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Shocks and very strong vertical diffusion

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1 Introduction

In [15] we have introduced the notion of proper viscosity solutions to solve the Cauchy problem for a single nonlinear first order equation of the form

$$\partial_t u + H(u, \nabla u) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty), \tag{1.1}$$

$$u|_{t=0} = u_0 \quad \text{in } \mathbf{R}^n \tag{1.2}$$

globally-in-time allowing jump discontinuities of solutions. If the quation (1.1) is a conservation law, there is a notion of the entropy solution (which is a special distributional weak solution) so that the Cauchy problem is uniquely solvable globally-in-time at least for bounded initial data (see e.g. [6]). However, there are a couple of interesting examples of (1.1) which is not a conservation law. Typical examples include

$$\partial_t u - a(u)|\nabla u| = 0, \tag{1.3}$$

$$\partial_t u - b(u)(1 + |\nabla u|^2)^{1/2} = 0, \tag{1.4}$$

where a and b are not nonincreasing. The conventional theory of viscosity solutions [5] does not apply for such problems including conservation laws. As explained in §3 the notion of proper viscosity solution is more restrictive than usual viscosity solution; the proper viscosity soloution requires some control on the speed of shocks (jump discontinuities) while the conventional viscosity solution does not require such a control. In [15] we have established various comparison principles for proper viscosity solutions and constructed a unique global proper viscosity solution for various situations. We also proved, in various setting, that the solution of a regularized problem

$$\partial_t u^\varepsilon + H(u^\varepsilon, \nabla u^\varepsilon) = \varepsilon \Delta u^\varepsilon$$

with (1.2) converges to the proper viscosity solution of (1.1), (1.2) as $\varepsilon \to 0$ in the sense of convergence of closed sets:

$$\text{sg } u^\varepsilon \to \text{sg } u,$$

where sg u^ε denotes the subgraph defined by

$$\text{sg } u^\varepsilon = \{(x', x_{n+1}, t);\ x' \in \mathbf{R}^n,\ x_{n+1} \le u^\varepsilon(x', t),\ t \in [0, \infty)\}.$$

It is a kind of Hausdorff distance convergence.

In this paper we show that the graph of our proper solution can be regarded as a solution of a surface evolution equation in $\mathbf{R}^{n+1}$ whose vertical diffusion is very strong so that its effect is nonlocal. The equations with very strong diffusivity has been proposed by S. Angenent and M. Gurtin [2] and J. Taylor [22] as crystalline flow and studied for many years; the reader is referred to [16] for the state of arts. Such an interpretation turns to be useful to calculate the evolution of the graph of proper solutions by the level set approach developed by [17], [18]. If sg u is regarded as the set $\{\psi > 0\}$ for an auxiliary (continuous) function $\psi(x', x_{n+1}, t)$ in $\mathbf{R}^n \times \mathbf{R} \times (0, \infty)$, (1.1) can be written as

$$\partial_t \psi + (-\partial_{x_{n+1}} \psi) H(x_{n+1}, \nabla_{x'} \psi / (-\partial_{x_{n+1}} \psi)) = 0. \tag{1.5}$$

In the level set approach we consider (1.5) in $\mathbf{R}^n \times \mathbf{R} \times (0, \infty)$ rather than on the zero level of ψ. When $r \mapsto H(r, p)$ is not nondecreasing, there is a chance that the zero level set $\{\psi = 0\}$ may overhang, i.e., $\{\psi = 0\} \cap \{x'_0\} \times \mathbf{R} \times \{t_0\}$ has more than two connected components at some (x'_0, t_0). Since the graph of a function does not have such a property even the function is discontinuous, the level set $\{\psi = 0\}$ does not corresponds to the graph of proper solution. The question is what is the reasonable reinterpretation of (1.5) so that $\{\psi = 0\}$ is the graph of a proper viscosity solution. Instead of (1.5) we propose to consider

$$\partial_t \psi + (-\partial_{x_{n+1}} \psi) H(x_{n+1},\ \nabla_{x'} \psi / (-\partial_{x_{n+1}} \psi)) = D|\nabla \psi| (\partial_{x_{n+1}} (\text{sgn } \partial_{x_{n+1}} \psi))/2. \tag{1.6}$$

for sufficiently large $D > 0$. In §2 we study the interface version of (1.6) and give a formal reason why $\{\psi = 0\}$ for (1.6) is the graph of a proper viscosity solution.

Although we do not discuss in the present paper, our formal reasoning is useful to define the notion of solution of (1.6) for general $D > 0$.

In this paper we also extend the notion of proper viscosity solutions so that it applies to some second order problems including

$$\partial_t u - a(u)|\nabla u| = \sigma|\nabla u| \operatorname{div}(\nabla u/|\nabla u|), \tag{1.7}$$

$$\partial_t u - b(u)(1 + |\nabla u|^2)^{1/2} = \sigma(1 + |\nabla u|^2)^{1/2} \operatorname{div}(\nabla u/(1 + |\nabla u|^2)^{1/2}) \tag{1.8}$$

with $\sigma > 0$. If $\sigma = 0$, (1.7) and (1.8) is nothing but (1.3) and (1.4) respectively. The equation (1.7) requires that each y-level set of u moves by $a(y)$ plus its mean curvature. The equation (1.8) requires that the graph of u moves by $b(y)$ plus its upward mean curvature. As already noted by [14], [3] the solution of (1.8) may cease to be continuous in a finite time when b is not nonincreasing. Thus the notion of proper solution is expected to be useful to extend the solution for such problems. We do not pursue such problems. There are several interesting examples of parabolic equations whose solution may cease to be continuous. The reader is referred to [1], [14], [20], [19], [21] and papers cited there.

In the last part of this paper we give several examples of solutions. In particular, we point out that our proper solution distinguish admissible shocks from non admissible one when (1.1) is a conservation law.

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2 Very strong vertical diffusion

We consider a surface evolution equation of a hypersurface $\Gamma_t \subset \mathbf{R}^{n+1}$ of the form:

$$V = v(x_{n+1}, \mathbf{n}) - \operatorname{div}_{\Gamma_t}\xi(\mathbf{n}) \quad \text{on } \Gamma_t. \tag{2.1}$$

Here V denotes the normal velocity of Γ_t in the direction of the unit normal vector $\mathbf{n}$ of Γ_t and $\operatorname{div}_{\Gamma_t}$ denotes the surface divergence on Γ_t. The function v is a given function of $n+1$-th component x_{n+1} of $x \in \mathbf{R}^{n+1}$ and $\mathbf{n}$. The function ξ is the gradient of $\gamma(p) = D|p_{n+1}|/2$ with a positive parameter D, i.e.,

$$\xi(p) = (\partial_{p_1}\gamma(p), \ldots, \partial_{p_n}\gamma(p),\ \partial_{p_{n+1}}\gamma(p)) = (0, \ldots, 0, D\ (\operatorname{sgn} p_{n+1})/2),$$
$$p = (p_1, \ldots, p_n, p_{n+1}) \in \mathbf{R}^{n+1}.$$

At the place where $\mathbf{n}$ is orthogonal to $(0,\dots,0,1)$, the curvature term $\mathrm{div}_{\Gamma_t}\xi(\mathbf{n})$ is not well-defined quantity in a usual sense even if Γ_t is small. The diffusion effect is too strong so that the quantity $\mathrm{div}_{\Gamma_t}\xi(\mathbf{n})$ turns to be nonlocal. If $n=1$ and v is independent of x_{n+1}, such a type of problems is well-studied in a series of papers [8], [10], [11], [12], [13]. Their assumptions on γ exclude that of (2.1); however, the results of these papers easily extend to (2.1). In this case if Γ_0 is given as a boundary of subgraph sg u_0 of a function of $x' \in \mathbf{R}^n$, then its evolution by (2.1) turns to agree with evolution by $V=v(\mathbf{n})$ and Γ_t stays a boundary of sg $u(\cdot,t)$ of a function of $x' \in \mathbf{R}^n$. In other words, graph-like property of Γ_t is preserved and no overhanging occurs; moreover, the curvature term plays no role. It is not difficult to check these properties by using definition of solutions in [12]; however, we do not give its proof here.

If $x_{n+1} \mapsto v(x_{n+1},\mathbf{n})$ is not nonincreasing, solution of

$$V = v(x_{n+1},\mathbf{n}) \tag{2.2}$$

is expected to be overhanged even for graph-like initial data and the curvature term really play a role so that solutions of (2.1) and (2.2) may be different each other.

Following suggestions going back to [7] and [9] (see also [16]) it is reasonable to define speed of Γ_t including the place at which $\mathbf{n}$ is orthogonal to $(0,\dots,0,1)$ in the followsing way:

$$V = V(x,t) = v(x_{n+1},\mathbf{n}) - \mathrm{div}_{\Gamma_t}\eta \tag{2.3}$$

$$\eta \in \partial\gamma(\mathbf{n}) \quad \text{almost everywhere on } \Gamma_t \tag{2.4}$$

and η minimizes

$$\int_{\Gamma_t} |v(x_{n+1},\mathbf{n}) - \mathrm{div}_{\Gamma_t}\eta|^2 dS, \tag{2.5}$$

where dS is the surface element and $\partial\gamma$ denotes the subdifferential of γ. We are fully aware that one has to prove that the choice of the speed is actually reasonable by approximating γ by smoother one; moreover, one has to specify a class of Γ_t to define evolution by (2.3)–(2.5). However, we do not pursue such problems in the present paper.

We now specify Γ_t and calculate its speed. We consider a shock profile

$$u(x',t) = \begin{cases} u_1(x',t), & x' \in U_t \\ u_2(x',t), & x' \notin U_t \end{cases} \tag{2.6}$$

where U_t is an open set in $\mathbf{R}^n$ and the boundary S_t of U_t is a smooth one-parameter family of smooth hypersurfaces. The functions u_1 is C^1 in $\overline{U}$ when $U = \cup_{t>0} U_t \times \{t\}$ and u_2 is C^1 in the complement of U. To fix idea we assume that the value of u_2 on S_t is always greater than that of u_1. Let Γ_t be the boundary of the subgraph sg u in $\mathbf{R}^{n+1}$ and $\mathbf{n}$ be the unit outward normal of sg u. We are interested in the velocity of Γ_t at (x_0, t_0) when x_0' is on the shock S_{t_0}. By (2.4) we see that

$$\eta = (0, \ldots, 0, D/2) \quad \text{and} \quad \mathrm{div}_{\Gamma_{t_0}} \eta = 0 \quad \text{on} \quad \Gamma_{t_0} \setminus (S_{t_0} \times \mathbf{R}). \tag{2.7}$$

By (2.4) on $\Gamma_{t_0} \cap (S_{t_0} \times \mathbf{R})$ the function η is of the form

$$\eta(x) = (0, \ldots, 0, \eta_{n+1}(x)), \ |\eta_{n+1}(x)| \le D/2. \tag{2.8}$$

Since

$$\mathrm{div}_{\Gamma_t} \eta = \frac{\partial \eta_{n+1}}{\partial x_{n+1}} = \partial_{x_{n+1}} \eta_{n+1} \quad \text{on} \quad \Gamma_t \cap (S_{t_0} \times \mathbf{R}),$$

the integral (2.5) is minimized if and only if

$$\int_{u_1(x', t_0)}^{u_2(x', t_0)} |v(x_{n+1}, \mathbf{n}(x, t_0)) - \partial_{x_{n+1}} \eta_{n+1}|^2 dx_{n+1} \tag{2.9}$$

is minimized at every $x' \in S_{t_0}$. We set $x' = x_0'$ and observe that the problem (2.7)–(2.9) can be interpreted as an obstacle problem: find $\tilde{\eta} : [a, b] \to \mathbf{R}$ which minimizes

$$\int_a^b |z(y) + \tilde{\eta}'(y)|^2 dy \tag{2.10}$$

subject to

$$|\tilde{\eta}(y)| \le D/2 \quad \text{for} \quad y \in [a, b], \tag{2.11}$$

$$\tilde{\eta}(a) = \tilde{\eta}(b) = D/2. \tag{2.12}$$

Here we set $a = u_1(x_0', t_0)$, $b = u_2(x_0', t_0)$, $z(y) = -v(y, \mathbf{n}(x_0', t_0))$ and $\tilde{\eta} = \eta_{n+1}$. The boundary condition (2.12) comes from (2.7) and the constraint (2.11) comes from (2.8). Such a type of obstacle problems is derived in [9] for Lipschitz continuous graph-like solution when $n = 1$. As in [9] we transform dependent variable $\tilde{\eta}$ by

$$\zeta(y) = \tilde{\eta}(y) + Z(y), \ Z(y) = \int_a^y z(\sigma) d\sigma.$$

Then (2.10)–(2.12) is equivalent to find a minimizer ζ of the set of values

$$(2.13)\quad \{\int_a^b |\zeta'|^2 dy;\ \zeta(a) = D/2,\ \zeta(b) = Z(b) + D/2,\ Z - D/2 \le \zeta \le Z + D/2\}.$$

Theorem 2.1. *Let Z_I denote the convexification of Z in $I = [a,b]$. Then $\zeta_0 = Z_I + \frac{D}{2}$ is the unique minimizer of (2.13) if and only if $Z_I \ge Z - D$ on $[a,b]$.*

Proof. Since the problem is convex, the unique existence of a minimizer is clear. If $\tilde{\zeta}$ is the minimizer, then $\tilde{\zeta}$ is convex outside the set where $\tilde{\zeta} = Z - D/2$, since otherwise one can deform $\tilde{\zeta}$ so that it decreases the energy $\int_I |\zeta'|^2 dy$. If $Z_I \ge Z - D$, then $\tilde{\zeta} \ge Z_I + \frac{D}{2}$ since otherwise it decreases the energy. Since $\tilde{\zeta}$ is now convex, by definition $Z_I + \frac{D}{2} \ge \tilde{\zeta}$. Thus $\frac{D}{2} + Z_I = \tilde{\zeta}$. If $Z_I \ge Z - D$ does not hold, then ζ_0 does not satisfy the constraint $Z - D/2 \le \zeta_0$ so ζ_0 cannot be the minimizer. □

It is clear that there is a threshold value of D for the property $Z_I \ge Z - D$ on I.

Corollary 2.2. *Let $D_0 = D_0(I)$ be te number defined by*

$$D_0 = \inf\{D;\ Z_I \ge Z - D \quad \text{on} \quad I\}.$$

Then $\zeta_0 = \frac{D}{2} + Z_I$ is the unique minimizer of (2.13) for $D \ge D_0$ and it is not the minimizer of (2.13) for $D < D_0$. Moreover, $D_0(I) \le D_0(J)$ if $I \subset J$.

The monotonicity of $D_0(I)$ in I is clear by definition. By these observations the speed at (x_0, t_0) in (2.3) equals $-\zeta_0'$ i.e. $-\partial_{x_{n+1}} Z_I$ for sufficiently large D, say $D \ge D_0(I)$ with $I = [u_1(x_0', t_0), u_2(x_0', t_0)]$.

We now consider

$$(2.14)\qquad \frac{\partial u}{\partial t} + H(u, \nabla u) = 0$$

where $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$, $\partial_{x_j} u = \partial u / \partial x_j$. Let Γ_t be the boundary of sg $u(\cdot, t)$. The unit normal $\mathbf{n}$ is taken outward so that its explicit form is

$$\mathbf{n} = (-\nabla u, 1)/(1 + |\nabla u|^2)^{1/2}.$$

Since $V = \partial_t u/(1 + |\nabla u|^2)^{1/2}$, (2.14) is equivalent to (2.2) if

$$(2.15)\qquad v(x_{n+1}, p_1, \dots, p_n, p_{n+1}) = -p_{n+1} H(x_{n+1}, (p_1, \dots, p_n)/(-p_{n+1})).$$

provided u is C^1. We consider (2.1) with interpretation of (2.3) and calculate the speed of shocks for a function u of the form (2.6) under the assumption that u is bounded. We need the value of v at $p_{n+1}=0$, which is formally derived by sending $p_{n+1}\downarrow 0$ in (2.15). Its explicit form is

$$v(x_{n+1},p_1,\dots,p_n,0)=-H_\infty(x_{n+1},-p_1,\dots,-p_n)$$

where H_∞ is the recession function defined by

$$H_\infty(r,p_1,\dots,p_n)=\lim_{\lambda\downarrow 0}\lambda H(r,(p_1,\dots,p_n)/\lambda).$$

For sufficiently large D, say $D\geq D_0(I)$ with $I=[\inf\ u,\sup\ u]$, the speed $V=V(x_0,t_0)$ at shock S_{t_0} is provided by $-\partial_{x_{n+1}}Z_I$ by Corollary 2.2. By definition

$$Z_I(x_{n+1})=\Big(\int^{x_{n+1}}H_\infty(r,-\hat{\mathbf{n}})dr\Big)_I$$

where $\hat{\mathbf{n}}=(n_1,\dots,n_n)$. Thus the speed V of S_t in the direction of $\hat{\mathbf{n}}$ agrees with the speed appeared in the speed of shocks in the definition of proper solutions; see §3 and [15]. If $r\mapsto H(r,p)$ is nonincreasing, $-\partial_{x_{n+1}}Z_I$ is constant on $\Gamma_{t_0}\cap\{x_0'\}\times\mathbf{R}$, $x_0'\in S_{t_0}$. Its value agrees with the one obtained by the Rankine–Hugoniot condition when (1.1) is a conservation law.

We have thus observed that a shock profile is resulted from very strong vertical diffusion.

3 Proper solutions

We extend the notion [15, §2] of a proper subsolution for a class of second order equation of the form

$$\partial_t u+H(u,\nabla u,\nabla\nabla u)=0, \tag{3.1}$$

where $\nabla\nabla u$ denotes the Hesse matrix of u. We assume that

$$H_\infty(r,p,X)=\lim_{\lambda\downarrow 0}\lambda H(r,p/\lambda,X/\lambda)$$

exists and $(p,X)\mapsto H_\infty(r,p,X)$ is geometric in the sense of [4], i.e.,

$$H_\infty(r,\lambda p,\lambda X+\sigma p\otimes p)=\lambda H(r,p,X)$$

for all $p\in\mathbf{R}^n\setminus\{0\}$, $X\in\mathbf{S}^n$, $\sigma\in\mathbf{R}$, $\lambda>0$ where $\mathbf{S}^n$ denotes the space of $n\times n$ real symmetric matrices. Let Ω be an open set in $\mathbf{R}^n$. We set $Q=\Omega\times(0,T)$.

Definition 3.1 (Proper subsolution). Let $u : Q \to \mathbf{R}$ be a subsolution [5] of (3.1) in Q. We say that u is a proper subsolution if for any $(x_0, t_0) \in Q$ and any upper test surface $\{S_t\}$ of u^* at (x_0, t_0) with level $\mu(< u^*(x_0, t_0))$ the inequality

$$V(x_0,t_0) + H^I(u^*(x_0,t_0),\ -\mathbf{n}(x_0,t_0),\ -R_{\mathbf{n}}\nabla\mathbf{n}R_{\mathbf{n}}) \le 0$$

holds with $I = [\mu, u^*(x_0, t_0)]$. Here u^* represents the upper semicontinuous envelope of u.

Here $V = V(x_0, t_0)$ denotes the normal velocity (in the direction of $\mathbf{n}$) of $\{S_t\}$ at (x_0, t_0) in the direction of $\mathbf{n}(x_0, t_0)$; $R_{\mathbf{n}} = I - \mathbf{n} \otimes \mathbf{n}$ which is the orthogonal projection to the space orthogonal to $\mathbf{n}$. The quantity $\nabla\mathbf{n}$ depends on extension of $\mathbf{n}$ outside S_t; however, $R_{\mathbf{n}}\nabla\mathbf{n}R_{\mathbf{n}}$ is independent of the extension. The relaxed function H^I is defined by

$$H^I(r,p,X) = \partial_r\Big(\int^r H_\infty(\rho,p,X)d\rho\Big)_I.$$

We recall the definition of an upper test surface; this notion is defined in [15, §2]. We say that a smooth family $\{S_t\}$ of hypersurfaces defined near $(x_0, t_0) \in \mathbf{R}^n \times (0, T)$ is an upper test surface of u^* at (x_0, t_0) with level μ if $S_t = \partial U_t$ and U_t is a smoothy family of open sets and

$$u^*(x,t) \le \mu \quad \text{for} \quad x \in U_t \quad \text{near} \quad (x_0,t_0).$$

We have given an orientation of S_t by taking inward unit normal $\mathbf{n}$ of ∂U_t.

A proper supersolution is defined in a symmetric way as described in [15, §2]. As usual if u is simultaneously proper sub- and supersolution, we say that u is a proper solution of (3.1).

Example 3.2 (First order problem). Assume that $r \mapsto H(r,p)$ is either strictly monotone increasing or decreasing (depending on $p \in \mathbf{R}^n$). We consider a shock profile of the form (2.6). Let $\mathbf{n}$ be the unit normal vector field of S_t pointing to U_t. Assume that the normal velocity of S_t at (x', t) equals

$$c = -\frac{1}{b-a}\int_a^b H(r, -\mathbf{n}(x',t))dr \tag{3.3}$$

with $b = u_2(x', t)$, $a = u_1(x', t)$. This is the speed determined by the Rankine–Hugoniot condition when (1.1) is a conservation law. If $r \mapsto H(r, -\mathbf{n}(x', t))$ is strictly decreasing for every point $x' \in S_t$, $t > 0$, then it is easy to see that u in (2.6) is a proper solution of (1.1) provided that u_1 and u_2 solve (1.1) off S_t. If $r \mapsto$

$H(r, -\mathbf{n}(x', t))$ is strictly increasing for some point $x' \in S_t$ for some $t > 0$, u in (2.6) is not a proper viscosity solution. The solution with shocks with speed satisfying (3.3) always satisfies (1.1) in distribution sense when (1.1) is of conservation type. To be an entropy solution it is known (e.g. [6]) that every characteristic line near shock is merging to the shock as time develop. This is equivalent to $r \mapsto H(r, -\mathbf{n}(x', t))$ is strictly decreasing. Thus our proper viscosity solution really distinguish admissible shock (entropy solution) from non admissible one when (1.1) is a conservation law.

Example 3.3 (Equation (1.7)). We shall give a special radial proper viscosity solution of (1.7) when $a(r)$ is increasing. Consider two-valued function

$$u(x', t) = \begin{cases} a, & |x'| < R(t), \\ b, & |x'| \geq R(t) \end{cases}$$

with $b > a$. It is easy to see that u is a proper viscosity solution of (1.7) if

$$\frac{dR}{dt}(t) = \frac{1}{b-a} \int_a^b a(r)dr - \frac{\sigma(n-1)}{R}.$$

For (1.8) it is not easy to give an explicit solution with jump discontinuities. However, we note that our Definition 3.1 is applicable to define the notion of proper viscosity solution for (1.8) since

$$H_\infty(r, p, X) = -b(r)|p| - \sigma \text{ trace}[(I - (p \otimes p)/|p|^2)X]$$

is geometric.

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