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Three phase boundary motion by surface diffusion

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1 Introduction

We study three phase boundary motion by surface diffusion in $\Omega\subset\boldsymbol{R}^{2}$, where Ω is a bounded domain which will be specified below when the global-in-time motions are to be investigated. The purpose of this note is to exhibit our previous results in [4, 5] and is also to announce our recent results in [6, 7] concerning the topics on global existence results and self-intersection.

Let Ω be a bounded domain in $\boldsymbol{R}^{2}$ which has a (possibly piecewise) smooth boundary $\partial\Omega$. We consider the situation that a binary or a ternary nonequilibrium alloy system is contained in Ω and three different phases of the alloy system are separated by three evolving phase boundaries $\Gamma^{i}(t)$, $i=1,2,3$, depending on time $t\geq 0$. Moreover one end points of $\Gamma^{i}(t)$ are connected at a triple junction $m(t)$ with a prescribed angle condition and other end points $b^{i}(t)$ are on $\partial\Omega$ where $\Gamma^{i}(t)$ and $\partial\Omega$ intersect perpendicularly, see Figure 1.

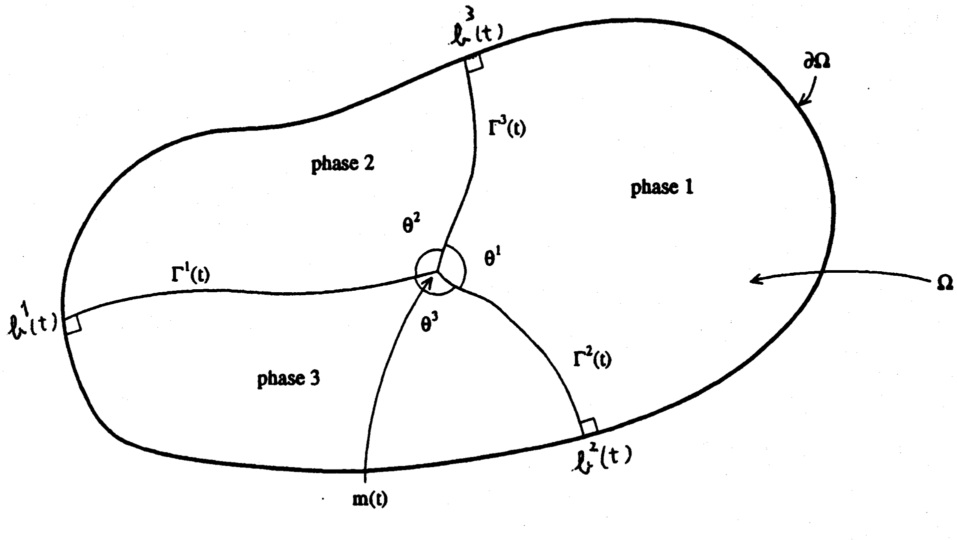

Figure 1

In this note we consider two types of mathematical models describing evolutions of three phase boundaries.

The first model to be considered is of the form for $i = 1, 2, 3$:

$$\left.\begin{array}{rl} \text{for } t > 0, & \\ \text{along } \Gamma^i(t): & V^i = -l^i\sigma^i\kappa^i_{ss} \text{ (surface diffusion flow equations)}, \\ \text{at } b^i(t): & \Gamma^i(t) \perp \partial\Omega, \\ & \kappa^i_s = 0 \text{ (no flux condition)}, \\ \text{at } m(t): & \angle(\Gamma^i(t), \Gamma^j(t)) = \theta^k \quad \text{for } i, j, k \in \{1, 2, 3\} \text{ different mutually}, \\ & \sigma^1\kappa^1 + \sigma^2\kappa^2 + \sigma^3\kappa^3 = 0 \text{ (continuity of the chemical potential)}, \\ & l^1\sigma^1\kappa^1_s = l^2\sigma^2\kappa^2_s = l^3\sigma^3\kappa^3_s \text{ (balance of fluxes)}, \\ \text{at } t = 0: & \Gamma^i(0) = \Gamma^i_0. \end{array}\right\} \tag{1}$$

Here V^i, κ^i, and s are the normal velocity, the curvature, and the arclength parameter of $\Gamma^i(t)$, respectively. The convention with respect to the direction used here is as follows: s runs from $m(t)$ to $b^i(t)$ along $\Gamma^i(t)$; V^i and κ^i are computed in the direction of the unit normal vector N^i of $\Gamma^i(t)$, where N^i is obtained by rotating the unit tangent vector T^i of $\Gamma^i(t)$ with $+\pi/2$. Moreover, l^i, σ^i, and θ^i $(i = 1, 2, 3)$ are positive constants with the constraints $0 < \theta^i < \pi$, $\theta^1 + \theta^2 + \theta^3 = 2\pi$ and

$$\frac{\sigma^1}{\sin\theta^1} = \frac{\sigma^2}{\sin\theta^2} = \frac{\sigma^3}{\sin\theta^3} : \text{ Young's law.} \tag{2}$$

The model (1) was derived by H. Garcke and A. Novick-Cohen [2] to describe evolutions of three interphase boundaries in a ternary alloy system.

The second model we are going to study is the following:

$$\left.\begin{array}{rl} \text{for } t > 0, & \\ \text{along } \Gamma^1(t): & V^1 = l^1\sigma^1\kappa^1 \text{ (curvature flow equation)}, \\ \text{along } \Gamma^i(t): & V^i = -l^i\sigma^i\kappa^i_{ss}, \quad i = 2, 3, \text{ (surface diffusion flow equations)}, \\ \text{at } b^i(t): & \Gamma^i(t) \perp \partial\Omega, \quad i = 1, 2, 3, \\ & \kappa^i_s = 0, \quad i = 2, 3, \text{ (no flux condition)}, \\ \text{at } m(t): & \angle(\Gamma^i(t), \Gamma^j(t)) = \theta^k \quad \text{for } i, j, k \in \{1, 2, 3\} \text{ different mutually}, \\ & \sigma^2\kappa^2 + \sigma^3\kappa^3 = 0 \text{ (continuity of the chemical potential)}, \\ & l^2\sigma^2\kappa^2_s = l^3\sigma^3\kappa^3_s \text{ (balance of fluxes)}, \\ \text{at } t = 0: & \Gamma^i(0) = \Gamma^i_0, \ i = 1, 2, 3. \end{array}\right\} \tag{3}$$

As in (1), l^i, σ^i and θ^i $(i = 1, 2, 3)$ are positive constants with $0 < \theta^i < \pi$, $\theta^1 + \theta^2 + \theta^3 = 2\pi$ and in the constraint (2). This model was derived by A. Novick-Cohen [10] to describe evolutions of one antiphase boundary and two interphase boundaries in a binary alloy system.

Now we set $\Gamma(t) = \bigcup_{i=1}^3 \Gamma^i(t)$ and $\Gamma_0 = \bigcup_{i=1}^3 \Gamma^i_0$. Then the problem for (1) ((3)) is stated as follows: given initial data Γ_0, find an unknown evolving phase boundary $\{\Gamma(t)\}_{t\geq 0}$ that solves (1) ((3)).

Our purpose here is to study two kinds of topics for (1) and (3): (A) a global-in-time solvability, and (B) self-intersection in a short time. As is expected, the configuration of the solution of (1) ((3)) as $t \to \infty$ may strongly depend on the configuration of $\partial\Omega$. We study (A) when Ω is a rectangular domain or a triangular domain; both cases will be studied for (1), and the only former case will be studied for (3). We investigate (A) for a class of initial data which are close to an equilibrium state. On the other hand, for (B) we consider a class of initial data which are far from an equilibrium state. Here we say that $\Gamma_e := \bigcup_{i=1}^{3} \Gamma_e^i$ is *equilibrium* (or *stationary*) for (1) ((3)) if Γ_e solves (1) ((3)) without initial condition and satisfies that $V^i \equiv 0$ on Γ_e^i for $t \geq 0$ and $i = 1, 2, 3$. It is easy to see that the equilibrium solution Γ_e for (1) consists of line segments or circular arcs, or mixture of them, and the equilibrium solution $\Gamma_e = \bigcup_{i=1}^{3} \Gamma_e^i$ for (3) is of the form $\Gamma_e^1 =$ line segment and $\Gamma_e^i =$ line segment or circular arc ($i = 2, 3$). We check this fact for (1) (the check for (3) can be done in a similar way). We denote by κ_e^i the curvature of Γ_e^i. From the definition of equilibrium it follows that $\kappa_{e,ss}^i = 0$ on Γ_e^i, which implies that $\kappa_e^i = a^i s + b^i$ with some constants a^i and b^i. Then, due to no flux condition at $\partial\Omega$ in (1), a^i, $i = 1, 2, 3$, must vanish. Thus we have $\kappa_e^i = b^i$, as desired.

When Ω is rectangular or triangular, we shall show that if initial data Γ_0 is close to an equilibrium state in some sense, then (1) ((3)) admits a unique global solution that converges to the equilibrium state as $t \to \infty$. We shall also show that if Γ_0 is far from equilibrium, then the solution of (1) ((3)) develops a self-intersection in finite time. The main difficulty in the proof of the global existence result for (1) in triangular Ω lies in the establishment of a priori estimates for both the $C^{2+\alpha}$-norm of the solution with some $\alpha \in (0, 1)$ and the distance between the triple junction and $\partial\Omega$. This can be overcome by finding a "vanishing property" for both curvatures and normal velocities, which strongly reflects both the peculiarity of the motion by (1) and the configuration of $\partial\Omega$.

In the remainder of this note we proceed as follows. In Section 2 we state our results on global existence and self-intersection in a precise manner. In Section 3 we then explain known results and fundamental properties for (1) and (3) which play an important role throughout the proof of our results. Finally, in Section 4 we present a brief outline of the proof only for our global existence results for (1) when Ω is triangular.

2 Main results

We first state our results when Γ_0 is close to an equilibrium state. For this purpose we pay our special attention to two types of domains $\Omega_\star$ and Ω_*; here $\Omega_\star$ is the rectangular domain defined by

$$\Omega_\star := \{(x, y); -a < x < 0, \ -b < y < b\}$$

with positive constants a and b; Ω_* is the triangular domain with three vertices p^i $(i = 1,2,3)$ being located counterclockwise whose interior angles are $\pi - \theta^i$ $(i = 1,2,3)$. We then consider two special equilibrium states. The first one is Γ_* which is contained in rectangular Ω_*, where $\Gamma_* = \bigcup_{i=1}^{3} \Gamma_*^i$ is the union of one line segment Γ_*^1 and two circular arcs Γ_*^i $(i = 2,3)$ in rectangular Ω_*. Moreover, Γ_* is assumed to be mirror symmetric, which means that Γ_*^2 and Γ_*^3 are symmetric with respect to the x-axis and Γ_*^1 is a line segment on the x-axis, and Γ_* is also assumed to have the area enclosed by the x, y-axes and Γ_*^3 which is the same as the one enclosed by the x, y-axes and Γ_0^3, see Figure 2. The second equilibrium state is Γ_* which is contained in triangular Ω_*, where $\Gamma_* = \bigcup_{i=1}^{3} \Gamma_*^i$ is the union of three line segments Γ_*^i $(i = 1,2,3)$ in triangular Ω_*, see Figure 3.

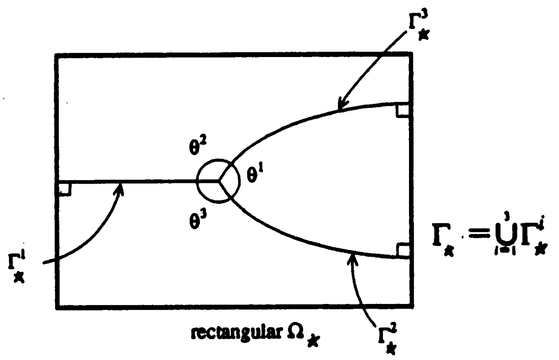

Figure 2

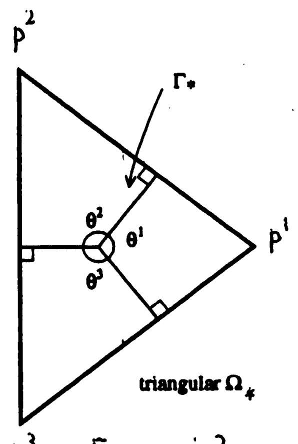

Figure 3

Then our global existence and stability results are stated as follows.

Theorem 2.1 *(Global existence and stability results). (i) (Rectangular Ω [5]). Let $\Omega = \Omega_*$ and let $l^2 = l^3$, $\sigma^2 = \sigma^3 =: \sigma$ and $\theta^2 = \theta^3$ (in this case, $\sigma^1 = 2\sigma\cos(\theta^1/2)$ due to Young's law (2)). Assume that Γ_0 belongs to C^3 with compatibility conditions for (1) ((3)) and have a mirror symmetry in the sense that Γ_0^2 and Γ_0^3 are symmetric with respect to the x-axis and Γ_0^1 is a line segment on the x-axis. Assume also that Γ_0 is close to a stationary solution Γ_* in the sense that*

$$\rho_0^2 := \|\kappa_{0,s}^3\|_{L^2(\Gamma_0^3)}^2 + C_0(E[\Gamma_0] - E[\Gamma_*])$$

is sufficiently small. Here C_0 is a constant depending on the area enclosed by the x, y-axes and Γ_0^3, and

$$E[\Gamma_0] := 2\{(a - \xi_0)\sigma\cos(\theta^1/2) + (\textit{the length of } \Gamma_0^3)\},$$
$$E[\Gamma_*] := 2\{(a - \xi_*)\sigma\cos(\theta^1/2) + (\textit{the length of } \Gamma_*^3)\},$$

where $-\xi_0$ and $-\xi_*$ are the x-coordinates of the triple junctions of Γ_0 and Γ_*, respectively. Then the model (1) ((3)) admits a unique smooth global-in-time solution $\{\Gamma(t)\}_{t\geq 0}$ which always stays mirror symmetric as in Γ_0. Moreover the solution $\{\Gamma(t)\}_{t\geq 0}$ converges to Γ_* uniformly as $t \to \infty$.

(ii) (Triangular Ω [6]). Let $\Omega = \Omega_*$ and let $l^1\sigma^1 = l^2\sigma^2 = l^3\sigma^3$. Assume that Γ_0 belongs to C^3 with compatibility conditions for (1) and fulfills the area condition:

$$\mu(D^i[\Gamma_0, \partial\Omega]) = \mu(D^i[\Gamma_*, \partial\Omega]), \quad i = 1, 2, 3,$$

where $D^i[\Gamma_0, \partial\Omega]$ denotes the domain enclosed by Γ_0^j, Γ_0^k, and $\partial\Omega$ without containing Γ_0^i (i, j, k: different mutually) and $\mu(D)$ denotes the area of domain D. Assume also that Γ_0 is close to a stationary solution Γ_* in the sense that

$$K_0^2 := \sum_{i=1}^{3}\left(\sigma^i \|\kappa_0^i\|^2_{L^2(\Gamma_0^i)} + l^i(\sigma^i)^2 \|\kappa_{0,s}^i\|^2_{L^2(\Gamma_0^i)}\right)$$

is sufficiently small. Then the model (1) admits a unique smooth global-in-time solution $\{\Gamma(t)\}_{t\geq 0}$. Moreover the solution $\{\Gamma(t)\}_{t\geq 0}$ converges to Γ_* uniformly as $t \to \infty$.

Remark 2.2 In Theorem 2.1 (i), due to the symmetry to be required, we should seek the solution satisfying that for $t \geq 0$, $m(t)$ always stays on the x-axis, $\kappa^1 \equiv 0$ on $\Gamma^1(t)$, and $\kappa^2(t, s) = -\kappa^3(t, s)$ for the common arclength parameter s of $\Gamma^2(t)$ and $\Gamma^3(t)$. In this case both (1) and (3) are reduced to the common problem only for $\Gamma^3(t)$:

$$\left.\begin{array}{rl}
\text{for } t > 0, & \\
\text{along } \Gamma^3(t): & V^3 = -l^3\sigma^3\kappa^3_{ss} \text{ (surface diffusion flow equation)}, \\
\text{at } b^3(t): & \Gamma^3(t) \perp \partial\Omega, \\
 & \kappa^3_s = 0 \text{ (no flux condition)}, \\
\text{at } m(t): & \angle(\Gamma^3(t), \text{the } x\text{-axis}) = \theta^1/2, \\
 & \kappa^3_s = 0 \text{ (balance of flux)}, \\
\text{at } t = 0: & \Gamma^3(0) = \Gamma^3_0.
\end{array}\right\} \qquad (4)$$

Hence (1) and (3) become the common problem (4) to find a single evolving curve $\Gamma^3(t)$ with two free end points.

We will show a brief outline of the proof of Theorem 2.1 (ii) in Section 4; for its details, see [6]. For the proof for the remaining part of Theorem 2.1, see [5].

Let us next state our result for the case that the initial data Γ_0 are far from equilibrium state. For this purpose we only treat the case $\Omega = \Omega_*$ and we consider initial data Γ_0 contained in Ω_* with the configuration as in Figure 4.

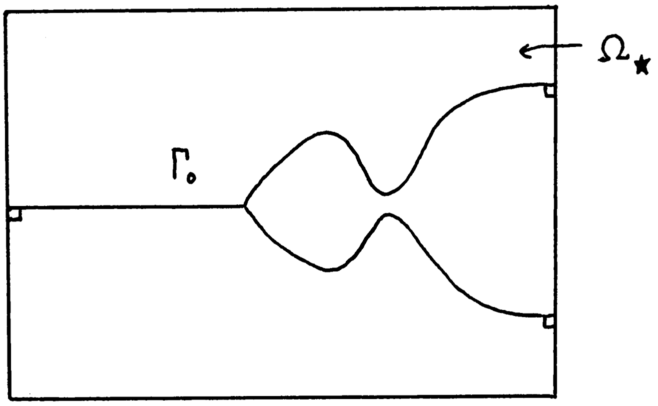

Figure 4

Then we have the following interesting motion of the solution of (1) ((3)).

Theorem 2.3 *(Self-intersection [7]). There is an initial data* $\Gamma_0 \in C^{2+\alpha}$ *(*$\alpha \in (0,1)$*) with the compatibility conditions for (1) as in Figure 4 such that the solution* $\{\Gamma(t)\}_{t\in[0,T]}$ *with a* $T > 0$ *of (1) ((3)) starting from* Γ_0 *develops a self-intersection in finite time smaller than* T*, although it stays smooth.*

Theorem 2.3 shows that the motions by (1) and (3) do not preserve the order of the interfaces in general. This strongly contrasts with the motion by curvature (see [11, Lemma 3] as a corresponding situation). The formation of self-intersections in two phase case is already known (see [3, 9])

As is pointed out in Remark 2.2, our task in the proof of Theorem 2.3 is reduced to investigate only the motion of $\Gamma^3(t)$ by assuming the mirror symmetry of the motion of $\Gamma(t)$ with respect to the x-axis. Then the main part of the proof is devoted to show that for a curve Γ_0^3 in $\Omega_\star$ away from the x-axis except one end point m_0, the solution $\{\Gamma^3(t)\}_{t\in[0,T]}$ starting from Γ_0^3 intersects the x-axis at a point on the x-axis between $m(t_1)$ and the origin $(0,0)$ in a finite time $t_1 \in (0,T]$. The precise description of its proof will appear in [7].

Remark 2.4 We here make a simple observation concerning the motions by (1) and (3) around the triple junction $m(t)$. For this purpose, we first note that Young's law (2) is equivalent to the condition

$$\sum_{i=1}^{3} \sigma^i N^i = 0 \quad \text{at } m(t), \tag{5}$$

where N^i are the unit normal vectors of $\Gamma^i(t)$ at $m(t)$ for $i = 1, 2, 3$.

In fact, we observe the equalities due to the angle condition at $m(t)$ in (1) ((3)):

$$\Big(\sum_{i=1}^{3} \sigma^i N^i, T^1\Big) = -\sigma^2 \sin\theta^3 + \sigma^3 \sin\theta^2,$$
$$\Big(\sum_{i=1}^{3} \sigma^i N^i, T^2\Big) = \sigma^1 \sin\theta^3 - \sigma^3 \sin\theta^1,$$

where $(\cdot,\cdot)$ denotes the usual inner product in $\boldsymbol{R}^2$ and T^i are the unit tangent vectors of $\Gamma^i(t)$ at $m(t)$ for $i = 1, 2, 3$. So, if we suppose (2), then $\sum_{i=1}^{3} \sigma^i N^i$ vanishes because $\{T^1, T^2\}$ is a basis in $\boldsymbol{R}^2$ and $(\sum_{i=1}^{3} \sigma^i N^i, T^j)$, $j = 1, 2$, are covariant components of $\sum_{i=1}^{3} \sigma^i N^i$ with respect to the basis $\{T^1, T^2\}$. Conversely, if we suppose (5), then we immediately obtain (2).

Now the inner product of (5) and $dm(t)/dt$ yields

$$\sum_{i=1}^{3} \sigma^i V^i = 0 \quad \text{at } m(t). \tag{6}$$

This gives us a useful information that (1) and (3) have no rotating solutions around $m(t)$ such as a spiral travelling wave solution if $m(t)$ moves. In fact, if such a solution exists, then three V^i at $m(t)$ may have the same sign, but it contradicts to (6). We emphasize that this remark is also valid for the three phase boundary motion by curvatures presented by L. Bronsard and F. Reitich [1].

3 Known results

We here summarize known fundamental results concerning (1) and (3) because they are the key to the proof of our results.

The sharp interface model (1) was first derived by H. Garcke and A. Novick-Cohen [2] via a formal singular limit from a degenerate Cahn-Hilliard system which is a model to describe diffuse interfaces in a ternary (or possibly multi-component) alloy system. H. Garcke and A. Novick-Cohen [2] also established a unique local-in-time existence result for (1) by applying solvability theory for initial boundary value problems of the system of linear parabolic equations due to V. A. Solonnikov [12]. Their result requires $C^{4+\alpha}$-regularity ($0 < \alpha < 1$) for initial data Γ_0. Another their significant contribution is to establish the two fundamental properties for the motion by (1), i.e., the energy-decreasing property and the area-preserving property. Here the energy associated with the model (1) is defined by

$$E[\Gamma(t)] := \sum_{i=1}^{3} \sigma^i L[\Gamma^i(t)], \tag{7}$$

where, for curve γ, $L[\gamma]$ denotes the length of γ. Then the energy-decreasing property for (1) is stated as

$$\frac{d}{dt}E[\Gamma(t)] = -\sum_{i=1}^{3} l^i(\sigma^i)^2 \int_0^{L[\Gamma^i(t)]} (\kappa_s^i)^2 ds \leq 0, \quad t \geq 0. \tag{8}$$

The area-preserving property for (1) is described as

$$\mu(D^i[\Gamma(t), \partial\Omega]) = \mu(D^i[\Gamma_0, \partial\Omega]), \quad t \geq 0, \ i = 1, 2, 3, \tag{9}$$

where $D^i[\Gamma(t), \partial\Omega]$ and μ are as in Theorem 2.1 (ii).

From the above we see that the flow by (1) is a gradient flow of the energy E. Hence, as conjectured in [2], it is expected that (1) admits a unique global solution that converges to a minimizer Γ_M of E subject to the area-constraint as (9) provided the initial data Γ_0 is close to the minimizer Γ_M and fulfills the area condition

$$\mu(D^i[\Gamma_0, \partial\Omega]) = \mu(D^i[\Gamma_M, \partial\Omega]), \quad i = 1, 2, 3.$$

Our Theorem 2.1 (i) and (ii) for (1) can be viewed as an answer for this conjecture.

The sharp interface model (3) was first derived by A. Novick-Cohen [10] via a formal singular limit from an Allen-Cahn/Cahn-Hilliard system which is a model to describe diffuse interfaces in a binary alloy system. In this case the energy associated with (3) is also defined by (7). She showed the energy-decreasing property for (3):

$$\frac{d}{dt}E[\Gamma(t)] = -l^1(\sigma^1)^2 \int_0^{L[\Gamma^1(t)]} (\kappa^1)^2 ds - \sum_{i=2}^{3} l^i(\sigma^i)^2 \int_0^{L[\Gamma^i(t)]} (\kappa_s^i)^2 ds \leq 0 \quad t \geq 0;$$

and the area-preserving property:

$$\mu(D^1[\Gamma(t), \partial\Omega]) = \mu(D^1[\Gamma_0, \partial\Omega]), \quad t \geq 0. \tag{10}$$

This means that (3) is a gradient flow of E and it can be expected that (3) admits a unique global solution that converges to a minimizer Γ'_M of E subject to the area-constraint as (10) provided the initial data Γ_0 is close to the minimizer Γ'_M and fulfills the area condition

$$\mu(D^1[\Gamma_0, \partial\Omega]) = \mu(D^1[\Gamma'_M, \partial\Omega]).$$

Our Theorem 2.1 (i) for (3) can be viewed as an answer for this conjecture.

Remark 3.1 E in Theorem 2.1 (i) is the same as (7) as is seen by Young's law (2).

4 Outline of the proof of Theorem 2.1 (ii)

We here briefly explain an outline of the proof of Theorem 2.1 (ii). The details of the most part of the proof are shown in [6] (the proof in the step (b) below will appear in [7]).

The proof proceeds via four steps.

Outline of the proof of Theorem 2.1 (ii). (a) *Existence of a unique minimizer of E subject to area-constraint.* The first step is to investigate the energy-minimizing problem subject to prescribed area-constraint which can be viewed as a stationary version of the time-dependent problem (1). More precisely, we consider the problem:

$$\text{minimize } E[\Gamma] \text{ for } \Gamma \in C^1_*.$$

Here C^1_* is the set of union of three C^1-curves $\Gamma = \bigcup_{i=1}^3 \Gamma^i$ contained in Ω_* such that one end point of each Γ^i is connected at a triple junction and its another end point intersects with $\partial\Omega_*$ between p^j and p^k perpendicularly (i, j, k: different mutually) and also satisfies the area-constraint

$$\mu(D^i[\Gamma, \partial\Omega_*]) = \mu(D^i[\Gamma_*, \partial\Omega_*]), \quad i = 1, 2, 3. \tag{11}$$

(The notations μ, D^i, Γ_* are explained in Section 2.) It is then proved by a direct method that the union of three line segments $\Gamma_* \in C^1_*$ is the unique global minimizer of E on C^1_*. Γ_* is of course an equilibrium solution of (1).

(b) *Local existence.* The second step is to establish a unique local existence result for less regular initial data than the one treated by Garcke and Novick-Cohen [2]. More precisely, we construct a unique local solution $\{\Gamma(t)\}_{t\in[0,T_0]}$ with an existence time $T_0 > 0$ which depends on both $C^{2+\alpha}$-norm of the initial data Γ_0 with $\alpha \in (0,1)$ and the distance between the triple junction of Γ_0 and $\partial\Omega_*$. We emphasize that this result is also valid even for general bounded domain Ω having piecewise smooth boundary $\partial\Omega$. This result can be proved by using an optimal regularity result in the theory of analytic semigroup as in [8]. This work is needed when we try to extend the local solution to the global one because $C^{2+\alpha}$-norm of the solution is seen to be a priori bounded, which is explained in the next step.

(c) *Global existence.* In the third step we establish both an a priori upper bound of the H^1-norm of the curvature of the solution $\{\Gamma(t)\}_{t\geq 0}$ and an a priori lower bound of the distance between its triple junction $m(t)$ and $\partial\Omega_*$ when the assumption of Theorem 2.1 (ii) is fulfilled. The main idea is in the "vanishing property" of both the curvatures and the normal velocities of the solution. This means that for any time $t > 0$, one of the curvatures κ^i and one of the normal velocities V^i ($i = 1, 2, 3$) must vanish somewhere on the solution curve. The advantage of this property is to yield a Poincaré-type inequality for the derivatives of the curvatures of the solution up to second order. Then it can be shown that this Poincaré-type inequality yields the desired a priori bound of the curvature in H^1 provided the assumption of Theorem 2.1 (ii) is fulfilled. Consequently, we can obtain both an a priori upper bound of the $C^{2+\alpha}$-norm of the solution (with $\alpha \in (0, 1/2])$ and an a priori lower bound of the distance between its triple junction and $\partial\Omega_*$. Thus we can extend the unique local solution of (1) obtained in (b) to the one globally in time.

(d) *Stability of the minimizer.* In the final step we show the convergence of the solution toward the minimizer Γ_* of the energy E subject to (11) as time goes to infinity. This can be done by introducing an auxiliary three line segments $\bar{\Gamma}(t) = \bigcup_{i=1}^3 \bar{\Gamma}^i(t)$ in Ω_*; $\bar{\Gamma}^i(t)$, $i = 1, 2, 3$, are connected at the same triple junction as that of the solution $\Gamma(t)$ and each $\bar{\Gamma}^i(t)$ intersects with $\partial\Omega_*$ between p^j and p^k perpendicularly at another end point (as usual, i, j, k are mutually different). We prove with the aid of the energy-decreasing property (8) that the asymptotic profile of the solution $\Gamma(t)$ as $t \to \infty$ is described by $\bar{\Gamma}(t)$. In addition, due to the area-preserving property (9), we can eventually show that $\bar{\Gamma}(t)$ must converge to Γ_* as $t \to \infty$. From these facts we conclude that $\Gamma(t)$ converges to Γ_* as $t \to \infty$.

This is the brief explanation of the proof for Theorem 2.1 (ii).

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