数理解析研究所講究録 1210 巻 2001 年 54-67

The Curve Shortening Flow –
The Classical Approach

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For a given smooth curve γ, we denote by $\mathbf{n}$ its unit normal (it is the inner unit normal when γ is closed), and k its curvature with respect to $\mathbf{n}$. The initial value problem for the *curve shortening flow* (*CSF*) is

$$\begin{cases} \dfrac{\partial\gamma}{\partial t}(\cdot,t) = k(\cdot,t)\mathbf{n}(\cdot,t)\ , \\ \gamma(\cdot,0) = \gamma_0 \quad \text{given}\ . \end{cases} \tag{1}$$

The CSF is the negative L^2-gradient flow of the length of the curve. It has been studied rigorously and extensively in the past two decades. On one hand, it may be viewed as the simplest curvature flow in geometry. On the other hand, it is related to various models for interfacial motions in phase transition and wave fronts in excitable media. In this talk we describe some of our, as well as others' results on (1) and its ramifications. We shall touch upon the following four topics:

- The CSF
- The anisotropic generalized CSF
- The CSF in Klein geometry
- Integrable flows in Klein geometry

We shall stay in the realm of the classical theory of (1). For the very fruitful theory of viscosity solutions of the mean curvature flow, which contains (1) as a special case, the reader should consult Chen-Giga-Goto [13] and Evans-Spruck [23].

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§1 The CSF: Basic Properties

We point out three basic properties of (1).

First, (1) is of parabolic nature. To see this, let's fix a smooth curve Γ near γ_0 and represent $\gamma(\cdot, t)$ as $\Gamma(p) + d(p,t)\mathbf{n}_\Gamma(p)$, where p parametrizes Γ and $\mathbf{n}_\Gamma$ is the unit normal of Γ. Then (1), which is a system of two equations, can be shown to be equivalent to a single parabolic equation for d (see, e.g., Chou-Zhu [17]),

$$\frac{\partial d}{\partial t} = \frac{(1-\kappa d)}{[(1-\kappa d)^2 + d_p^2]} d_{pp} + \text{ lower order terms },$$

where κ is the curvature of Γ (with respect to $\mathbf{n}_\Gamma$). It follows from a standard argument that (1) admits a unique solution, and it persists as long as its curvature is bounded. When the flow $\gamma(\cdot, t)$ is representable as a family of local graphs $\{(x, u(x,t))\}$, (1) is equivalent to a simple parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{u_{xx}}{1+u_x^2} \\ &= (\tan^{-1} u_x)_x \ . \end{aligned} \tag{2}$$

From standard parabolic regularity theory, a curvature bound follows from a bound on the gradient of u. Together with the geometric nature of (1), which, in particular, means this equation is satisfied in any cartesian coordinates, one can use powerful tools such as the Sturm Oscillation Theorem and the construction of foliations (Angenent [8], [9] and Chou-Zhu [19]) to control the curvature of the flow.

Second, we note that many geometric quantities are monotonic decreasing along the flow or the flow obtained by recaling (1) so that the enclosed area is always constant ("normalization"). These include the length, the area, the total absolute curvature, the number of inflection points, the entropy, the isoperimetric ratio (when γ is convex), ... and so on. These monotonicity properties play crucial roles in the study of the asymptotic behavior and the singularities of (1). Their presence makes (1) very special among parabolic equations.

Third, Euclidean invariance of (1) leads to the existence of special invariant solutions such as traveling waves, spirals and expanding/contracting self-similar solutions. The traveling waves (*Grim Reapers*) and the closed contracting self-similar solutions (*Abresch-Langer curves*) are described as follows. In suitable coordinates, the Grim Reaper with speed c is the graph of

$$u(x,t) = \frac{1}{c} \log \sec cx + ct \; , \quad x \in (-\pi/2c, \pi/2c) \; , \; t \in \mathbb{R} \; .$$

The Abresch-Langer curves are of the form $(-t)^{1/2}\gamma(\cdot)$, $t < 0$. It can be shown that circles are the only embedded self-similar solutions. For immersed ones, Abresch and Langer [1] show that, for any given positive, relatively prime integers n and m, satisfying $n/m \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$, there exists a unique (up to homothety) closed, contracting self-similar solutions with m leaves and total curvature $2n\pi$.

§2 The CSF: Results

First of all, the following two results characterize (1) for embedded, closed initial curves.

Theorem 1 (Gage-Hamilton [25]). *Let γ_0 be a closed, convex curve. Then (1) preserves convexity and shrinks to a point smoothly as $t \uparrow \omega$, where ω is equal to the initial enclosed area divided by 2π. After normalizing (1) so that it encloses constant area for all time, the normalized flow converges to a circle smoothly and exponentially.*

Theorem 2 (Grayson [28]). *The CSF turns any closed, embedded curve into a convex curve before it shrinks to a point.*

Recently, we have the following result.

Theorem 3 (Chou-Zhu [18]). *Let γ_0 be a complete, embedded curve which divides the plane into two regions of infinite area. Then (1) has a solution for all time* ≥ 0.

For earlier results in this direction, see Ecker-Huisken [22] and Huisken [33].

A question, under the assumption on γ_0 as stated in Theorem 3, is: Is the solution unique? The answer is yes if the two ends of γ_0 are graphs over some axes.

When γ_0 has self-intersections, it is easy to see that singularities may form in finite time. For instance, the small loop in a cardioid contracts to form a cusp while the large loop still exists. Naturally one would like to study the singularities of (1). For a closed, immersed γ_0 there are always at most finitely many singularities as t tends to ω, the blow-up time. We may localize the study of singularities. A singularity Q is of *type I* if

$$\sup_{\mathcal{U}} |k(\cdot,t)|(\omega - t)^{1/2} \leq C \ ,$$

for some constant C, where $\mathcal{U}$ is a small neighborhood of Q disjoint from other singularities. A singularities is of *type II* if it is not of type I.

Theorem 4 (Altschuler [3]). *Let γ_0 be a closed, immersed curve.*

(a) *If all singularities are of type I as $t \uparrow \omega$, then $\gamma(\cdot,t)$ shrinks to a point and, after normalization, converges to an Abresch-Langer curve.*

(b) *If Q is a type II singularity, any sequence $\{\gamma(p_j, t_j)\}$ satisfying $t_j \uparrow \omega$, $Q = \lim_{j\to\infty} \gamma(p_j, t_j)$ contains a subsequence which, after suitable normalization, converges to a Grim Reaper.*

Singularities formation is also studied in Angenent [10] for locally convex curves. In Huisken [32] a monotoncity formula, which is crucial in the study of type I singularities, is obtained for the mean curvature flow.

Specified examples of type II singularities are discussed in Grayson [28] (figure-eight curves) and Angenent-Velazquez [11] (the cardoid). Especially in the latter work, the precise blow-up rate for a class of cardoids is determined. It is worth to point out that in [28] it is shown that a figure-eight curve with equal enclosed area exists until the area vanishes. However, it is not known whether the curve always shrinks to a point, or sometimes collapses into an arc.

The proof of Theorem 2 in [27] is elementary, but difficult to read. An elegant, shorter proof which establishes Theorems 1 and 2 at one stroke was outlined by Hamilton [31]. A new ingredient is the monotoncity property of a new isoperimetric ratio. We refer to Zhu [37] and [19] for further discussion of the proof.

§3 Anisotropic Generalized CSF

The flow can be written in the following form,

$$\frac{\partial\gamma}{\partial t} = \Phi(\theta)|k|^{\sigma-1}k + \Psi(\theta)\ , \quad \Phi > 0\ ,\ \sigma > 0\ , \tag{3}$$

where Φ and Ψ are 2π-periodic functions of the normal angle θ of the curve. Much is known for (3) when γ_0 is convex. Let's describe works concerning convex flows first. First of all, when $\sigma = 1$ and $\Psi(\theta) \equiv 0$, (3) can be interpreted as a CSF in

Minkowski geometry. Results corresponding to Theorem 1 were obtained in Gage [24] and Gage-Li [26]. When $\sigma = 1$, the flow arises from the theory of phase transition (Gurtin [30]), and a rather complete understanding is achieved in Chou-Zhu [17]. When $\sigma > 0$ and $\Psi(\theta) \equiv 0$, the flow is called the (anisotropic) generalized CSF. Many results are established in Andrews [6], except for$\sigma = 1/3$. In general, the behavior of (3) for σ between 1/3 and 1 is very similar to (1). For instance, the flow shrinks to a point smoothly in finite time, and, after normalization, subconverges to a self-similar solution of the flow. When $\sigma = 1/3$, $\Phi(\theta) \equiv 1$ and $\Psi(\theta) \equiv 0$, the flow is called the *affine* CSF. It is affine invariant and was proposed in the context of image processing (Alvarez-Guichard-Lions [4] and Sapiro-Tannenbaum [36]). In Andrews [5] (see also Sapiro-Tannenbaum [37]) it was shown that the flow shrinks smoothly to an elliptical point.

In studying the asymptotic shape of the normalized flow, one is naturally led to self-similar solutions of the anisotropic generalized CSF. In short, the support function of the flow $\gamma(\cdot, t)$, $h(\theta, t)$, satisfies the ODE

$$(4) \qquad h_{\theta\theta} + h = \frac{\Phi^{1/\sigma}(\theta)}{h^{1/\sigma}}, \quad h > 0 \text{ and } 2\pi\text{-periodic}.$$

Existence and uniqueness of solutions of (4) can be found in Gage [24], Gage-Li [26], Dohmen-Giga-Mizoguchi [21], Dohmen-Giga [20], Andrews [6], Ai-Chou-Wei [2] and Chou-Zhang [15]. Among many open problems in this topic, we mention only one: It was proved by Gage [24] that (4) has a unique solution when $\sigma = 1$ and $\Phi(\theta + \pi) = \Phi(\theta)$. Does uniqueness still hold when the symmetry condition is removed?

There are very few results concerning (3) when γ_0 is non-convex, aside from Theorem 2. In [18], we show that Theorem 2 continues to hold for any anisotropic

CSF satisying $\sigma = 1$, $\Phi(\theta+\pi) = \Phi(\theta)$, and $\Psi(\theta+\pi) = -\Psi(\theta)$. In [12], Angenent, Sapiro and Tannenbaum prove that the affine CSF shrinks a closed, embedded curve into a point smoothly. Furthermore, the total absolute curvature tends to 2π at the end. In the last chapter of [19], we show that for the generalized CSF ($\sigma \in (0,1), \Phi \equiv 1$ and $\Psi \equiv 0$), the flow either collapses to a line segment, or converges a point. In both cases the total absolute curvature approaches 2π.

§4 The CSF in Klein Geometry

Here the description of Klein geometry is extremely brief. We refer to Olver-Sapiro-Tannenbaum [35] for a very readable account of the theory.

According to the Erlanger Programme, every Lie group G acting effectively and transitively on the plane gives rise to a Klein geometry. First of all, its Lie algebra can be realized as a subalgebra of vector fields under the Poisson bracket. All Lie algebras of planar vector fields were classified by Lie (see Olver [34] for further discussion) up to local diffeomorphisms. In the most interesting case of primitive Lie algebras, there are exactly eight types of them (see Table 1). In particular, they include the Euclidean geometry, affine geometry, conformal geometry and projective geometry.

Name	Generators	Dimension	Structure
$E_\alpha(2)$	$\partial_x, \partial_u, u\partial_x - x\partial_u + \alpha(x\partial_x + u\partial_u)$	3	$\mathbb{R} \ltimes \mathbb{R}^2$
$SL(2)$	$x\partial_u, u\partial_x, x\partial_x - u\partial_u$	3	$\mathfrak{sl}(2)$
$SO(3)$	$u\partial_x - x\partial_u, (1+x^2-u^2)\partial_x + 2xu\partial_u,$ $2xu\partial_x + (1-x^2+u^2)\partial_u$	3	$\mathfrak{so}(3)$
$Sim(2)$	$\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u$	4	$\mathbb{R}^2 \ltimes \mathbb{R}^2$
$SA(2)$	$\partial_x, \partial_u, x\partial_x - u\partial_u, x\partial_u, u\partial_x$	5	$\mathfrak{sa}(2)$

$A(2)$	$\partial_x, \partial_u, x\partial_x, u\partial_u, x\partial_u, u\partial_x$	6	$\mathfrak{a}(2)$
$SO(3,1)$	$\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u,$ $(x^2-u^2)\partial_x + 2xu\partial_u, 2xu\partial_u + (u^2-x^2)\partial_u$	6	$\mathfrak{so}(3,1)$
$SL(3)$	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u,$ $x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u$	8	$\mathfrak{sl}(3)$

Table 1. Primitive Lie algebras of vector fields in $\mathbb{R}^2$

Let γ be a curve, which is a local graph $\{(x, u(x)) : x \in [a, b]\}$. We let $\gamma' = \{(y, v(y)) : y \in [c, d]\}$ be its image under a typical element g in G. A *differential invariant* of G is a function Φ depending on x, u and its derivatives such that $\Phi(x, u, \dots, u^{(n)}) = \Phi(y, v, \dots, v^{(n)})$ whenever γ' is related to γ in the above manner. An *invariant 1-form* $d\omega = \Phi(x, u, \dots, u^{(n)})dx$ satisfies

$$\int_a^b \Phi(x, u(x), \dots, u^{(n)}(x))dx = \int_c^d \Phi(y, v(y), \dots, v^{(n)}(y))dy \ ,$$

for all γ'. For example, take $G = E_0(2)$, the Euclidean group composed of translations and rotations. One can verify that the Euclidean curvature $k = u_{xx}(1+u_x^2)^{-3/2}$ and the arc-length element $ds = (1+u_x^2)^{-1/2}dx$ are respectively a differential invariant and an invariant 1-form respectively. In fact, it is true that any other differential invariants are functions of k and its derivatives with respect to ds. Moreover, any invariant 1-form can be expressed as Φds where Φ is a differential invariant. In general, such "generating" differential invariants and invariant 1-forms exist in other Klein geometries and they play the roles of curvature and arc-length element. For primitive Klein geometries, they are specified in Table 2.

Name	Arc-length	Curvature
$E_\alpha(2)$	$\exp(\alpha \arctan u_x)\sqrt{1+u_x^2}dx$	$\exp(-\alpha \arctan u_x)(1+u_x^2)^{-3/2}u_{xx}$
$SL(2)$	$d\tilde{s} = (xu_x - u)^{-2}u_{xx}dx$	$\phi = (xu_x - u)^3 u_{xx}^{-1}$
$SO(3)$	$2(1+x^2+u^2)^{-1}\sqrt{1+u_x^2}dx$	$\frac{(1+x^2+u^2)u_{xx}}{(1+u_x^2)^{3/2}} + \frac{2(u-xu_x)}{\sqrt{1+u_x^2}}$

$Sim(2)$	$d\theta = (1+u_x^2)^{-1}u_{xx}dx$	$\chi = [(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2]u_{xx}^{-2}$
$SA(2)$	$d\rho = u_{xx}^{1/3}dx$	$\mu = u_{xx}^{-8/3}P_4$
$A(2)$	$dl = u_{xx}^{-1}\sqrt{P_4}dx$	$\kappa = P_4^{-3/2}P_5$
$SO(3,1)$	$\frac{\sqrt{(1+u_x^2)u_{xxx}-3u_x u_{xx}^2}}{1+u_x^2}dx$	Q_5
$SL(3)$	$u_{xx}^{-1}P_5^{1/3}dx$	$P_7P_5^{-8/3}$

Table 2. The arc-length and curvature for the geometries in Table 1

In this table,

$$P_4 = 3u_{xx}u_{xxxx} - 5u_{xxx}^2,$$

$$Q_5 = k_s^{-2}k_{sss} - \frac{5}{4}k_s^{-3}k_{ss}^2 - k^2k_s^{-1},$$

$$P_5 = 9u_{xx}^2u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^2,$$

$$P_7 = \frac{1}{3}u_{xx}^2\left[6P_5D_x^2P_5 - 7(D_xP_5)^2\right] + 2u_{xx}u_{xxx}P_5D_xP_5$$
$$- \left(9u_{xx}u_{xxxx} - 7u_{xxx}^2\right)P_5^2,$$

where $k = u_{xx}(1+u_x^2)^{-3/2}$ and ds denote the Euclidean curvature and arc-length respectively.

With well-defined notions of curvature and arc-length, we can talk about curvature flows.

Olver, Sapiro and Tannenbaum [35] propose the CSF in Klein geometry as follows. First, define the *group tangent* and *group normal* by $\mathbf{T} = \gamma_\sigma$ and $\mathbf{N} = \gamma_{\sigma\sigma}$ respectively, where $d\sigma$ is the group arc-length. Both the group tangent and group normal are invariantly defined under G. The *group CSF* is simply given by

(5) $$\frac{\partial\gamma}{\partial t} = \pm\mathbf{N},$$

where the sign "+" or "−" must be chosen to ensure parabolicity. Although (5) is a Kleinian analogue of (1), the reader should keep in mind that it may not always decrease the group perimeter of a closed curve. When $G = E(2)$, we have $\mathbf{N} = \mathbf{T}_\sigma = k\mathbf{n}$ by the Frenet formulas. Hence (5) reduces to (1). When $G = SA(2)$,

$$\begin{aligned}\mathbf{N} &= k^{-1/3}(k^{-1/3}\gamma_s)_s \\ &= k^{1/3}\mathbf{n} + k^{-1/3}(k^{-1/3})_s\mathbf{t}\ .\end{aligned}$$

Since tangential velocity does not affect the shape of the flow, (5) is equivalent to the affine CSF. The flow (5) for other geometries are discussed in [35].

We point out another naturnal generalization of (1), yielding flows of higher order. In fact, the variation of the group perimeter usually assumes the form

$$\frac{dL}{dt} = -\text{constant} \int \kappa\phi d\sigma\ ,$$

where κ is the group curvature and ϕ is the variation along the group normal. We may study the "variational" group CSF,

$$\gamma_t = \pm\kappa\mathbf{N}\ . \tag{6}$$

When $G = SA(2)$, (6) was studied in Andrews [7] (with minus sign taken in (6)), where it is shown that the flow expands to infinity and is asymptotic to an ellipse. Besides, there are no results on (6) for other geometries.

§5 Integrable Flows in Klein Geometry

In some physical models one prefers flows which preserve both the perimeter and the area. The simplest model of this kind is

$$\frac{\partial\gamma}{\partial t} = -k_s\mathbf{n} - \frac{k^2}{2}\mathbf{t}\ , \tag{7}$$

where the tangential velocity is added so that the arc-length is independent of time. Physicists (Goldstein-Petrich [29]) discovered the remarkable fact that the curvature of (7) satisfies the KdV equation

$$k_t + k_{sss} + \frac{3}{2}k^2 k_s = 0 .$$

Recently we look for similar flows in Klein geometry under the requirement that they preserve the group arc-length pointwisely. It turns out that many integrable equations arise in the same ways (Chou-Qu [14]). We give some examples.

EXAMPLE 1. Take $G = SL(2)$ and

$$\gamma_t = -(\log\phi)_s \mathbf{N} - 2\phi\mathbf{T} .$$

Then the $SL(2)$-curvature ϕ satisfies the KdV equation,

$$\phi_t + \phi_{sss} + 6\phi\phi_{ss} = 0 .$$

EXAMPLE 2. Take $G = \text{Sim}(2)$ and

$$\gamma_t = -\mathbf{N} - 2\chi\mathbf{T} .$$

Then the Sim(2)-curvature χ satisfies the Burgers equation,

$$\chi_t = \chi_{\theta\theta} - 2\chi\chi_\theta .$$

EXAMPLE 3. Take $G = SA(2)$ and

$$\gamma_t = -\frac{1}{3}\mu_\rho \mathbf{N} - \frac{1}{3}(\mu_{\rho\rho} - \mu^2)\mathbf{T} .$$

Then the affine curvature μ satisfies the Sawada-Kotera equation

$$\mu_t + \mu_{\rho\rho\rho\rho\rho} + 5\mu\mu_{\rho\rho\rho} + 5\mu_\rho\mu_{\rho\rho} + 5\mu^2\mu_\rho = 0.$$

It is well-known that integrable equations are well-posed for all time. In view of the fact that the group curvature determines the curve up to an isometry, the corresponding flow exists for all time, too. This is in sharp constrast with the CSF and its companion flows that singularities occur frequently. It will be interesting to investigate the long time behavior of these flows.

Acknowledgement. This article is based on a talk given by the author in the Conference "Free Boundary Problems" in RIMS, October 10-13, 2000. He would like to thank the organizers of the conference, especially Prof. Y. Giga and Prof. H. Ishii, for their support and hospitality.

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