

# Isomorphic factorization of the Kronecker product of generalized de Bruijn digraphs

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## Abstract

We study the Kronecker product of generalized de Bruijn digraphs. It is shown that the binary generalized de Bruijn digraph of order  $n$  factorizes the Kronecker product of binary generalized de Bruijn digraphs of orders corresponding to the factors factoring the integer  $n$ .

*Key words:* interconnection network, isomorphic factorization, Kronecker product, de Bruijn digraph, generalized de Bruijn digraph

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## 1 Introduction

The de Bruijn digraph denoted by  $B(d, D)$  possesses good properties as interconnection networks for designing massively parallel computers (see [2]). There are several kinds of definitions of the de Bruijn digraph such as using words on an alphabet, the line digraph operation, or the congruence. The generalized de Bruijn digraph denoted by  $G_B(n, d)$  is a generalization of the de Bruijn digraph obtained by extending the modulo to any natural number, that is,  $V(G_B(n, d))$ ,  $n \geq d \geq 2$ , is the set of integers modulo  $n$  (which is denoted by  $\mathbb{Z}_n$ ) and a vertex  $x$  is adjacent to vertices  $(dx + r) \bmod n$  with  $0 \leq r \leq d - 1$  (see [4,6]). In [7], by means of a relation between the Kronecker product and the line digraph operation, it is shown that the Kronecker product of de Bruijn digraphs has an isomorphic factorization into the de Bruijn digraphs corresponding to the prime factorization of its order. We will extend the result to binary generalized de Bruijn digraphs.

A *functional digraph* is a digraph in which adjacency is defined by functions. More formally, given a set of functions  $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_{d-1}\}$  on a set  $V$ , a functional digraph denoted  $(V, \Lambda)$  is a  $d$ -out regular (every vertex has out-degree  $d$ ) digraph with a vertex set  $V$  and an arc set  $\{(v, \lambda_i(v)) \mid 0 \leq i \leq d-1\}$ . A functional digraph is also called a transformation graph [1]. A generalized de Bruijn digraph is an example of a functional digraph. There are many operations on digraphs such as graph products or the line digraph operation. It is of use to consider relations between some families of digraphs and graph operations in order to investigate structural properties of interconnection networks. Several relations among the Kronecker product and the line digraph operation, and the de Bruijn digraph are obtained in [7]. Here, we consider a part of relations between functional digraphs and the Kronecker product.

Let  $G$  be a digraph. A *factor* of  $G$  is a subdigraph of  $G$  whose vertex set is  $V(G)$ . If  $H \cong H_1 \cong \dots \cong H_n$  are pairwise arc-disjoint factors of  $G$  such that  $E(G) = \bigcup_{i=1}^n E(H_i)$ , then  $G$  has an *isomorphic factorization* into  $H$  and denoted by  $H \mid G$ . The Kronecker product of digraphs  $G$  and  $H$ , denoted by  $G \otimes H$ , is a digraph with a vertex set  $V(G) \times V(H)$  and an arc set  $\{((u, v), (w, x)) \mid (u, w) \in E(G), (v, x) \in E(H)\}$ . Let  $\sigma$  be a bijection from  $V(G)$  to a set  $S$ . We define  $\sigma(G)$  a digraph with a vertex set  $V(\sigma(G)) = S$  and an arc set  $E(\sigma(G)) = \{(\sigma(u), \sigma(v)) \mid (u, v) \in E(G)\}$ . In other words, the digraph  $\sigma(G)$  is a digraph obtained by relabeling  $G$ . Other terminology and notation, we refer [3].

**Lemma 1** *Let  $G$  be a functional digraph  $(V, \Lambda)$  and  $\sigma$  a bijection from  $V$  to  $V'$ , where  $\Lambda = \{f_0, f_1, \dots, f_{d-1}\}$ . Then*

$$\sigma(G) = (V', \Lambda'), \text{ where } \Lambda' = \{\sigma f_i \sigma^{-1} \mid 0 \leq i \leq d-1\}.$$

As it is shown in [7], the de Bruijn digraph, the Kronecker product, and the line digraph operation are closely related. A result on isomorphic factorization is shown as follows.

**Theorem 2** [7]

$$B(d, kD) \mid \bigotimes_{1 \leq i \leq k} B(d, D) \cong B(d^k, D)$$

Let  $n, d$  be integers and  $f_i$  a bijection from  $\mathbb{Z}_n$  onto  $\mathbb{Z}_n$  such that  $f_i : x \mapsto (dx+i) \bmod n$ . Then, the generalized de Bruijn digraph  $G_B(n, d)$  is represented by  $(\mathbb{Z}_n, \{f_0, f_1, \dots, f_{d-1}\})$  as a functional digraph. We extend Theorem 2 to binary generalized de Bruijn digraphs.

## 2 Isomorphic factorizations of the Kronecker product of the binary generalized de Bruijn digraphs

We first show that the Kronecker product of two generalized de Bruijn digraphs contains a generalized de Bruijn digraph as a factor. For real  $x$ ,  $\lfloor x \rfloor$  is the greatest integer less than or equal to  $x$ .

**Proposition 3** *Let  $m, n \geq d \geq 2$  be integers. Then*

$$G_B(mn, d) \subseteq G_B(m, d) \otimes G_B(n, d).$$

**PROOF.** We define a bijection  $\sigma$  from  $\mathbb{Z}_{mn}$  to  $\mathbb{Z}_m \times \mathbb{Z}_n$  so that

$$\sigma : x \mapsto \left( \left\lfloor \frac{x}{n} \right\rfloor, x \bmod n \right).$$

Note that  $\sigma^{-1}((a, b)) = na + b$ . Let  $H = \sigma(G_B(mn, d)) = (\mathbb{Z}_m \times \mathbb{Z}_n, \{\sigma f_i \sigma^{-1} \mid 0 \leq i \leq d-1\})$ . Then,

$$\begin{aligned} \sigma f_i \sigma^{-1}((a, b)) &= \sigma f_i(na + b) \\ &= \sigma((d(na + b) + i) \bmod mn) \\ &= \left( \left\lfloor \frac{(d(na + b) + i) \bmod mn}{n} \right\rfloor, (db + i) \bmod n \right) \\ &= \left( \left( da + \left\lfloor \frac{db + i}{n} \right\rfloor \right) \bmod m, (db + i) \bmod n \right). \end{aligned}$$

Since  $db + i \leq d(n-1) + (d-1)$ ,  $\left\lfloor \frac{db + i}{n} \right\rfloor \leq \left\lfloor \frac{dn-1}{n} \right\rfloor = d-1$ . That is,  $H$  is a factor of  $G_B(m, d) \otimes G_B(n, d)$ .  $\square$

We obtain that  $G_B(mn, d)$  is a factor of  $G_B(m, d) \otimes G_B(n, d)$  for any  $d \geq 2$ , but the next question is what is the structure of  $G_B(m, d) \otimes G_B(n, d)$  other than  $G_B(mn, d)$ . We show the case for  $d = 2$ , the binary generalized de Bruijn digraphs.

**Theorem 4** *Let  $m, n \geq 2$  be integers. Then,*

$$G_B(mn, 2) \mid G_B(m, 2) \otimes G_B(n, 2).$$

**PROOF.** We define  $h$  as a function from  $\mathbb{Z}_{mn}$  onto  $\mathbb{Z}_{mn}$  so that

$$h : x \mapsto mn - n - x + 2(x \bmod n).$$

Note that  $h$  is a bijection such that  $h(h(x)) = x$  and  $h(x \bmod mn) = h(x) \bmod mn$ . Let  $H = h(G_B(mn, 2))$ . We now show that  $G_B(m, 2) \otimes G_B(n, 2)$  is decomposable into  $H$  and  $G_B(mn, 2)$ . On  $H$ , a vertex  $h(x) = mn - n - x + 2(x \bmod n)$  is adjacent to vertices  $A_i = mn - n - \{(2x + i) \bmod mn\} + 2\{(2x + i) \bmod n\}$ ,  $0 \leq i \leq 1$ , while a vertex  $h(x)$  is adjacent to vertices  $B_j = \{-2(x + n) + 4(x \bmod n) + j\} \bmod mn$ ,  $0 \leq j \leq 1$ , on  $G_B(mn, 2)$ . We have

$$\begin{aligned} A_i - B_j &\equiv -n - 2x - i + 2\{(2x + i) \bmod n\} + 2n + 2x - 4(x \bmod n) - j \\ &\equiv n - i - j + 2\{(2x + i) \bmod n - 2(x \bmod n)\} \pmod{mn}. \end{aligned}$$

Let  $r = x \bmod n$ . Since

$$\begin{aligned} (2x + i) \bmod n - 2(x \bmod n) &= 2x + i - n \left\lfloor \frac{2x + i}{n} \right\rfloor - 2 \left\{ x - n \left\lfloor \frac{x}{n} \right\rfloor \right\} \\ &= i - n \left\{ \left\lfloor \frac{2x + i}{n} \right\rfloor - 2 \left\lfloor \frac{x}{n} \right\rfloor \right\} \\ &= i - n \left\lfloor \frac{2r + i}{n} \right\rfloor \\ &= \begin{cases} i & \text{if } 0 \leq r \leq \frac{n - i - 1}{2}; \\ i - n & \text{if } \frac{n - i}{2} \leq r \leq n - 1, \end{cases} \end{aligned} \quad (1)$$

we have

$$A_i - B_j \equiv \begin{cases} n + i - j & \pmod{mn}, \text{ or} \\ i - n & \pmod{mn}. \end{cases}$$

It means  $A_i \neq B_j$  and thus  $H$  and  $G_B(mn, 2)$  are arc-disjoint with each other. Next, we will show that the digraph  $\sigma(H)$  is a factor of  $G_B(m, 2) \otimes G_B(n, 2)$ , where  $\sigma$  is the bijection defined in Proposition 3.  $H$  is represented by  $(\mathbb{Z}_{mn}, \{hf_0h^{-1}, hf_1h^{-1}\})$ . For  $0 \leq i \leq 1$ ,

$$\begin{aligned} hf_ih^{-1}(x) &= hf_ih(x) \\ &= hf_i(mn - n - x + 2(x \bmod n)) \\ &= h((2(mn - n - x + 2(x \bmod n)) + i) \bmod mn) \\ &= (n + 2x - i + 2((2x + i) \bmod n - 2(x \bmod n))) \bmod mn. \end{aligned}$$

From equation (1), we have

$$hf_ih^{-1}(x) = \begin{cases} (2x + i + n) \bmod mn & \text{if } 0 \leq r \leq \frac{n - i - 1}{2}; \\ (2x + i - n) \bmod mn & \text{if } \frac{n - i}{2} \leq r \leq n - 1. \end{cases} \quad (2)$$

$$\sigma h f_i h^{-1}(x) = \begin{cases} \left( \left\lfloor \frac{2x+i}{n} + 1 \right\rfloor \bmod m, (2x+i) \bmod n \right) & \text{if } 0 \leq r \leq \frac{n-i-1}{2}; \\ \left( \left\lfloor \frac{2x+i}{n} - 1 \right\rfloor \bmod m, (2x+i) \bmod n \right) & \text{if } \frac{n-i}{2} \leq r \leq n-1. \end{cases}$$

On the digraph  $G_B(m, 2) \otimes G_B(n, 2)$ , a vertex  $(\lfloor \frac{x}{n} \rfloor, x \bmod n)$  is adjacent to vertices

$$\left( \left\{ 2 \left\lfloor \frac{x}{n} \right\rfloor + k \right\} \bmod m, (2x+l) \bmod n \right), \quad 0 \leq k, l \leq 1.$$

Since  $\lfloor \frac{2x+i}{n} \rfloor + 1 - \{ 2 \lfloor \frac{x}{n} \rfloor + k \}$  is equal to  $1 - k$  or  $-k$ ,  $\sigma(H)$  is a factor of  $G_B(m, 2) \otimes G_B(n, 2)$ .  $\square$

The Kronecker product has a good property with respect to isomorphic factorizations as shown in [7]; if  $G, G', H$ , and  $H'$  are digraphs such that  $G' | G$  and  $H' | H$ , then  $G' \otimes H' | G \otimes H$ . Applying this relation to Theorem 4, we have the next corollary as a generalization of Theorem 2.

**Corollary 5** *Let  $N$  be an integer and assume that  $N$  is factorized into  $p_1 p_2 \cdots p_k$  for  $p_i \geq 2$  ( $1 \leq i \leq k$ ). Then*

$$G_B(N, 2) | \bigotimes_{1 \leq i \leq k} G_B(p_i, 2)$$

While we have seen that  $G_B(m, 2) \otimes G_B(n, 2)$  is decomposed into  $\sigma(G_B(mn, 2))$  and  $\sigma h(G_B(mn, 2))$ , there is another isomorphism from  $G_B(mn, 2)$  to  $h(G_B(mn, 2))$ .

**Corollary 6** *Let  $h$  and  $h'$  be bijections from  $\mathbb{Z}_{mn}$  onto  $\mathbb{Z}_{mn}$  such that*

$$\begin{aligned} h : x &\mapsto mn - n - x + 2(x \bmod n), \text{ and} \\ h' : x &\mapsto x + n - 1 - 2(x \bmod n), \end{aligned}$$

*respectively. Then,  $h(G_B(mn, 2)) = h'(G_B(mn, 2))$ .*

**PROOF.** Note that  $h'(h'(x)) = x$ ,

$$\begin{aligned}
h' f_i h'^{-1}(x) &= h' f_i h'(x) \\
&= h' f_i(x + n - 1 - 2(x \bmod n)) \\
&= h'((2(x + n - 1 - 2(x \bmod n)) + i) \bmod mn) \\
&= (2x + 3n - 3 + i \\
&\quad - 2(2(x \bmod n) + (-2x - 2 + i) \bmod n)) \bmod mn.
\end{aligned}$$

We have

$$h' f_i h'^{-1}(x) = \begin{cases} (2x + 1 - i + n) \bmod mn & \text{if } 0 \leq r \leq \frac{n+i}{2} - 1; \\ (2x + 1 - i - n) \bmod mn & \text{if } \frac{n+i-1}{2} \leq r \leq n-1, \end{cases} \quad (3)$$

where  $r = x \bmod n$ , and thus equation (3) is equivalent to equation (2) in the proof of Theorem 4.

The generalized Kautz digraph (or Imase Itoh digraph) denoted  $G_I(n, d)$  is a functional digraph  $(\mathbb{Z}_n, \{g_i | 1 \leq i \leq d\})$ , where  $g_i(x) = (-dx - i) \bmod n$ . Similarly, we may argue about the generalized Kautz digraph using the same functions.

**Corollary 7** *Let  $N$  be an integer and assume that  $N$  is factorized into  $p_1 p_2 \cdots p_k$  for  $p_i \geq 2$  ( $1 \leq i \leq k$ ). Then*

$$G_I(N, 2) \mid \bigotimes_{1 \leq i \leq k} G_I(p_i, 2).$$

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