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Some models for shape memory alloys

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0 Introduction

In this paper we consider a shape memory material occupying the one-dimensional interval $(0,1)$. We put $Q(T) := (0,T) \times (0,1)$, $0 < T < \infty$. It is a crucial step to describe the relationship between the temperature field θ, the stress σ and the shear strain ε, in the analysis of the dynamics of shape memory alloys as a system of differential equations. By some experiments we have already obtained the following load-deformation curves (Fig. 0.1).

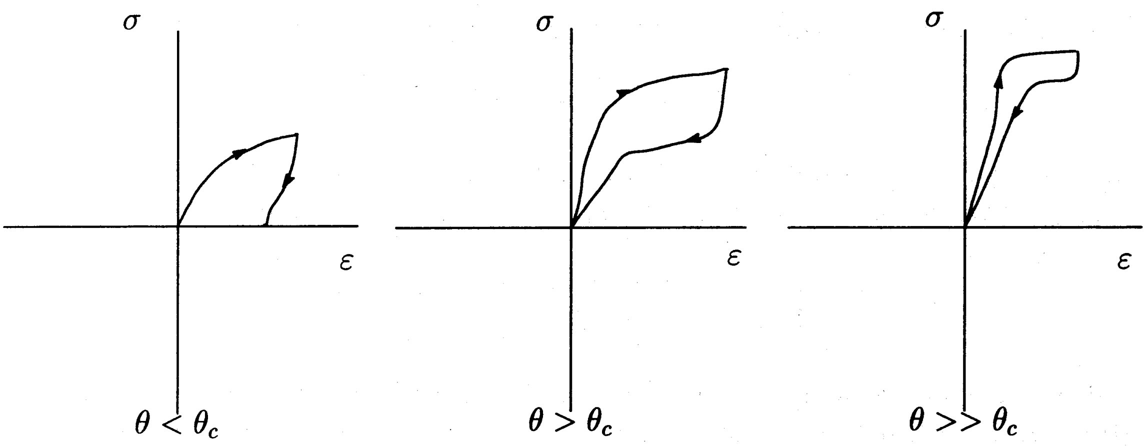

Figure 0.1. Load-deformation curves.

There are a lot of papers dealing with one-dimensional shape memory alloy problems (see [3]). In the most of papers Landau-Devonshire form was employed, which is one of acceptable approximations of the load deformation curves. The idea is as follows. Let $\Psi := \Psi(\theta, \varepsilon)$ be the Helmholtz free energy function given by

$$\Psi(\theta,\varepsilon) = \Psi_0(\theta) + \kappa_1(\theta - \theta_0)\varepsilon^2 - \kappa_2\varepsilon^4 + \kappa_3\varepsilon^6,$$

where

$$\Psi_0(\theta) = -C_V\theta\log(\theta/\tilde{\theta}) + C_V\theta + C_0,$$

κ_1, κ_2, κ_3, θ_0, C_0 and C_V are positive constants. Moreover, we assume that

$$\sigma = \frac{\partial\Psi}{\partial\varepsilon}.$$

In such an approximation the relationship corresponding to the load-deformation curves is able to be mathematically described. Now, we recall the following Falk's model which is based on the Landau-Devonshire form and thermodynamics theory (cf. [6]).

$$u_{tt} + \gamma u_{xxxx} - (f_1(u_x)\theta + f_2(u_x))_x = f \quad \text{in } Q(T),$$

$$\theta_t - k\theta_{xx} - f_1(u_x)\theta u_{xt} = g \quad \text{in } Q(T),$$

$$u(t,0) = u(t,1) = 0 \quad \text{for } 0 \le t \le T,$$

$$u_{xx}(t,0) = u_{xx}(t,1) = 0 \quad \text{for a.e. } t \in [0,T],$$

$$\theta_x(t,0) = \theta_x(t,1) = 0 \quad \text{for a.e. } t \in [0,T],$$

$$u(0,x) = u_0(x), u_t(0,x) = v_0(x) \text{ and } \theta(0,x) = \theta_0(x) \quad \text{for } x \in (0,1),$$

where f_1 and f_2 are continuous functions on $\mathbf{R}$, f and g are given functions on $Q(T)$, and u_0, v_0 and θ_0 are initial functions. In this system we denote by u the displacement and assume that $\varepsilon = u_x$. This problem was already discussed in [11, 3, 4, 5, 1]. Also, the above system with viscosity terms was studied by Hoffmann-Zochowski [7] and Sprekels-Zheng-Zhu [12].

Recently, some types of hysteresis operators were characterized by the differential equations including the subdifferential operators of the indicator functions of closed intervals in $\mathbf{R}$ (cf. [13]). In this paper we consider an ordinary differential equation of the form:

$$\sigma_t + \partial I(\theta, \varepsilon; \sigma) \ni c\varepsilon_t, \tag{0.1}$$

where c is a non-negative constant and I is the indicator function of the closed interval $[f_a(\theta,\varepsilon), f_d(\theta,\varepsilon)]$ for given continuous functions f_a and f_d on $\mathbf{R} \times \mathbf{R}$, that is,

$$I(\theta,\varepsilon;\sigma) = \begin{cases} 0 & \text{if } f_a(\theta,\varepsilon) \le \sigma \le f_d(\theta,\varepsilon), \\ +\infty & \text{otherwise.} \end{cases}$$

In case $c = 0$, the hysteresis operator (see Fig. 0.2) with unti-clockwise trend is characterized by (0.1). Kenmochi-Koyama-Meyer discussed parabolic PDEs and quasivariational inequalities with hysteresis operators in the case of $c = 0$ (cf. [8]). Their system contains an approximation equation for (0.1) with $c = 0$.

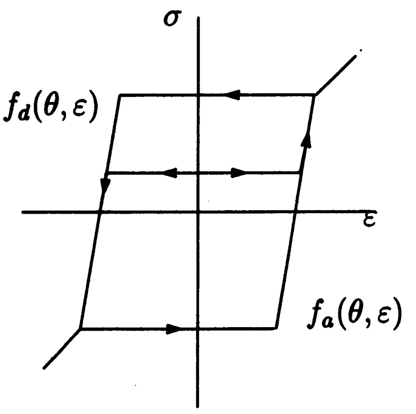

Figure 0.2

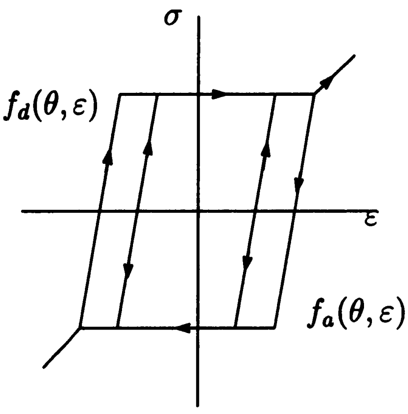

Figure 0.3

We can describe hysteresis operators (see Fig. 0.3) with clockwise trend as the differential equation (0.1) in the case that $c > 0$ and

$$0 \leq \frac{\partial f_a}{\partial \varepsilon}, \frac{\partial f_d}{\partial \varepsilon} \leq c \text{ on } \mathbf{R} \times \mathbf{R}.$$

In this setting σ is determined by the hysteresis operator illustrated by Fig. 0.3 if and only if σ is a solution of (0.1). This idea was already found by Krejci (cf. [9]) in case I is independent of ε.

In this paper by using this characterization of hysteresis operators we discuss the following system: Our problem is to find functions u, θ and σ on $Q(T)$ satisfying

$$u_{tt} + \gamma u_{xxxx} - \mu u_{xxt} - \sigma_x = 0 \quad \text{in } Q(T), \tag{0.2}$$

$$\theta_t - \kappa\theta_{xx} = \sigma u_{xt} \quad \text{in } Q(T), \tag{0.3}$$

$$\sigma_t - \nu\sigma_{xx} + \partial I(\theta, \varepsilon; \sigma) \ni c u_{xt} \quad \text{in } Q(T), \tag{0.4}$$

$$u(t,0) = u(t,1) = 0 \text{ and } u_{xx}(t,0) = u_{xx}(t,1) = 0 \quad \text{for } 0 < t < T, \tag{0.5}$$

$$\theta_x(t,0) = \theta_x(t,1) = 0 \quad \text{for } 0 < t < T, \tag{0.6}$$

$$\sigma_x(t,0) = \sigma_x(t,1) = 0 \quad \text{for } 0 < t < T, \tag{0.7}$$

$$u(0) = u_0, u_t(0) = v_0, \theta(0) = \theta_0, \sigma(0) = \sigma_0 \quad \text{on } [0,1], \tag{0.8}$$

where $\varepsilon = u_x$, γ, μ, κ, ν and c are positive constants and u_0, v_0, θ_0 and σ_0 are initial functions. Throughout this paper we denote by (P) := (P)(u_0, v_0, θ_0, σ_0) the above system (0.2) $\sim$ (0.8).

The momentum balance law with viscosity yields (0.2) (cf. [7]). From the physical point of view it is natural to add $-\nu(u_{xt})^2$ to the left hand of the balance equation (0.3) of the internal energy, when we consider the viscosity for the stress. Our system (P) is regarded as a mathematical model for the shape memory alloys by using the hysteresis operators instead of the Landau-Devonshire form. Also, we approximate (0.1) by (0.4) so as to control mathematically σ_x in (0.2).

The plan of this paper is as follows. In section 1 we list the assumptions for data and give the definition of a solution of (P) and an existence and uniqueness theorem for problem (P). The brief proof of the theorem will be given in section 2.

We refer to the book by Brezis ([2]) for the definitions and basic properties of subdifferential operators.

1 Main result

We begin with the precise assumptions for data. Throughout this paper we assume that (A1) $f_a, f_d \in C^2(\mathbf{R} \times \mathbf{R}) \cap W^{2,\infty}(\mathbf{R} \times \mathbf{R})$ and $f_a \leq f_d$ on $\mathbf{R} \times \mathbf{R}$. Here, we put

$$L = \max\{|f_a|_{W^{2,\infty}(\mathbf{R}\times\mathbf{R})}, |f_d|_{W^{2,\infty}(\mathbf{R}\times\mathbf{R})}\}.$$

(A2) $u_0 \in H^4(0,1)$ with $u_0(0) = u_0(1) = u_{0xx}(0) = u_{0xx}(1) = 0$, $v_0 \in H_0^1(0,1)$, $\theta_0 \in H^1(0,1)$ and $\sigma_0 = 0$. Moreover, $f_a(\theta_0, \varepsilon_0) \leq \sigma_0 \leq f_d(\theta_0, \varepsilon_0)$ a.e. on $(0,1)$.

In order to apply the abstract theory of evolution equations, we recall the definition of the indicator functions. Now, for any given $\theta \in L^2(0,1)$ and $\varepsilon \in L^2(0,1)$ we denote by $I(\theta, \varepsilon; \cdot)$ the function on $L^2(0,1)$ defined by

$$I(\theta, \varepsilon; \sigma) = \begin{cases} 0 & \text{if } \sigma \in K(\theta, \varepsilon), \\ +\infty & \text{otherwise,} \end{cases}$$

where $K(\theta, \varepsilon) = \{\sigma \in L^2(0,1) : f_a(\theta, \varepsilon) \leq \sigma \leq f_d(\theta, \varepsilon) \text{ a.e. on } (0,1)\}$. Clearly, $I(\theta, \varepsilon; \cdot)$ is proper, l.s.c. and convex on $L^2(0,1)$, $D(I(\theta, \varepsilon; \cdot)) = K(\theta, \varepsilon)$, and its subdifferential $\partial I(\theta, \varepsilon; \cdot)$ is a multivalued operator in $L^2(0,1)$ which satisfies the following property: $\xi \in \partial I(\theta, \varepsilon; \sigma)$ if and only if $\sigma \in L^2(0,1)$ with $f_a(\theta, \varepsilon) \leq \sigma \leq f_d(\theta, \varepsilon)$ a.e. on $(0,1)$ and $\xi \in L^2(0,1)$ such that

$$\int_0^1 \xi(z - \sigma) dx \leq 0 \quad \text{for any } z \in K(\theta, \varepsilon); \tag{1.1}$$

this variational inequality is equivalent to

$$\begin{cases} \xi(x) \leq 0 & \text{for a.e. } x \in (0,1) \text{ with } \sigma(x) = f_d(\theta, \varepsilon), \\ \xi(x) = 0 & \text{for a.e. } x \in (0,1) \text{ with } f_a(\theta, \varepsilon) < \sigma(x) < f_d(\theta, \varepsilon), \\ \xi(x) \geq 0 & \text{for a.e. } x \in (0,1) \text{ with } \sigma(x) = f_a(\theta, \varepsilon). \end{cases}$$

By using the above notation we define a solution of (P) as follows:

Definition 1.1. We call that a triplet $\{u, \theta, \sigma\}$ of functions u, θ and σ on $Q(T)$ is a solution of (P) on $[0,T]$, if the following conditions hold.
(S1) $u \in L^\infty(0,T; H^4(0,1)) \cap W^{1,2}(0,T; H^3(0,1)) \cap W^{1,\infty}(0,T; H^2(0,1))$,
$\theta \in W^{1,2}(0,T; L^2(0,1)) \cap L^\infty(0,T; H^1(0,1))$, $\sigma \in W^{1,2}(0,T; L^2(0,1)) \cap L^\infty(0,T; H^1(0,1))$.
(S2) (0.2) and (0.3) hold for a.e. $(t,x) \in Q(T)$, there exists $\xi \in L^2(Q(T))$ such that $\xi(t) \in \partial I(\theta(t), \varepsilon(t); \sigma(t))$ for a.e. $t \in [0,T]$ and

$$\sigma_t(t) - \nu \sigma_{xx}(t) + \xi(t) = c u_{xt}(t) \quad \text{in } L^2(0,1) \text{ and for a.e. } t \in [0,T], \tag{1.2}$$

and (0.5) $\sim$ (0.8) hold.

Remark 1.1. It is easy to see that (1.2) with (0.7) holds if and only if $\sigma(t) \in K(\theta(t), \varepsilon(t))$ for a.e. $t \in [0,T]$ and

$$\int_0^1 \sigma_t(t)(\sigma(t) - z) dx + \nu \int_0^1 \sigma_x(t)(\sigma_x(t) - z_x) dx \leq c \int_0^1 \varepsilon_t(t)(\sigma(t) - z) dx$$

for any $z \in K(\theta(t), \varepsilon(t))$ and a.e. $t \in [0,T]$.

Our main result is stated as follows.
Theorem 1.1. *Assume that (A1) and (A2) hold. Then, there exists one and only one solution* $\{u, \theta, \sigma\}$ *of (P)*$(u_0, v_0, \theta_0, \sigma_0)$ *on* $[0,T]$.

In our existence proof we shall use the following approximation of the indicator function. For $\lambda > 0$ let $I_\lambda(\theta, \varepsilon; \sigma)$ be the Yosida approximation of $I(\theta, \varepsilon; \sigma)$. We have already known the certain expression of I_λ and ∂I_λ as mentioned below.

Lemma 1.1. *(cf. [8; Section 4]) For each $\lambda > 0$ it holds that*

$$I_\lambda(\theta, \varepsilon; \sigma) = \frac{1}{2\lambda}\{|[\sigma - f_d(\theta, \varepsilon)]^+|^2_{L^2(0,1)} + |[f_a(\theta, \varepsilon) - \sigma]^+|^2_{L^2(0,1)}\} \text{ for } \theta, \varepsilon, \sigma \in L^2(0,1),$$

$$\partial I_\lambda(\theta, \varepsilon; \sigma) = \frac{1}{\lambda}\{[\sigma - f_d(\theta, \varepsilon)]^+ - [f_a(\theta, \varepsilon) - \sigma]^+\} \text{ for } \theta, \varepsilon, \sigma \in L^2(0,1).$$

Later, we consider the following problem $(\mathrm{P})_\lambda$ for each $\lambda > 0$:

$$u_{tt} + \gamma u_{xxxx} - \mu u_{xxt} - \sigma_x = 0 \quad \text{in } Q(T), \tag{1.3}$$

$$\theta_t - \kappa\theta_{xx} = \sigma u_{xt} \quad \text{in } Q(T), \tag{1.4}$$

$$\sigma_t - \nu\sigma_{xx} + \partial I_\lambda(\theta, \varepsilon; \sigma) = c u_{xt} \quad \text{in } Q(T), \tag{1.5}$$

$$u(t,0) = u(t,1) = 0 \text{ and } u_{xx}(t,0) = u_{xx}(t,1) = 0 \quad \text{for } 0 < t < T, \tag{1.6}$$

$$\theta_x(t,0) = \theta_x(t,1) = 0 \quad \text{for } 0 < t < T, \tag{1.7}$$

$$\sigma_x(t,0) = \sigma_x(t,1) = 0 \quad \text{for } 0 < t < T, \tag{1.8}$$

$$u(0) = u_0, u_t(0) = v_0, \theta(0) = \theta_0, \sigma(0) = \sigma_0 \quad \text{on } [0,1]. \tag{1.9}$$

2 Proof of Theorem 1.1.

The purpose of this section is to give a brief proof of Theorem 1.1. The proof is rather long so that the complete proof will be given in authors' forthcoming paper. Throughout this section we assume (A1) and (A2), use the same notation as in the previous section.

First, we prove the uniqueness of solutions of (P). In order to show the uniqueness we provide several lemmas. We denote by $\{u_1, \theta_1, \sigma_1\}$ and $\{u_2, \theta_2, \sigma_2\}$ two solutions of $(\mathrm{P})(u_0, v_0, \theta_0, \sigma_0)$ on $[0,T]$. Here, for simplicity we put

$$M(s) = \max\{|f_a(\theta_1, \varepsilon_1) - f_a(\theta_2, \varepsilon_2)|_{L^\infty(Q(s))}, |f_d(\theta_1, \varepsilon_1) - f_d(\theta_2, \varepsilon_2)|_{L^\infty(Q(s))}\} \text{ for } 0 < s \le T.$$

Lemma 2.1. *For each $s \in (0,T]$ the following inequality holds:*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_0^1 |[\sigma_1(t) - \sigma_2(t) - M(s)]^+|^2 dx + \int_0^1 |[\sigma_2(t) - \sigma_1(t) - M(s)]^+|^2 dx\Big) \\
&+\nu\Big(\int_0^1 |[\sigma_1(t) - \sigma_2(t) - M(s)]^+_x|^2 dx + \int_0^1 |[\sigma_2(t) - \sigma_1(t) - M(s)]^+_x|^2 dx\Big) \\
&+\frac{\hat{c}}{2}\frac{d}{dt}\Big(\int_0^1 |u_{1t}(t) - u_{2t}(t)|^2 dx + \gamma\int_0^1 |u_{1xx}(t) - u_{2xx}(t)|^2 dx\Big) \\
&+\frac{\hat{c}\mu}{2}\int_0^1 |u_{1tx}(t) - u_{2tx}(t)|^2 dx \\
\le\ & \frac{\hat{c}}{2\mu}M(s)^2 \quad \textit{for a.e. } t \in [0,s],
\end{aligned} \tag{2.1}$$

where $\hat{c} = c$ *if* $c > 0$, $= 1$ *if* $c = 0$.

Proof. We can prove the lemma in a similar way to that of [8; Lemma 3.1]. So we omit its proof. □

Next, we give some estimates for $\theta_1 - \theta_2$. Before the statement, we note that for $i = 1, 2$, $u_{itx} \in L^\infty(Q(T))$ and $\sigma_i \in L^\infty(Q(T))$ because $u_i \in W^{1,\infty}(0,T;H^2(0,1))$ and $\sigma_i \in L^\infty(0,T;H^1(0,1))$.

Lemma 2.2. *There exists a positive constant* C_1 *depending only on* μ, c, κ, $|u_{1tx}|_{L^\infty(Q(T))}$ *and* $|\sigma_2|_{L^\infty(Q(T))}$ *such that*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\theta_1(t) - \theta_2(t)|^2_{L^2(0,1)} + \kappa|\theta_{1x}(t) - \theta_{2x}(t)|^2_{L^2(0,1)} \\ \le\ & C_1(|\theta_1(t) - \theta_2(t)|^2_{L^2(0,1)} + |\sigma_1(t) - \sigma_2(t)|^2_{L^2(0,1)}) + \frac{\hat{c}\mu}{4}|u_{1tx}(t) - u_{2tx}(t)|^2_{L^2(0,1)}, \quad (2.2)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\theta_{1x}(t) - \theta_{2x}(t)|^2_{L^2(0,1)} + \frac{\kappa}{2}|\theta_{1xx}(t) - \theta_{2xx}(t)|^2_{L^2(0,1)} \\ \le\ & C_1(|\sigma_1(t) - \sigma_2(t)|^2_{L^2(0,1)} + |u_{1tx}(t) - u_{2tx}(t)|^2_{L^2(0,1)}) \quad \textit{for a.e. } t \in [0,T]. \quad (2.3)\end{aligned}$$

Proof. The proof of (2.2) and (2.3) are elementary. □

For simplicity, we introduce the following notations: $u := u_1 - u_2$, $\theta := \theta_1 - \theta_2$, $\sigma := \sigma_1 - \sigma_2$,

$$\begin{aligned}E_0(t) &:= \frac{1}{2}\left(\int_0^1 |[\sigma(t) - M(s)]^+|^2 dx + \int_0^1 |[-\sigma(t) - M(s)]^+|^2 dx\right) \\ &\quad + \frac{\hat{c}}{2}\left(\int_0^1 |u_t(t)|^2 dx + \gamma \int_0^1 |u_{xx}(t)|^2 dx\right) + \frac{1}{2}\int_0^1 |\theta(t)|^2 dx,\end{aligned}$$

and

$$\begin{aligned}E_1(t) &:= \nu\left(\int_0^1 |[\sigma(t) - M(s)]^+_x|^2 dx + \int_0^1 |[-\sigma(t) - M(s)]^+_x|^2 dx\right) \\ &\quad + \frac{\hat{c}\mu}{4}\int_0^1 |u_{tx}(t)|^2 dx + \kappa\int_0^1 |\theta_x(t)|^2 dx.\end{aligned}$$

Lemma 2.3. *There exists a positive constant* C_2 *such that*

$$\sup_{0 \le t \le s} E_0(t) + \int_0^s E_1(t)dt \le C_2 s M(s)^2 \quad \textit{for } 0 \le s \le T.$$

Proof. From Lemma 2.1 and (2.2) it follows that

$$\frac{d}{dt}E_0(t) + E_1(t) \le \frac{\hat{c}}{2\mu}M(s)^2 + C_1(|\theta(t)|^2_{L^2(0,1)} + |\sigma(t)|^2_{L^2(0,1)}) \text{ for a.e. } t \in [0,s].$$

We note that

$$|\sigma| \le [\sigma - M(s)]^+ + [-\sigma - M(s)]^+ + M(s) \quad \text{on } Q(s).$$

Then, we see that

$$\int_0^1 |\sigma(t)|^2 dx \le 18E_0(t) + 9M(s)^2 \quad \text{for } t \in [0,s]. \tag{2.4}$$

Therefore, we obtain the following inequality:

$$\frac{d}{dt}E_0(t) + E_1(t) \le (\frac{\hat{c}}{2\mu} + 9C_1)M(s)^2 + 20C_1E_0(t) \quad \text{for a.e. } t \in [0,s].$$

By applying the Gronwall's inequality to the above inequality we have

$$E_0(t) + \int_0^t E_1(\tau)d\tau \le e^{20C_1t}(\frac{\hat{c}}{2\mu} + 9C_1)M(s)^2 s \quad \text{for } t \in [0,s].$$

Put $C_2 = 2e^{20C_1T}(\frac{\hat{c}}{2\mu} + 9C_1)$. Then, we have proved this lemma. □

On account of (A1) it is clear that

$$\begin{aligned} M(s) &= \max\{|f_a(\theta_1,\varepsilon_1) - f_a(\theta_2,\varepsilon_2)|_{L^\infty(Q(s))}, |f_d(\theta_1,\varepsilon_1) - f_d(\theta_2,\varepsilon_2)|_{L^\infty(Q(s))}\} \\ &\le L_1(|\theta|_{L^\infty(Q(s))} + |\varepsilon|_{L^\infty(Q(s))}), \end{aligned} \tag{2.5}$$

where $L_1 = 2L$. In order to get an estimate for $M(s)$ we give the following two lemmas.

Lemma 2.4. *There exists a positive constant C_3 depending only on T such that*

$$|\varepsilon|^2_{L^\infty(Q(s))} \le C_3 \sup_{0\le t\le s} E_0(t) \quad \textit{for } 0 \le s \le T.$$

Next, we give an estimate for $|\theta|_{L^\infty(Q(s))}$.

Lemma 2.5. *There exists a positive constant C_4 such that*

$$|\theta|^2_{L^\infty(Q(s))} \le C_4 \int_0^s (|\sigma(t)|^2_{L^2(0,1)} + |u_{tx}(t)|^2_{L^2(0,1)})dt + C_4 \sup_{0\le t\le s} E_0(t) \quad \textit{for } 0 \le s \le T.$$

Proof. By using the Gagliardo-Nirenberg inequality we infer that Lemmas 2.4 and 2.5 are true. □

Using the above lemmas we give a proof of the uniqueness of (P).

Proof of uniqueness. First, we show that there is a positive constant C_5 satisfying

$$|\sigma(t)|^2_{L^2(0,1)} \le C_5\{\sup_{0\le\tau\le s} E_0(\tau) + \int_0^s (|\sigma(\tau)|^2_{L^2(0,1)} + E_1(\tau))d\tau\} \text{ for } 0 \le t \le s \le T. \tag{2.6}$$

In fact, from (2.4), (2.5) and Lemmas 2.4 and 2.5 it follows that

$$\begin{aligned} &|\sigma(t)|^2_{L^2(0,1)}dx \\ \le\ & 18E_0(t) + 9M(s)^2 \\ \le\ & 18\{E_0(t) + L_1^2(|\theta|^2_{L^\infty(Q(s))} + |\varepsilon|^2_{L^\infty(Q(s))})\} \\ \le\ & 18\{1 + L_1^2(C_3 + C_4)\} \sup_{0\le t\le s} E_0(t) + 18L_1^2C_4 \int_0^s (|\sigma(\tau)|^2_{L^2(0,1)} + \frac{4}{\hat{c}\mu}E_1(\tau))d\tau \end{aligned}$$

for $0 \le t \le s \le T$. Hence, by putting $C_5 = 18\{1 + L_1^2(C_3 + C_4) + \frac{4}{c}\mu\}$ we get (2.6).

Next, by Lemmas 2.3 $\sim$ 2.5 and (2.5) we see that

$$\begin{aligned}
&\sup_{0\le t\le s} E_0(t) + \int_0^s E_1(\tau)d\tau \\
\le\ & 2C_2L_1^2C_3 s \sup_{0\le t\le s} E_0(t) \\
&+2C_2L_1^2C_4 s\{\int_0^s (|\sigma(t)|^2_{L^2(0,1)} + |u_{tx}(t)|^2_{L^2(0,1)})dt + \sup_{0\le t\le s} E_0(t)\} \quad \text{for } 0 \le s \le T.
\end{aligned}$$

Here, we take a number $T_1 \in (0, T]$ such that

$$2C_2L_1^2C_3T_1 + 2C_2L_1^2C_4T_1 \le \frac{1}{2}.$$

Then, we have

$$\begin{aligned}
&\frac{1}{2}\sup_{0\le t\le s} E_0(t) + \int_0^s E_1(\tau)d\tau \\
\le\ & 2C_2L_1^2C_4 s \int_0^s (|\sigma(t)|^2_{L^2(0,1)} + |u_{tx}(t)|^2_{L^2(0,1)})dt \\
\le\ & C_6 s(\int_0^s |\sigma(\tau)|^2_{L^2(0,1)}d\tau + \int_0^s E_1(\tau)d\tau) \quad \text{for } 0 \le s \le T_1,
\end{aligned}$$

where C_6 is a suitable positive constant. Now, choose a number $T_2 \in (0, T_1]$ with $C_6T_2 \le \frac{1}{2}$. Then, we get the following inequality:

$$\sup_{0\le t\le s} E_0(t) + \int_0^s E_1(\tau)d\tau \le sC_7 \int_0^s |\sigma(\tau)|^2_{L^2(0,1)}d\tau \quad \text{for } 0 \le s \le T_2, \tag{2.7}$$

where $C_7 = 2C_6$. (2.6) and (2.7) imply that

$$\begin{aligned}
A(s) \ &:= \sup_{0\le t\le s} E_0(t) + \int_0^s E_1(\tau)d\tau \\
&\le s^2C_7C_5\{A(s) + \int_0^s |\sigma(\tau)|^2_{L^2(0,1)}d\tau\} \\
&\le C_5C_7s^2A(s) + C_7C_5^2s^3\{A(s) + \int_0^s |\sigma(\tau)|^2_{L^2(0,1)}d\tau\}
\end{aligned}$$

for $0 \le s \le T_2$. Recursively, we obtain

$$\begin{aligned}
&\sup_{0\le t\le s} E_0(t) + \int_0^s E_1(\tau)d\tau \\
\le\ & C_7(C_5s^2 + C_5^2s^3 + \cdots + C_5^n s^{n+1})A(s) + C_7C_5^n s^{n+1} \int_0^s |\sigma(\tau)|^2_{L^2(0,1)}d\tau
\end{aligned}$$

for $0 \le s \le T_2$ and $n = 1, 2, \cdots$. By choosing a number $T_3 \in (0, T_2]$ satisfying

$$sC_7(C_5 s + C_5^2 s^2 + \cdots + C_5^n s^n \le \frac{1}{2} \text{ for } 0 \le s \le T_3 \text{ and each } n,$$

we infer that

$$A(s) \le 2C_7 C_5^n s^{n+1} \int_0^s |\sigma(\tau)|^2_{L^2(0,1)} d\tau \quad \text{for } 0 \le s \le T_3 \text{ and } n. \tag{2.8}$$

Finally, take a number $T_4 \in (0, T_3]$ with $C_5 T_4 \le \frac{1}{2}$. Then, letting $n \to \infty$ in (2.8) yields that

$$\sup_{0 \le t \le s} E_0(t) + \int_0^s E_1(\tau) d\tau = 0 \quad \text{for } 0 \le s \le T_4.$$

Thus we have proved the uniqueness of solutions of (P). □

Next, we prove the existence of a solution of (P). To do so we discuss the approximate problem $(P)_\lambda$ for each $\lambda > 0$, which was defined in section 1. The following lemma guarantees the existence of a solution of $(P)_\lambda$.

Lemma 2.6. *For each $\lambda > 0$ there exist $T_\lambda > 0$ and a unique solution $\{u_\lambda, \theta_\lambda, \sigma_\lambda\}$ of $(P)_\lambda$ on $[0, T_\lambda]$, that is, $u_\lambda \in L^\infty(0, T_\lambda; H^4(0,1)) \cap W^{1,2}(0, T_\lambda; H^3(0,1)) \cap W^{1,\infty}(0, T_\lambda; H^2(0,1))$, $\theta_\lambda \in W^{1,2}(0, T_\lambda; L^2(0,1)) \cap L^\infty(0, T_\lambda; H^1(0,1))$, $\sigma_\lambda \in W^{1,2}(0, T_\lambda; L^2(0,1)) \cap L^\infty(0, T_\lambda; H^1(0,1))$ satisfy (1.3) $\sim$ (1.9) with $T = T_\lambda$ in the usual sense.*

By using the Banach's fixed point theorem we can easily prove this lemma, because ∂I_λ is Lipschitz continuous. So, we omit its proof. Form now on, we give some uniform estimates for approximate solutions with respect to $\lambda \in (0, 1]$ and $t \in (0, T]$, $0 < T < \infty$.

Lemma 2.7. *Let $T > 0$ and $\hat{T}_\lambda = \min\{T, T_\lambda\}$. Then, there exists a positive constant M_1 depending only on T, L, γ, μ, c, $|u_0|_{H^2(0,1)}$, $|v_0|_{L^2(0,1)}$ and $|\sigma_0|_{L^2(0,1)}$ such that*

$$|u_{\lambda t}(t)|^2_{L^2(0,1)} + |u_{\lambda xx}(t)|^2_{L^2(0,1)} \le M_1 \quad \textit{for } t \in (0, \hat{T}_\lambda] \textit{ and } \lambda \in (0, 1],$$

$$\int_0^{\hat{T}_\lambda} |u_{\lambda \tau x}(\tau)|^2_{L^2(0,1)} d\tau \le M_1 \quad \lambda \in (0, 1],$$

$$|\sigma_\lambda(t)|^2_{L^2(0,1)} \le M_1 \quad \textit{for } t \in (0, \hat{T}_\lambda] \textit{ and } \lambda \in (0, 1].$$

Proof. Similarly to that to [8; Lemma 4.1] we can obtain the above estimates. □

The following lemma shows the uniform estimate for L^∞-norm of σ_λ by using the classical result (cf. [10]).

Lemma 2.8. *(cf. [10; Theorem 7.1, Chapter 3]) There exists a positive constant M_2 independent of $\lambda \in (0, 1]$ such that*

$$|\sigma_\lambda(t, x)| \le M_2 \quad \textit{for } (t, x) \in Q(\hat{T}_\lambda) \textit{ and } \lambda \in (0, 1].$$

Lemma 2.8 is not a direct application of [10; Theorem 7.1, Chapter 3]. But, in a similar way to that of [10; Theorem 7.1, Chapter 3] we can obtain the L^∞-estimate for σ_λ. The following lemma is easily proved thanks to Lemmas 2.7 and 2.8.

Lemma 2.9. *There exists a positive constant M_3, depending only on M_2, κ and $|u_0|_{H^1(0,1)}$, such that*

$$|\theta_\lambda|_{W^{1,2}(0,\hat{T}_\lambda;L^2(0,1))} + |\theta_\lambda|_{L^\infty(0,\hat{T}_\lambda;H^1(0,1))} \le M_3 \text{ for } \lambda \in (0,1].$$

Lemma 2.10. *There exists a positive constant M_4 such that*

$$|\sigma_{\lambda x}|_{L^2(Q(\hat{T}_\lambda))} \le M_4 \quad \text{for } \lambda \in (0,1].$$

Proof. Let $\lambda \in (0,1]$. Multiplying (1.5) by $\sigma_\lambda - f_d(\theta_\lambda, \varepsilon_\lambda)$ and integrating it over $[0,1]$ yield the conclusion of this lemma. □

Next, we give a lemma on the other uniform estimates for u_λ.

Lemma 2.11. *There exists a positive constant M_5 such that*

$$|u_{\lambda tx}(t)|^2_{L^2(0,1)} + |u_{\lambda xxx}(t)|^2_{L^2(0,1)} \le M_5 \quad \text{for } 0 \le t \le \hat{T}_\lambda \text{ and } \lambda \in (0,1],$$

$$\int_0^{\hat{T}_\lambda} |u_{\lambda txx}(t)|^2_{L^2(0,1)} dt \le M_5 \quad \text{for } \lambda \in (0,1].$$

This lemma is quite easy, so we omit the proof.

Lemma 2.12. *There exists a positive constant M_6 such that*

$$|\sigma_\lambda(t)|^2_{H^1(0,1)} \le M_6 \quad \text{for } 0 \le t \le \hat{T}_\lambda \text{ and } \lambda \in (0,1],$$

$$\int_0^{\hat{T}_\lambda} |\sigma_{\lambda t}(t)|^2_{L^2(0,1)} dt \le M_6 \quad \text{for } \lambda \in (0,1],$$

$$\int_0^{\hat{T}_\lambda} |\partial I_\lambda(\theta_\lambda(t), \varepsilon_\lambda(t); \sigma_\lambda(t))|^2_{L^2(0,1)} dt \le M_6 \quad \text{for } \lambda \in (0,1].$$

We can prove this lemma in a similar way to that of [8; Lemma 4.2]. Hence, we omit the proof. At the end of this section we show the uniform estimates for $u_{\lambda xxxx}$.

Lemma 2.13. *There exists a positive constant M_7 such that*

$$|u_{\lambda txx}(t)|^2_{L^2(0,1)} + |u_{\lambda xxxx}(t)|^2_{L^2(0,1)} \le M_7 \quad \text{for } 0 \le t \le \hat{T}_\lambda \text{ and } \lambda \in (0,1],$$

$$\int_0^{\hat{T}_\lambda} |u_{\lambda txxx}(t)|^2_{L^2(0,1)} dt \le M_7 \quad \text{for } \lambda \in (0,1].$$

By Lemma 2.12 it is easy to prove Lemma 2.13. Next, we show a global existence of approximate solutions.

Lemma 2.14. *Let $T > 0$ and $\lambda \in (0,1]$. Then, (P)$_\lambda$ admits a unique solution on $[0,T]$.*

Proof. Let $\lambda \in (0,1]$, $T > 0$ and $[0, T_\lambda)$ be a maximal interval of existence of a solution of (P)$_\lambda$. We assume that $T_\lambda < T$. Lemma 2.6 with the help of Lemmas 2.7 $\sim$ 2.13 implies that we can extend the solution beyond T_λ. This is a contradiction. □

Finally, we give a proof of the existence of a solution of (P). The most important part of the proof is quite similar to that of [8; section 5].

Proof of existence. Let $\lambda \in (0,1]$, $T > 0$ and $\{u_\lambda, \theta_\lambda, \sigma_\lambda\}$ be a solution of $(\mathrm{P})_\lambda$ on $[0,T]$ because of Lemma 2.6. The argument of the previous section implies that there are a subsequence $\{\lambda_j\}$ of $\{\lambda\}$, and functions u, θ, σ and ξ on $Q(T)$ such that

$$\begin{aligned} u_j := u_{\lambda_j} \to u \quad & \text{weakly* in } L^\infty(0,T;H^4(0,1)),\\ & \text{weakly in } W^{1,2}(0,T;H^3(0,1)),\\ & \text{weakly* in } W^{1,\infty}(0,T;H^2(0,1)),\end{aligned}$$

$$\begin{aligned} \sigma_j := \sigma_{\lambda_j} \to \sigma \quad & \text{weakly in } W^{1,2}(0,T;L^2(0,1)),\\ & \text{weakly* in } L^\infty(0,T;H^1(0,1)),\\ & \text{in } C(\overline{Q(T)}),\end{aligned}$$

$$\begin{aligned} \theta_j := \theta_{\lambda_j} \to \theta \quad & \text{weakly in } W^{1,2}(0,T;L^2(0,1)),\\ & \text{weakly* in } L^\infty(0,T;H^1(0,1)),\end{aligned}$$

$$\xi_j := \partial I_{\lambda_j}(\theta_j, \varepsilon_j; \sigma_j) \to \xi \quad \text{weakly in } L^2(Q(T)) \text{ as } j \to \infty,$$

where $\varepsilon_j = u_{jx}$. The above convergences guarantee that $\{u, \theta, \sigma\}$ satisfy (S1) and (S2) except for (1.2). Hence, by Remark 1.1 it is sufficient to show that $f_a(\theta,\varepsilon) \le \sigma \le f_d(\theta,\varepsilon)$ a.e. on $Q(T)$ where $\varepsilon = u_x$ and (1.1). On account of Lemma 1.1 we have

$$[\sigma_j - f_d(\theta_j,\varepsilon_j)]^+ - [f_a(\theta_j,\varepsilon_j) - \sigma_j]^+ = \lambda_j \xi_j \to 0 \text{ in } L^2(Q(T)) \text{ as } j \to \infty.$$

This convergence yields that $f_a(\theta,\varepsilon) \le \sigma \le f_d(\theta,\varepsilon)$ a.e. on $Q(T)$.

Next, let z be any function in $L^2(Q(T))$ satisfying $f_a(\theta,\varepsilon) \le z \le f_d(\theta,\varepsilon)$ a.e. on $Q(T)$ and put

$$z_j = \max\{\min\{f_d(\theta_j,\varepsilon_j), z\}, f_a(\theta_j,\varepsilon_j)\}.$$

It is clear that $f_a(\theta_j,\varepsilon_j) \le z_j \le f_d(\theta_j,\varepsilon_j)$ a.e. on $Q(T)$ and $z_j \to z$ in $L^2(Q(T))$ as $j \to \infty$. Accordingly,

$$\int_{Q(T)} \xi_{\lambda_j}(z_j - \sigma_j)\,dxdt \to \int_{Q(T)} \xi(z - \sigma)\,dxdt \quad \text{as } j \to \infty.$$

On the other hand,

$$\int_{Q(T)} \xi_{\lambda_j}(z_j - \sigma_j)\,dxdt \le 0 \quad \text{for each } j.$$

Therefore, we obtain (1.1). □

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