, we study the problem in the context of online computation and competitive analysis. Researchers in the field of finance have examined the advantage of diversified investments using some stochastic parameters and verified that the correlations among securities make diversification beneficial. Instead of the stochastic parameters, we use a graph-based method, (Δ, G)-Adversary, to describe the correlations. Against the adversary, we propose the online portfolio selection algorithm BLNC investing to the diversified portfolio. Then, we seek its competitive ratio against a specified class of instances of (Δ, G)-Adversary, and make sure that BLNC is a money making algorithm against this class. Finally, we show the optimality of BLNC against this class.

1 Introduction

In most financial problems, we are forced to make current decisions without any complete knowledge of the future. For example, when we intend to invest to some stocks, we have to make decisions about how much quantity of what stock we should buy with our scarce information on the future stock price movements. This characteristic has urged a number of researchers to study financial problems such as Trading Problem [4][5] and Portfolio Selection Problem [7] in the online computation framework.

Portfolio Selection Problem is one of traditional financial problems, wherein we are asked to study a variety of investment policies that decide asset allocations, which are called "*portfolios*", to several securities such as bonds, stocks and foreign currencies. Originally, the problem has been studied by researchers in the field of finance. Harry Markowitz, a Nobel Prize winner in economics, published seminal articles on the problem in the 1950s. His work provided a way to methodically deal with security return and risk through stochastic parameters. In this framework, he succeeded in verifying the empirical knowledge that "*diversification*", that is, investing to a portfolio composed of a variety of securities, has an advantage of reducing the total risk of the portfolio [8]. This advantage results from the correlations between pairs of securities. As to the correlation between a pair of securities, we can often observe that the increasing value of a domestic currency leads the decrease of stock prices of the companies in export business. In this case, we can guess that there are negative correlations between the domestic currency and the stocks. Considering security returns to be stochastic, Markowitz formalized the correlation by the covariance or the correlation coefficient between a pair of return variables. In fact, he verified that diversified investments are beneficial as long as the portfolio includes some pair of securities between which the correlation coefficient is not equal to +1.

In this paper, we study Portfolio Selection Problem along the context of online compuation and competitive analysis. We regard an investment policy as an online algorithm and evaluate it against an adversary. The goal in this paper is to verify that the diversified investment also performs well in the framework without the use of any stocastic parameter.

In Section 2, we formalize Portfolio Selection Problem along the context of online computation and competitive analysis. This formalization follows [2][6].

In Section 3, we accommodate the correlations contributing to the advantage of diversification into the adversary. Now we propose the adversary with some restrictions, "(Δ, G)-Adversary". Under (Δ, G)-Adversary, we assume that each security price has its own fixed fluctuation prescribed by Δ. The assumption enables us to specify feasible combinations of security price movements, because each security moves either upward or downward by the constant quantity. Considering each combination to be a vertex and the price transition to be an edge, we can represent the correlations by the graph G. Then, (Δ, G)-Adversary is forced to choose the worst sequence against an online portfolio selection algorithm among all walks in the graph G.

In Section 4, we present the online portfolio selection algorithm BLNC. BLNC always selects the "*balanced*" portfolios yielding the same return no matter which adjacent vertex (Δ, G)-Adversary chooses. We demonstrate that BLNC makes diversified investments under the market composed of two securities correlated negatively. We use (Δ, G)-Adversary to specify this market in the context of competitive analysis. Then, we evaluate the competitive ratio of BLNC against this class of instances of (Δ, G)-Adversary. In the process of the investigation, we can make sure that BLNC is a "*money making*" algorithm [4], that is, it returns positive profit in any sequence of relative price vectors for which the optimal offline algorithm accrues positive profit. Finally, we show that BLNC is an optimal online portfolio selection algorithm against the class of instances of (Δ, G)-Adversary. In the proof, we can confirm that BLNC actually has the same behavior as the well-known maxmin strategy in game theory. The optimality concludes that the diversified investment policy has the advantage under the market including correlated securities.

In Section 5, we give the conclusion and brief mention of other interesting results and future works.

2 Portfolio Selection Problem

An investor with his initial wealth w_0 in cash is faced with a market consisting of m securities. These can be stocks, bonds or currencies. Let $\mathbf{p}_i = (p_{i1}, p_{i2}, \cdots, p_{im})$ denote a vector of the m security prices, where for each $j = 1, 2, \cdots, m$, p_{ij} denotes the number of units of the jth security that can be bought for one unit of cash at the start of the ith period, $(i = 1, 2, \cdots)$. Thus, the small quantity of p_{ij} means that the jth security at the start of the ith period is expensive. The change in security prices during the ith period is represented as a relative price vector $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{im})$ where for each i and j, $x_{ij} = p_{ij}/p_{(i+1)j}$. An investment of d units of cash to the jth security at the ith period yields dx_{ij} by the end of the ith period. A sequence of relative price vectors is denoted by $X = \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$.

According to his investment policy, an investor makes online decisions about each portfolio $\mathbf{b}_i = (b_{i1}, b_{i2}, \cdots, b_{im})$ $(i = 1, 2, \cdots)$ where for any i and j, $b_{ij} \geq 0$ and $\sum_{j=1}^{m} b_{ij} = 1$. Namely, the portfolio $\mathbf{b}_i$ is the proportion of his wealth invested to each of the m securities at the ith period. The investment to the portfolio $\mathbf{b}_i$ will multiply his wealth by

$$\mathbf{b}_i \cdot \mathbf{x}_i = (b_{i1}x_{i1} + b_{i2}x_{i2} + \cdots + b_{im}x_{im})$$

during the ith period. At this stage, the portfolio $\mathbf{b}_i$ is cashed and adjusted by reinvesting the entire current wealth to the portfolio $\mathbf{b}_{i+1}$ according to the investment policy. Given a sequence of relative price vectors $X = \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$, a portfolio selection algorithm decides a sequence of portfolios $B = \mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$. An online portfolio selection algorithm is required to decide each portfolio $\mathbf{b}_i$ with incomplete knowledge of the future price movements, $\mathbf{x}_i, \cdots, \mathbf{x}_n$, whereas an offline algorithm makes decisions with the entire information on X.

Without loss of generality, we can assume that an online portfolio selection algorithm ALG begins investments with an initial wealth of 1. The return $\mathrm{R}_{\mathrm{ALG}}(X)$ of ALG with respect to a sequence of relative price vectors $X = \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ is defined to be

$$\mathrm{R}_{\mathrm{ALG}}(X) = \prod_{i=1}^{n} \mathbf{b}_i \cdot \mathbf{x}_i.$$

In the framework of competitive analysis, we evaluate an online algorithm through the following competitive ratio; The competitive ratio c_{ALG} of ALG is

$$c_{\mathrm{ALG}} = \sup_{X} \frac{\mathrm{R}_{\mathrm{OPT}}(X)}{\mathrm{R}_{\mathrm{ALG}}(X)},$$

where OPT is an optimal offline portfolio selection algorithm. It is obvious that the competitive ratio c_{ALG} is more than or equal to 1 and that the closer to 1 c_{ALG} is, the better ALG performs. This definition indicates that the performance of ALG is evaluated through the comparison with the performance of the optimal offline algorithm with complete knowledge of the future. In that sense, the competitive ratio c_{ALG} is the measure of the performance of ALG for the worst case input X. The goal in Portfolio Selection Problem is to acquire an optimal online portfolio selection algorithm, that is, the online algorithm with the smallest competitive ratio.

3 (Δ, G)-Adversary

In real life, we can observe that there is some pair of assets whose values have some correlated movements; The upward movement of a domestic currency often reduces the stock prices of the companies in export business. The decline of interest rates, in other words, the rise of bond prices raises the stock prices of the companies with heavy debt. In order to keep our wealth, we have learned through these observations that we had better diversify to several assets whose values have distinct movements rather than invest to just one. This advantage of diversification was verified in a rigorous method by Markowitz. He formalized the correlation of price movements between a pair of securities by means of the correlation coefficient. Using the correlation coefficients, he pointed out that the offsets of price movements among securities in a portfolio contribute to the reduction of the entire portfolio risk. Along the context of competitive analysis, we propose the adversary accommodating the correlations without using any stochastic parameter.

3.1 Δ-Fixed fluctuation restriction

Let Δ_j be a constant real value greater than or equal to 1 for each j $(1 \leq j \leq m)$. Suppose that all relative prices x_{ij} are either Δ_j^{-1} (downward) or Δ_j (upward). The value Δ_j, which is called the fluctuation ratio of jth security, is inherent on the security. Without loss of generality, we can assume

$$1 \leq \Delta_1 \leq \Delta_2 \leq \cdots \leq \Delta_m.$$

Also, a vector $\Delta = (\Delta_1, \Delta_2, \cdots, \Delta_m)$ is called a vector of fluctuation ratios. As the result, the adversary must choose a vector $\mathbf{x}_i$ of relative prices among 2^m candidate vectors.

Although the Δ restriction seems to be too simplified, it enables us to have an useful apparatus for representing the correlations without any stochastic parameter. Under the Δ restriction, we consider each relative price vector $\mathbf{x}_i$ to be a vertex of a graph. The graph has 2^m vertices because the number of the possible vectors is 2^m. Without any further restriction, a feasible sequence of relative price vectors would be a walk in the complete graph K_{2^m} with self-loop. Against an online portfolio selection algorithm ALG, the adversary must choose the walk X of length n in the graph that maximizes the ratio $\mathrm{R_{OPT}}(X)/\mathrm{R_{ALG}}(X)$.

3.2 G-Price transition restriction

We impose the adversary on an additional restriction for the purpose of representing the correlations of price movements between pairs of securities, which is formalized in Markowitz's model by means of the correlation coefficients. Assume that the kth security and the lth security always move to the opposite direction. Namely, whenever the kth security moves upward, the lth security moves downward, and vice versa. Under the assumption, the adversary is forced to choose a sequence of relative price vectors among the following 2^{m-1} vectors;

$$(x_{i1}, \cdots, x_{i(k-1)}, \Delta_{ik}, x_{i(k+1)}, \cdots \\ \cdots, x_{i(l-1)}, \Delta_{il}^{-1}, x_{i(l+1)}, \cdots, x_{im}),$$

$$(x_{i1}, \cdots, x_{i(k-1)}, \Delta_{ik}^{-1}, x_{i(k+1)}, \cdots \\ \cdots, x_{i(l-1)}, \Delta_{il}, x_{i(l+1)}, \cdots, x_{im}),$$

where each x_{ij} except the kth and lth elements may be either Δ_j or Δ_j^{-1}. In other words, the adversary is forced to choose some walk in the induced subgraph consisting of these 2^{m-1} vertices.

4 Optimal Online Portfolio Selection Algorithm against (Δ, G)-Adversary

4.1 Online algorithm: BLNC

In the previous section, we have defined (Δ, G)-Adversary in the most genelarized form. In this section, we focus on the restricted class of instances of the adversary representing the market consisting of two securities whose prices move to the opposite direction. The discussion on the generalized form of the adversary is mentioned in Section 5.

Let us specify (Δ, G)-Adversary representing this situation. Denote the vector of fluctuation ratios by $\Delta = (\Delta_1, \Delta_2)$, where $1 \leq \Delta_1 \leq \Delta_2$. Then the price transition restriction G is the complete graph with self-loops whose vertex set consists of the two elements $\{(\Delta_1, \Delta_2^{-1}), (\Delta_1^{-1}, \Delta_2)\}$. (See Figure 1.)

For each period, the online portfolio selection algorithm BLNC invests to the balanced portfolio $\mathbf{b}^* = (b_1^*, b_2^*)$ such that

$$b_1^*\Delta_1 + b_2^*\Delta_2^{-1} = b_1^*\Delta_1^{-1} + b_2^*\Delta_2.$$

By solving this formula and $b_1^* + b_2^* = 1$, we obtain b_1^* and b_2^*;

$$b_1^* = \frac{\Delta_2 - \Delta_2^{-1}}{(\Delta_1 - \Delta_1^{-1}) + (\Delta_2 - \Delta_2^{-1})} \tag{1}$$

$$b_2^* = \frac{\Delta_1 - \Delta_1^{-1}}{(\Delta_1 - \Delta_1^{-1}) + (\Delta_2 - \Delta_2^{-1})} \tag{2}$$

where $1 \leq \Delta_1 \leq \Delta_2$. (See Figure 2.) Note that the formulae (1) and (2) suggest that BLNC is a *"constant rebalanced"* algorithm investing to some fixed portfolio for all periods.

The formulae (1) and (2) suggest that BLNC makes diversified investments as long as both Δ_1 and Δ_2 is greater than 1. Furthermore, they has the implication that BLNC invests to more quantity of the first security with the smaller fluctuation ratio Δ_1 than that of the second security with the larger one Δ_2. This result may reflect the attitude toward our investment policy in real life.

4.2 The competitive ratio of BLNC

Let $\mathbf{x}'$ and $\mathbf{x}''$ be two relative price vectors $(\Delta_1, \Delta_2^{-1})$ and $(\Delta_1^{-1}, \Delta_2)$, respectively. No matter which of $\mathbf{x}'$ or $\mathbf{x}''$ the adversary chooses, the portfolio yields the unique return $\mathrm{r_{BLNC}}$ during one period;

$$\begin{aligned} \mathrm{r_{BLNC}} &= \frac{\Delta_1\Delta_2 - \Delta_1^{-1}\Delta_2^{-1}}{(\Delta_1 - \Delta_1^{-1}) + (\Delta_2 - \Delta_2^{-1})} \\ &= \frac{\Delta_1\Delta_2 + 1}{\Delta_1 + \Delta_2}. \end{aligned} \tag{3}$$

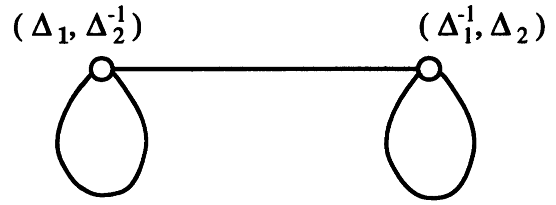

Figure 1: The price transition graph G of the class

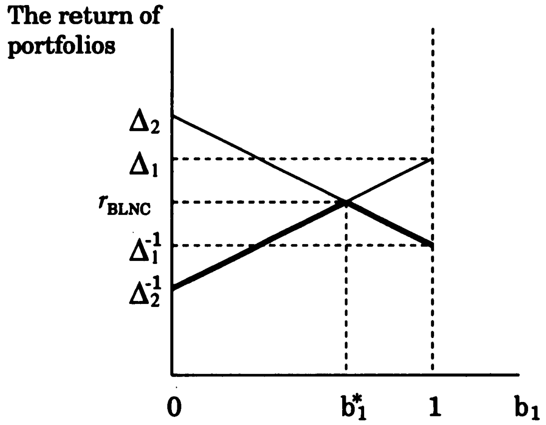

Figure 2: The return of portfolios

Thus, the return of BLNC, $\mathrm{R_{BLNC}}(X)$, is

$$\mathrm{R_{BLNC}}(X) = \left(\frac{\Delta_1\Delta_2 + 1}{\Delta_1 + \Delta_2}\right)^n \tag{4}$$

for any sequence X of relative price vectors of length n against the (Δ, G)-Adversary. We can easily make sure that the right-hand side of the formula (3) is always greater than 1 as long as $1 < \Delta_1 \leq \Delta_2$ holds. It follows that BLNC is a money making algorithm against the class of instances of (Δ, G)-Adversary in the sense that whenever the optimal offline algorithm accrues positive profit, BLNC also ends with positive profit.

On the other hand, the return of the optimum offline algorithm OPT, $\mathrm{R_{OPT}}(X)$, is

$$\mathrm{R_{OPT}}(X) = \prod_{i=1}^{n} \max\{x_{i1}, x_{i2}\} \tag{5}$$

for any sequence X of relative price vectors of length n. OPT can take such behavior that it invests the entire wealth to the security with the highest yield for each period because of its complete knowledge of the future price movements.

For the sequence B^* of balanced portfolios, the adversary chooses the sequence X of relative price vectors such that the ratio $\mathrm{R_{OPT}}(X)/\mathrm{R_{BLNC}}(X)$ is maximized. Since BLNC yields the unique return $\mathrm{R_{BLNC}}(X)$ regardless of the adversary's choice, the adversary must choose the relative price vector $\mathbf{x}'' = (\Delta_1^{-1}, \Delta_2)$ for all periods in order to maximize the numerator $\mathrm{R_{OPT}}(X)$. Consequently, the adversary would adopt the sequence $X^* = \mathbf{x}'', \mathbf{x}'', \cdots$ of length n as the worst-case sequence. The competitive ratio c_{BLNC} is;

$$c_{\mathrm{BLNC}} = \frac{\mathrm{R_{OPT}}(X^*)}{\mathrm{R_{BLNC}}(X^*)} = \left[\frac{\Delta_2}{(\Delta_1\Delta_2 + 1)/(\Delta_1 + \Delta_2)}\right]^n$$

Using $1 \leq \Delta_1 \leq \Delta_2$, we have $\Delta_1\Delta_2 \leq \Delta_1\Delta_2 + 1 \leq 2\Delta_1\Delta_2$. By the inequality, c_{BLNC} is bounded as follows;

$$\left[\frac{\Delta_2(\Delta_1 + \Delta_2)}{2\Delta_1\Delta_2}\right]^n \leq c_{\mathrm{BLNC}} \leq \left[\frac{\Delta_2(\Delta_1 + \Delta_2)}{\Delta_1\Delta_2}\right]^n.$$

Thus,

$$\left(1 + \frac{\Delta_2}{\Delta_1}\right)^n 2^{-n} \leq c_{\mathrm{BLNC}} \leq \left(1 + \frac{\Delta_2}{\Delta_1}\right)^n. \tag{6}$$

4.3 The optimality of BLNC

Next we show that BLNC is an optimal online portfolio selection algorithm against the instance of (Δ, G)-Adversary prescribed by Figure 1. In fact, we can confirm that BLNC always makes the optimal selection for each period in terms of the return. Let $\mathbf{b}$ be an arbitrary portfolio. The (Δ, G)-Adversary suppress the return of $\mathbf{b}$ on the bold line in Figure 2, because the adversary is able to choose the smaller return of the two candidates on the two straight lines. We can see that the balanced portfolio $\mathbf{b}^*$ yields the maximum return r_{BLNC} on the bold line. Thus, we have verified that BLNC is an optimal online portfolio selection algorithm against the instance prescribed by Figure 1. Note that BLNC takes substantially the same behavior as the well-known maxmin strategy in game theory.

5 Conclusion and Future Work

On Portfolio Selection Problem, we have conducted research in the context of online computation and competitive analysis. Assuming that the sucurities have their own unique fluctuations, we have used (Δ, G)-Adversary to describe correlations of price movements among pairs of securities in a market. While researchers in the field of finance often formalize the correlation using the covariance or the correlation coefficient, we make use of the graph whose vertex represents a feasible combination of security prices and whose edge prescribes the price transition. Then we have presented the online portfolio selection algorithm BLNC against the restricted class of instances of (Δ, G)-Adversary. In the market comprised of the two securities with the opposite movements, we have confirmed that BLNC makes diversified investments. Actually, BLNC selects the portfolio which contains relatively more quantity of the security with the smaller fluctuation. This result seems to be consistent with ordinary investment policies in real life. Along the context of competitive analysis, we have computed the competitive ratio and gained the upper and lower bounds of the ratio. In the process, we have made sure that BLNC is money making. Furthermore, we have proved that BLNC is an optimal online portfolio selection algorithm against the class of instances of (Δ, G)-Adversary corresponding with the market in question. Thus, we can conclude that diversification has the advantage on portfolio selections.

While this paper focuses on the quite simple situation represented by Figure 1, we expect to study more generalized classes of instances of (Δ, G)-Adversary. Until now, we already have some results on other genelarized classes. Especially, we gain an interesting result on the class of instances whose fluctuation vector Δ has three dimensions, in other words, on the market composed of three securities. As to the market, we consider the following situation; While BLNC knows that two securities of the three have the negative correlation each other, it has no information on the correlations between both of the pair and the remaining security. We have already acquired the result; Against this class of instances of (Δ, G)-Adversary representing this situation, BLNC invests to the partially diversified portfolios whose constituents are only the two correlated securities. Namely, BLNC never invests to the security whose correlations the algorithm has no information on. Thus, BLNC follows the ordinary belief that investors do not invest to the security on which they have little information.

Also, we intend to characterize the performance of BLNC in terms of graph configuration. In this paper, we have focused on the complete graph with self-loops. However, we have made sure that the adversary always adopts the same relative price vector as the worst-case input in this case. This results from the fact that the adversary is able to choose the favorable vector to itself among all the vertices because of the complete graph with self-loops. This fact urges us to characterize BLNC or other online algorithms from the point of graph configuration.

While (Δ, G)-Adversary is based on the assumption that security prices move either upward or downward for each period, they are any nonnegative values within some ranges in real market. Thus, we have to consider an adversary accommodating this situation.

References

[1] S.al-Binali. A risk-reward framework for the competitive analysis of financial games. *Algorithmica*, 25:pp.99-115, 1999.

[2] A.Borodin and R.El-Yaniv. *Online Computation and Competitive Analysis*. Cambridge University Press, 1998.

[3] A.Borodin, S.Irani, P.Raghavan, and B.Schieber. Competitive paging with locality of reference. *Journal of Computer and System Science*, 50(2):pp.244-258, 1995.

[4] A.Chou, J.Cooperstock, R.El-Yaniv, M.Klugerman, and T.Leighton. The statistical adversary allows optimal money-making trading strategies. In *Proceedings of the 6th Annual ACM-SIAM Symposium on Discrete Algorithms*, pp.467-476, 1995.

[5] R.El-Yaniv, A.Fiat, R.Karp, and G.Turpin. Competitive analysis of financial games. In *Proceedings of the 33rd Annual Symposium on Foundations of Computer Science*, pp.327-333, 1992.

[6] A.Fiat and G.J.Woeginger. *Online Algorithms, The State of the Art*. Springer, 1998.

[7] E.Ordentlich and T.M.Cover. On-line portfolio selection. In *Proceedings of the 9th Annual Conference on Computational Learning Theory*, pp.310-313, 1996.

[8] W.F.Sharpe and G.J.Alexander. *Investments*. Prentice-Hall, 1990.