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On a unified approach to resolvent expansions for Schrödinger operators

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1 Introduction

In this lecture I present a unified approach to asymptotic expansions of a resolvent of a Schrödinger-type operator around a point on the real line. This approach was introduced in [8]. Here I present the approach in an abstract framework.

The starting point is a self-adjoint operator H_0 on a Hilbert space $\mathcal{H}$, where we assume that its resolvent has an asymptotic expansion

$$R_0(\zeta)=\sum_{j=-2}^{N}(i\zeta^{1/2})^{j}G_j+o(|\zeta|^{N/2})$$

as $\zeta\to 0$, $\operatorname{Im}\zeta>0$. The expansion is assumed to hold in the norm topology of the bounded operators $\mathcal{B}(\mathcal{K},\mathcal{K}^*)$, where we assume given a scale of spaces $\mathcal{K}\hookrightarrow\mathcal{H}\hookrightarrow\mathcal{K}^*$. This assumption is modeled on Schrödinger operators in odd dimensional spaces with rapidly decaying potentials. We then consider $H=H_0+V$, and under suitable assumptions on the interaction we prove that an analogous expansion holds for the resolvent of H. The procedure is constructive, provided we can determine some projections, which in applications to Schrödinger operators are determined by solutions to $H\psi=0$ in a space larger than the original Hilbert space, e.g. $\mathcal{K}^*$.

It is also possible to state results modeled on Schrödinger operators in even dimensional spaces, but we have not included these results here, since

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they are rather involved to state, and require choice of the right asymptotic sequence, a choice that depends on the interaction V.

Let us briefly comment on the literature. Asymptotic expansions for Schrödinger operators have been obtained in [7, 5, 6, 4, 2, 3, 1, 8]. A general class of differential operators has been treated in [12]. The technique we use here is related to the one used in [12]. Schrödinger operators with radial long range potentials have been treated in [14]. Some results have been obtained on a multi-channel model in [10]. Results on magnetic Schrödinger operators in dimension two, with compactly supported magnetic fields, have been obtained in [13].

One of the advantages of the approach in [8] is that we can take any of the asymptotic expansions mentioned above as the starting point for our new expansions. For example, we can add a sufficiently short range perturbation to the slowly decreasing long range radial potentials considered in [14].

2 Preliminaries

We start with a few preliminary results, and then we introduce our assumptions, and briefly discuss them.

We recall some results from [8], in a notation adapted to this paper. Let $\mathcal{H}$ be a Hilbert space. The bounded operators on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$.

Lemma 2.1 ([8, Lemma 2.1]). *Let T be a closed operator, and P a projection, on a Hilbert space $\mathcal{H}$. Assume that $T+P$ has a bounded inverse. Then T has a bounded inverse, if and only if the operator*

$$A = P - P(T+P)^{-1}P \tag{2.1}$$

is invertible in $\mathcal{B}(P\mathcal{H})$. In the affirmative case we have

$$T^{-1} = (T+P)^{-1} + (T+P)^{-1}PA^{-1}P(T+P)^{-1}. \tag{2.2}$$

Lemma 2.2 ([8, Corollary 2.2]). *Let $T(z) \in \mathcal{B}(\mathcal{H})$, $|z| < \delta$, $\operatorname{Im} z > 0$. Assume that*

$$T(z) = T_0 + zT_1(z) \tag{2.3}$$

with $\|T_1(z)\| \leq c$ for $|z| < \delta$, $\operatorname{Im} z > 0$. Assume that 0 is an isolated point in the spectrum of T_0. Let P_0 denote the associated Riesz projection. Then there exists a $\delta_1 > 0$ such that for $|z| < \delta_1$, $\operatorname{Im} z > 0$, the operator $S(z) \in \mathcal{B}(P_0\mathcal{H})$ defined by

$$S(z) = \sum_{j=0}^{\infty}(-1)^j z^j P_0 \left[T_1(z)(T_0+P_0)^{-1}\right]^{j+1} P_0 \tag{2.4}$$

satisfies $\|S(z)\| \leq c < \infty$ *for* $|z| < \delta_1$, $\operatorname{Im} z > 0$. *The operator* $T(z)$ *has a bounded inverse in* $\mathcal{B}(\mathcal{H})$ *for* $|z| < \delta_1$, $\operatorname{Im} z > 0$, *if and only if* $S(z)$ *has a bounded inverse in* $\mathcal{B}(P_0\mathcal{H})$. *In the affirmative case we have*

$$\begin{aligned} T(z)^{-1} &= (T(z)+P_0)^{-1} \\ &\quad + \frac{1}{z}(T(z)+P_0)^{-1}P_0S(z)^{-1}P_0(T(z)+P_0)^{-1}. \end{aligned} \tag{2.5}$$

Remark 2.3. We will also use (2.5) in the trivial case, where 0 is in the resolvent set, and thus $P_0 = 0$, to avoid consideration of special cases in Section 3.

Let us now introduce our main assumptions. We choose 0 as the point of interest in the spectrum of our operators. Any other point can be considered after a simple change of variables. For $\zeta \in \mathbf{C}\backslash[0,\infty)$ we let $\zeta^{1/2}$ denote the branch of the square root with positive imaginary part.

Assumption 2.4. Let $\mathcal{H}$ denote a Hilbert space, and let H_0 denote a self-adjoint operator on $\mathcal{H}$. Let $R_0(\zeta) = (H_0 - \zeta)^{-1}$. Let $N \geq 0$ be an integer. Assume that there exists a Hilbert space $\mathcal{K}_N$, such that $\mathcal{K}_N \hookrightarrow \mathcal{H}$ densely and continuously. Assume there exist $G_j \in \mathcal{B}(\mathcal{K}_N, \mathcal{K}_N^*)$, $j = -2, -1, 0, \ldots, N$, such that

$$R_0(\zeta) = \sum_{j=-2}^{N} (i\zeta^{1/2})^j G_j + o(|\zeta|^{N/2}) \tag{2.6}$$

as $\zeta \to 0$, $\operatorname{Im}\zeta > 0$, in the norm topology of $\mathcal{B}(\mathcal{K}_N, \mathcal{K}_N^*)$.

Remark 2.5. The operator G_{-2} is the eigenprojection for eigenvalue zero, with the convention $G_{-2} = 0$, when 0 is not an eigenvalue.

This assumption is modeled on a Schrödinger operator on $L^2(\mathbf{R}^d)$ for d odd. Using ζ and $\ln\zeta$ in suitable combinations to form an asymptotic sequence, one can give an analogous definition applicable to Schrödinger operators on $L^2(\mathbf{R}^d)$ for d even. To simplify our presentation we will only discuss the case d odd.

Let us give a simple example of the type of expansion assumed in Assumption 2.4.

Example 2.6. Let

$$H_0 = -\frac{d^2}{dx^2} \quad \text{on} \quad L^2(\mathbf{R}).$$

Then the resolvent $R_0(\zeta)$ has the integral kernel

$$\frac{ie^{i\sqrt{\zeta}|x-y|}}{2\sqrt{\zeta}}.$$

Using the Taylor expansion of this kernel we obtain asymptotic expansions of the form (2.6). Fix $\varepsilon > 0$. We take

$$\mathcal{K}_N = L^2(\mathbf{R}, (1+|x|)^{N+\frac{3}{2}+\varepsilon}dx).$$

The operators G_j are defined by their integral kernels

$$G_j\colon\ -\frac{|x-y|^{j+1}}{2(j+1)!},\quad j=-1,0,1,\dots.$$

We also let $G_{-2}=0$. Then it is easy to verify that Assumption 2.4 holds for for all N.

For $d \geq 3$ odd and for $H_0 = -\Delta + W(x)$ it has been shown in [7, 5, 12] that expansions of the form (2.6) hold (with $\mathcal{K}_N$ a weighted L^2-space) up to an order N, depending on the decay rate of the potential $W(x)$. The case $d=1$ has been treated in [12, 2, 3, 4, 8]. The case d even (with asymptotic sequences based on ζ, $\ln\zeta$, and V) has been treated in [5, 12, 6, 1, 8]. Note that the cases $d=2$ and $d=4$ are rather complicated.

We want to study perturbations of operators satisfying Assumption 2.4. We use the same factorization technique as in [8], in an abstract framework. For simplicity we consider only bounded perturbations of H_0. Unbounded perturbations can be treated using a suitable scale of spaces.

Assumption 2.7. Let H_0 satisfy Assumption 2.4 for some N. Let $V \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Assume $V = vUv$ for some self-adjoint $v \in \mathcal{B}(\mathcal{H})$, such that also $v \in \mathcal{B}(\mathcal{H}, \mathcal{K}_N)$, and some self-adjoint $U \in \mathcal{B}(\mathcal{H})$, such that $U^2 = I$ (the identity). Let $H = H_0 + V$ and $R(\zeta) = (H-\zeta)^{-1}$.

Remark 2.8. We do not assume that v and U commute. One can generalize further and consider a perturbation factored as $V = w^*Uw$ with $w \in \mathcal{B}(\mathcal{H}, \mathcal{X})$, $U \in \mathcal{B}(\mathcal{X})$, $U^2 = I$, where $\mathcal{X}$ is an auxiliary Hilbert space. The decay condition implicit in Assumption 2.7 is then imposed by requiring that w extends to a bounded operator from $\mathcal{K}_N^*$ to $\mathcal{X}$. See [9] for the use of this factorization technique.

We restate another result from [8] in the current framework.

Lemma 2.9 ([8, Section 4]). *Let Assumption* 2.7 *hold. Assume that* $\zeta \in \mathbf{C}$, *with* $\operatorname{Im}\zeta \neq 0$. *Then the operator*

$$M(\zeta) = U + vR_0(\zeta)v \tag{2.7}$$

is invertible in $\mathcal{B}(\mathcal{H})$, *and the symmetrized second resolvent equation holds,*

$$R(\zeta) = R_0(\zeta) - R_0(\zeta)vM(\zeta)^{-1}vR_0(\zeta). \tag{2.8}$$

We have

$$UvR(\zeta)vU = U - M(\zeta)^{-1}, \tag{2.9}$$

and consequently

$$\sup_{|\zeta|<1, \mathrm{Im}\,\zeta\neq 0} |\zeta| \|M(\zeta)^{-1}\| < \infty. \tag{2.10}$$

3 Asymptotic Expansion of the Resolvent

We will now give our main results concerning the asymptotic expansion of the resolvent of the perturbed operator $H = H_0 + V$.

We will need some additional conditions on the coefficients in the asymptotic expansion of $R_0(\zeta)$, and a condition relating G_0 and the potential V. One possibility is the following assumption, which is adapted to 0 being a threshold of H_0, with $(-\delta, 0)$ in the resolvent set for some $\delta > 0$.

Assumption 3.1. Let Assumption 2.7 hold. Assume that $G_j = G_j^*$, $j = -2, \dots, N$. Assume furthermore that G_{-2} and G_{-1} in (2.6) are finite rank operators, and that the operator vG_0v is compact in $\mathcal{B}(\mathcal{H})$.

Theorem 3.2. *Let Assumption* 3.1 *hold, with* $N \geq 8$ *in* (2.6). *Then the resolvent* $R(\zeta)$ *has an asymptotic expansion*

$$R(\zeta) = \sum_{j=-2}^{N-8} (i\zeta^{1/2})^j R_j + o(|\zeta|^{(N-8)/2}) \tag{3.1}$$

as $\zeta \to 0$, $\mathrm{Im}\,\zeta > 0$, *in* $\mathcal{B}(\mathcal{K}_N, \mathcal{K}_N^*)$. *The coefficients* R_j, $j = -2, \dots, N-8$, *are computable in terms of the coefficients in* (2.6), *the operators* v *and* U, *and four projections. At most four equations determine whether* R_{-2} *and/or* R_{-1} *are non-zero.*

Proof. The proof is based on repeated application of Lemma 2.2. It follows from Lemma 2.9 that it suffices to obtain an asymptotic expansion for $M(\zeta)$. It is convenient to change variable to $\kappa = i\zeta^{1/2}$. In the sequel we will consider κ satisfying $-\delta < \mathrm{Re}\,\kappa < 0$, $0 < \mathrm{Im}\,\kappa < \delta$, $\kappa \neq 0$, for some sufficiently small $\delta > 0$. We will not repeat these restrictions below, and the value of δ will be adjusted during the proof.

We give the proof in the general case. If it is known that $G_{-2} = 0$ and/or $G_{-1} = 0$, the argument can be simplified considerably. In this context we also recall Remark 2.3.

As the first step in the proof we write

$$\begin{aligned} M(-\kappa^2) &= U + vR_0(-\kappa^2)v \\ &= \sum_{j=-2}^{N} \kappa^j M_j + o(|\kappa|^N), \end{aligned}$$

where

$$\begin{aligned} M_0 &= U + vG_0v, \\ M_j &= vG_jv, \quad j = -2, -1, 1, \dots, N. \end{aligned} \tag{3.2}$$

Due to our assumptions the operators M_j, $j = -2, \dots, N$ are self-adjoint operators in $\mathcal{B}(\mathcal{H})$. We rewrite $M(-\kappa^2)$ as

$$M(-\kappa^2) = \frac{1}{\kappa^2}\left(M_{-2} + \kappa A(\kappa)\right),$$

where

$$\begin{aligned} A(\kappa) &= \sum_{j=0}^{N+1} \kappa^j A_j + o(|\kappa|^{N+1}), \\ A_j &= M_{j-1}, \quad j = -1, \dots, N+1. \end{aligned} \tag{3.3}$$

Let P_0 denote the orthogonal projection onto $\ker_{\mathcal{H}} M_{-2}$. Since we assume that G_{-2} is a finite rank operator, it follows that 0 is an isolated point in the spectrum of M_{-2}. Thus we can use Lemma 2.2 to conclude that $M(-\kappa^2)$ is invertible in $\mathcal{B}(\mathcal{H})$, if and only if the operator $S_0(\kappa)$ given by

$$S_0(\kappa) = \sum_{j=0}^{\infty} (-)^j \kappa^j P_0 \left[A(\kappa)(M_{-2} + P_0)^{-1}\right]^{j+1} P_0 \tag{3.4}$$

is invertible in $\mathcal{B}(P_0\mathcal{H})$.

It follows from our assumptions and the definition above that we can write

$$S_0(\kappa) = P_0 A_0 P_0 + \kappa B(\kappa),$$

where

$$B(\kappa) = \sum_{j=0}^{N} \kappa^j B_j + o(|\kappa|^N).$$

The coefficients B_j can be found from (3.4) by straightforward computations, which we omit. We state the results for the first three coefficients. In order to shorten the expressions we introduce the notation

$$m = (M_{-2} + P_0)^{-1}.$$

We then find that

$$\begin{aligned} B_0 &= P_0A_1P_0 - P_0A_0mA_0P_0, \\ B_1 &= P_0A_2P_0 - P_0A_0mA_1P_0 - P_0A_1mA_0P_0 \\ &\quad + P_0A_0mA_0mA_0P_0, \\ B_2 &= P_0A_3P_0 - P_0A_0mA_2P_0 - P_0A_2mA_0P_0 \\ &\quad + P_0A_1mA_0mA_0P_0 + P_0A_0mA_1mA_0P_0 \\ &\quad + P_0A_0mA_0mA_1P_0. \end{aligned} \tag{3.5}$$

Since M_{-1} is assumed to be a finite rank operator, it follows that the point 0 is isolated in the spectrum of $P_0A_0P_0$ in $\mathcal{B}(P_0\mathcal{H})$. Let P_1 denote the orthogonal projection on $\ker_{P_0\mathcal{H}}(P_0A_0P_0)$. We consider P_1 both as an operator in $\mathcal{B}(P_0\mathcal{H})$, and in $\mathcal{B}(\mathcal{H})$, by extending it to be zero on the orthogonal complement of $P_0\mathcal{H}$. Using Lemma 2.2 we have that $S_0(\kappa)$ is invertible in $\mathcal{B}(P_0\mathcal{H})$, if and only if the operator $S_1(\kappa)$, defined by

$$S_1(\kappa) = \sum_{j=0}^{\infty}(-)^j\kappa^j P_1\left[B(\kappa)(P_0A_0P_0 + P_1)^{-1}\right]^{j+1}P_0,$$

is invertible in $\mathcal{B}(P_1\mathcal{H})$. We write

$$S_1(\kappa) = P_1B_0P_1 + \kappa C(\kappa),$$

where

$$C(\kappa) = \sum_{j=0}^{N-1}\kappa^jC_j + o(|\kappa|^{N-1}).$$

As above, the coefficients can be expressed in terms of the previously defined coefficients. We introduce

$$a = (P_0A_0P_0 + P_1)^{-1}$$

as an operator in $\mathcal{B}(P_0\mathcal{H})$. Straightforward computations yield

$$\begin{aligned} C_0 &= P_1B_1P_1 - P_1B_0P_0aP_0B_0P_1, \\ C_1 &= P_1B_2P_1 - P_1B_0P_0aP_0B_1P_1 - P_1B_1P_0aP_0B_0P_1 \\ &\quad + P_1B_0P_0aP_0B_0P_0aP_0B_0P_1. \end{aligned}$$

In order to use Lemma 2.2 on $S_1(\kappa)$ we need to look at the spectrum of $P_1B_0P_1$. Using (3.5) and $P_1P_0 = P_0P_1 = P_1$, we find that

$$P_1B_0P_1 = P_1A_1P_1 - P_1A_0mA_0P_1.$$

The last term is a finite rank operator, since $A_0 = M_{-1}$ is assumed to be a finite rank operator. The first term is rewritten using (3.3) and (3.2).

$$\begin{aligned} P_1A_1P_1 &= P_1UP_1 + P_1vG_0vP_1 \\ &= U - (1-P_1)U(1-P_1) - (1-P_1)UP_1 \\ &\quad - P_1U(1-P_1) + P_1vG_0vP_1. \end{aligned} \tag{3.6}$$

Now $1-P_1$ as an operator in $\mathcal{B}(\mathcal{H})$ is the projection onto $(\ker_{\mathcal{H}}(P_0A_0P_0))^{\perp} = \operatorname{ran}(P_0A_0P_0)$, since $P_0A_0P_0$ is self-adjoint and of finite rank. By assumption this range is finite dimensional. We have also assumed that vG_0v is compact. Since $U^2 = I$, we have $\sigma(U) \subseteq \{-1, 1\}$. It follows from the stability of the essential spectrum under compact perturbations that the point 0 is an isolated point in the spectrum of $P_1A_1P_1$ on $\mathcal{H}$, and therefore also in the spectrum of this operator on $P_1\mathcal{H}$. Thus we can again apply Lemma 2.2.

Let P_2 denote the orthogonal projection onto $\ker_{P_1\mathcal{H}}(P_1B_0P_1)$. By the above argument we know that $\operatorname{rank} P_2 < \infty$. Then $S_1(\kappa)$ is invertible in $\mathcal{B}(P_1\mathcal{H})$, if and only if $S_2(\kappa)$ is invertible in $\mathcal{B}(P_2\mathcal{H})$, where

$$S_2(\kappa) = \sum_{j=0}^{\infty} (-)^j \kappa^j P_2 \left[C(\kappa)(P_1B_0P_1 + P_2)^{-1}\right]^{j+1} P_1.$$

Once more we write

$$S_2(\kappa) = P_2C_0P_2 + \kappa D(\kappa).$$

We have

$$D(\kappa) = \sum_{j=0}^{N-2} \kappa^j D_j + o(|\kappa|^{N-2}).$$

We now apply Lemma 2.2 the last time. Since $\operatorname{rank} P_2 < \infty$, we have that 0 is an isolated point in the spectrum of $P_2C_0P_2$. Let P_3 denote the orthogonal projection onto $\ker_{P_2\mathcal{H}}(P_2C_0P_2)$. Then $S_2(\kappa)$ is invertible in $\mathcal{B}(P_2\mathcal{H})$, if and only if $S_3(\kappa)$ is invertible in $\mathcal{B}(P_3\mathcal{H})$, where

$$S_3(\kappa) = \sum_{j=0}^{\infty} (-)^j \kappa^j P_3 \left[D(\kappa)(P_2C_0P_2 + P_3)^{-1}\right]^{j+1} P_3.$$

Once more we write

$$S_3(\kappa) = P_3 D_0 P_3 + \kappa E(\kappa).$$

Now we claim that $\ker_{P_3\mathcal{H}}(P_3 D_0 P_3) = \{0\}$. Suppose that the kernel is nontrivial. Then we can repeat the application of Lemma 2.2. The iteration must stop with an invertible operator after a finite number of steps, since $\operatorname{rank} P_3 < \infty$. Then using (2.5) we conclude that the inverse $M(-\kappa^2)^{-1}$ exists and has a leading term $\kappa^{-n}X$ with $X \neq 0$ and $n > 3$. But this result contradicts (2.10) (recall that $\zeta = -\kappa^2$).

We conclude that $P_3 D_0 P_3$ is invertible in $\mathcal{B}(P_3\mathcal{H})$. We can then apply (2.5) repeatedly to conclude that we have an expansion

$$M(-\kappa^2) = \sum_{j=-2}^{N-4} \kappa^j F_j + o(|\kappa|^{N-4}).$$

We omit the details of this computation. Back-substitution of the various expressions derived above leads to computability of the coefficients in terms of the coefficients in the given asymptotic expansion and the projections introduced in the proof. We can then use (2.8) to get an expansion of $R(\zeta)$ of the form (3.1). In this context we use (2.10) once more to eliminate terms of order $(i\zeta^{1/2})^j$, $j < -2$. We use (2.9) to conclude that $F_{-2} \neq 0$ implies $R_{-2} \neq 0$, and similarly that $F_{-1} \neq 0$ implies $R_{-1} \neq 0$.

One final comment. At stated at the beginning of the proof, we have considered the general case. If $P_j = 0$ for some $j = 0, 1, 2$, then the iteration can be stopped at that point, since the leading term in the relevant $S_j(\kappa)$ at that point will be invertible. In those cases one can also obtain an expansion to a higher order than in the general case. □

We now give a result adapted to the case where H_0 has an eigenvalue embedded in the absolutely continuous spectrum. Thus we do not assume that coefficients in the expansion are self-adjoint. We use the following assumption.

Assumption 3.3. Let Assumption 2.7 hold, with $N = 2K$, $K \geq 4$. Assume that G_{-2} is of finite rank, and that vG_0v is compact in $\mathcal{B}(\mathcal{H})$. Assume furthermore that $G_j = 0$, $j < N$ odd.

Let us note that this assumption can be verified at an eigenvalue embedded in the absolutely continuous spectrum, using the generalized Mourre technique, see [11].

Theorem 3.4. *Let Assumption* 3.3 *hold. Then the resolvent* $R(\zeta)$ *has an asymptotic expansion*

$$R(\zeta) = \sum_{\ell=-1}^{K-4} (-\zeta)^{\ell} R_{2\ell} + o(|\zeta|^{K-4}) \tag{3.7}$$

as $\zeta \to 0$, $\operatorname{Im}\zeta > 0$, *in* $\mathcal{B}(\mathcal{K}_N, \mathcal{K}_N^*)$.

Proof. The proof is similar to the proof of Theorem 3.2. Note that we use the same symbols as above, but with different definitions. We start with the asymptotic expansion for $M(\zeta)$ from (2.7), using our assumptions on the structure of the expansion (2.6).

$$M(\zeta) = \sum_{\ell=-1}^{K} \zeta^{\ell} M_{\ell} + o(|\zeta|^{K}),$$

where

$$\begin{aligned} M_0 &= U + vG_0v \\ M_{\ell} &= (-1)^{\ell} vG_{2\ell}v, \quad \ell = -1, 1, 2, \dots, K. \end{aligned}$$

We write

$$M(\zeta) = \frac{1}{\zeta}\left(M_{-1} + \zeta A(\zeta)\right),$$

where

$$A(\zeta) = \sum_{\ell=0}^{K} \zeta^{\ell} A_{\ell} + o(|\zeta|^{K})$$

with

$$A_{\ell} = M_{\ell}, \quad \ell = 0, \dots, K.$$

By assumption M_{-1} is self-adjoint and of finite rank, hence 0 is an isolated point in the spectrum of M_{-1}. Let P_0 denote the orthogonal projection onto $\ker_{\mathcal{H}}(M_{-1})$. Then Lemma 2.2 shows that $M(\zeta)$ is invertible in $\mathcal{B}(\mathcal{H})$, if and only if $S_0(\zeta)$ is invertible in $\mathcal{B}(P_0\mathcal{H})$, where

$$S_0(\zeta) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \zeta^{\ell} P_0 \left[A(\zeta)(M_{-1} + P_0)^{-1}\right]^{\ell+1} P_0. \tag{3.8}$$

In order to apply the lemma once more we write

$$S_0(\zeta) = P_0 A_0 P_0 + \zeta B(\zeta), \tag{3.9}$$

and note that we have

$$B(\zeta) = \sum_{\ell=0}^{K-1} \zeta^\ell B_\ell + o(|\zeta|^{K-1}).$$

We also note that a simple computation using (3.9) and (3.8) yields

$$B_0 = P_0 A_1 P_0 - P_0 A_0 (M_{-1} + P_0)^{-1} A_0 P_0.$$

We have

$$P_0 A_0 P_0 = P_0 U P_0 + P_0 v G_0 v P_0.$$

The computation in (3.6) can be repeated here, and the argument following that computation can also be repeated, since we also here assume that vG_0v is compact. We conclude that 0 is an isolated point in the spectrum of $P_0A_0P_0$ in $\mathcal{B}(P_0\mathcal{H})$. Let P_1 denote the associated Riesz projection. Note that in the current context we are not assuming that G_0 is self-adjoint, hence P_1 need not be an orthogonal projection.

Now we use Lemma 2.2 once more. We have that $S_0(\zeta)$ is invertible in $\mathcal{B}(P_0\mathcal{H})$, if and only if $S_1(\zeta)$ is invertible in $\mathcal{B}(P_1\mathcal{H})$, where

$$S_1(\zeta) = \sum_{\ell=0}^{\infty} (-1)^\ell \zeta^\ell P_1 \left[B(\zeta)(P_0 A_0 P_0 + P_1)^{-1} \right]^{\ell+1} P_1.$$

Again we write

$$S_1(\zeta) = P_1 B_0 P_1 + \zeta C(\zeta).$$

Now we claim that $P_1 B_0 P_1$ is invertible in $\mathcal{B}(P_1\mathcal{H})$, since otherwise we can repeat the application of Lemma 2.2, leading to a singularity of the type ζ^{-j} with $j > 1$, contradicting (2.10). Back-substitution leads to the existence of an expansion

$$M(\zeta)^{-1} = \sum_{\ell=-1}^{K-2} \zeta^\ell F_\ell + o(|\zeta|^{K-2}),$$

with computable coefficients. The proof is then concluded using (2.8), as in the proof of Theorem 3.2. $\square$

4 Computation of Expansion Coefficients

The proofs of the theorems in the previous section give a systematic procedure to determine the expansion coefficients. The coefficients depend on the expansion coefficients G_j in (2.6), on $V = vUv$, and on the projections introduced in the proofs. In concrete cases these projections can be computed in terms of solutions to the equation $H\psi = 0$ in some auxiliary space. We should note that the determination of the projections can involve a substantial effort.

As an example, where everything has been carried out according to the procedure outlined here, we mention the computation of the expansion coefficients in the $d = 1$ case $H = -d^2/dx^2 + V(x)$, given in [8].

Let us note that the procedure in the proofs of the two theorems also allows one to set up a number of equations such that existence of nontrivial solutions to these equations leads to the existence of an eigenvalue at zero (or a zero-resonance).

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