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Operator Algebras with Hierarchies of Symbols

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Introduction

Partial differential equations in the "usual" set–up are given on an open set (or on a manifold) that a priori has a C^∞ structure. However, many applications in physics and engineering as well as in pure mathematics (geometry, topology) are from the very beginning connected with singular geometries of the configuration or with non–compact exits to infinity.

We will discuss some elements of the analysis on manifolds with geometric singularities, boundaries, edges, corners, etc. A typical novelty compared with the smooth case is that singularities lead to natural classes of degenerate operators (that include "non–degenerate" ones as well) and to new (in general operator–valued) principal and complete symbol structures that behave like "semi–quantised" objects and encode specific information from the singularities, especially asymptotics of solutions.

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1 The general program

Configurations with singularities that we have in mind are special stratified spaces (in fact, pseudo–manifolds), locally modelled by "singular" charts that map neighbourhoods on M to corresponding model cones or wedges under natural conditions for the transition maps. A cone with base X is the quotient space

$$X^{\Delta} := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X),$$

i.e., $\{0\} \times X$ represents the tip of the cone. A wedge with model cone X^{Δ} is a Cartesian product

$$X^{\Delta} \times \Omega \qquad \text{for open } \Omega \subseteq \mathbb{R}^q.$$

A manifold M with singularities contains a subset M' of singular points such that $M \setminus M'$ is a C^∞ manifold. M' itself is assumed to be singular in a similar sense, with a subset M'' of singular points, etc.

This gives us a chain of spaces $M := M^{(0)} \supset M' \supset M'' \supset \ldots \supset M^{(e)} \supset M^{(e+1)} = \emptyset$, such that $M^{(k)} \setminus M^{(k+1)}$ is a C^∞ manifold for every k.

An example is the closed unit cube M in $\mathbb{R}^3$, where $M' = \partial M$ is its surface, including edges and corners, M'' is the system of one–dimensional egdes, including corners, and M''' is the system of corner points.

The general program for the analysis of operators on M includes the following points:

- Establish a pseudo–differential calculus that contains the typical differential operators on M
- Characterise adequate scales of Sobolev spaces
- Study operator algebras with natural symbol hierarchies that determine ellipticity and additional conditions on the lower–dimensional skeleta
- Characterise asymptotics of solutions near the singular points.

Analogous problems are reasonable in the context of parabolicity and hyperbolicity.

As concrete realisations of problems with singularities we have:

- Boundary value problems on spaces with piecewise smooth geometry
- Mixed elliptic, transmission, and crack problems
- Problems on spaces with components of different dimensions
- Parabolic problems in the (infinite) space/time cylinder, where the spatial configularations are not smooth; here, $t \to \infty$ behaves like an exit to infinity
- Hyperbolic problems in domains with non–smooth geometry, such as propagation and reflection of singularities near conical points, edges, etc.
- Asymptotics of solutions to various kinds of non–linear problems

Each of these models has its own history; the analytic understanding in the hyperbolic case is far from being in a satisfactory state.

The same is true of the index theory of elliptic operators on manifolds with geometric singularities.

A common feature of these models is a specific interplay between microlocal and global information, expressed by the components of hierarchies of symbols.

In this survey we content ourselves with elliptic operators and illustrate typical elements of the calculus for conical and edge singularities as well as for boundary value problems. For the general background, cf. the monographs Schulze [25], [28] or Egorov and Schulze [6]. A general machinery to construct operator algebras on (pseudo–) manifolds with higher singularities is given in [30]. Applications to parabolic problems are given in the author's joint paper with Krainer [14]. Concerning edge space aspects in hyperbolicity, cf. Dreher and Witt [4]. Non–linear problems, with singularities of various kind have been studied by Witt [41] and in the author's joint paper with Vishik, Witt and Zelik [38].

2 Fuchs type operators

First recall that when X is a closed compact C^∞ manifold, $E, F \in \mathrm{Vect}(X)$, where $\mathrm{Vect}(\cdot)$ the set of smooth complex vector bundles on the space in the brackets, and $A : C^\infty(X, E) \to C^\infty(X, F)$ a (classical) pseudo–differential operator of order μ on X, we have the homogeneous principal symbol of order μ, that is a homomorphism

$$\sigma_\psi(A) : \pi_X^* E \to \pi_X^* F, \tag{1}$$

$\pi_X : T^*X \setminus 0 \to X$. Ellipticity of A means that (1) is an isomorphism. While this is a purely microlocal condition,

$$\dim \ker A, \qquad \dim \operatorname{coker} A$$

and index $\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A$ are globally defined. This is, of course, well–known, but operator families on X appear as a part of the symbol structure of operators on a manifold with conical singularities with base X which gives us a first impression on how ellipticity and the calculus as a whole are determined by local and global data.

A manifold B with conical singularities is a topological space that contains a finite subset S (the conical points) such that $B \setminus S$ can be regarded as the interior of a C^∞ manifold $\mathbb{B}$ with compact boundary. For simplicity, consider the case when S only consists of one point. Then $\partial\mathbb{B} =: X$ may be regarded as the base of a cone X^Δ that is the local model of B in a neighbourhood of $v \in S$. Then $\mathbb{B}$ is locally near $\partial\mathbb{B}$ modelled by $\overline{\mathbb{R}}_+ \times X$; in this connection $\mathbb{B}$ is called the stretched manifold associated with B. Let $(r, x) \in \mathbb{R}_+ \times X$ denote a splitting of points on $\mathbb{B}$ near S. Then the typical differential operators A on $\mathbb{B}$ are those of Fuchs type, that is

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \Big(-r\frac{\partial}{\partial r}\Big)^j, \tag{2}$$

for convenience, written with operator–valued symbols $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}($
Here, $\mathrm{Diff}^\nu(\cdot)$ denotes the space of all differential operators of order ν on the space in the brackets, with smooth coefficients (in local representations); $\mathrm{Diff}^\nu(\cdot)$ is Fréchet in a natural way in all cases that we are considering in this paper. (For the moment, we simply speak about operators acting in function spaces rather than in spaces of sections in bundles; this can be done as well and is straightforward.) In local coordinates $x \in \Sigma$ on X, $\Sigma \subseteq \mathbb{R}^n$ open, $n = \dim X$, the operator A can be locally described by a symbol of the form

$$p(r, x, \varrho, \xi) = r^{-\mu}\widetilde{p}(r, x, r\varrho, \xi),$$

where $\widetilde{p}(r, x, \widetilde{\varrho}, \xi)$ is smooth up to $r = 0$.

Let $p_{(\mu)}(r, x, \varrho, \xi)$ denote the homogeneous principal part of $p(r, x, \varrho, \xi)$ of order μ. Then

$$\sigma_\psi(A)(r, x, \varrho, \xi) = p_{(\mu)}(r, x, \varrho, \xi)$$

is the homogeneous principal symbol of A (near $(\partial\mathbb{B})$) in the usual sense. We also have a "compressed" variant, namely

$$\sigma_{\psi,\mathfrak{f}}(A)(r, x, \varrho, \xi) := \widetilde{p}_{(\mu)}(r, x, \varrho, \xi)$$

which is also invariant, now in the sense of a function on a so–called compressed cotangent bundle (realised by a simple singular modification of the standard cocycle for the cotangent bundle).

In addition, we have the principal conormal symbol

$$\sigma_M(A)(z) := \sum_{j=0}^{\mu} a_j(0) z^j. \tag{3}$$

$z \in \mathbb{C}$ is interpreted as a covariable for the Mellin transform $(Mu)(z) = \int_0^\infty r^{z-1} u(r)\, dr$; then $M^{-1}zM = -r\frac{\partial}{\partial r}$. Now $\sigma_M(A)(z)$ induces a family of continuous operators

$$\sigma_M(A)(z) : H^s(X) \to H^{s-\mu}(X)$$

between standard Sobolev spaces on X. The principal symbol "hierarchy" of operators of Fuchs type is

$$\sigma(A) := (\sigma_\psi(A), \sigma_M(A)).$$

Definition 1 *A is said to be elliptic with respect to a weight $\gamma \in \mathbb{R}$, if*

(i) *$\sigma_\psi(A) \neq 0$ on $T^*(\operatorname{int}\mathbb{B}) \setminus 0$ and $\sigma_{\psi,\mathfrak{f}}(A)(r, x, \varrho, \xi) \neq 0$ for $(\varrho, \xi) \neq 0$, up to $r = 0$,*

(ii) *$\sigma_M(A)(z) : H^s(X) \to H^{s-\mu}(X)$ are isomorphisms for all $z \in \mathbb{C}$, $\mathrm{Re}\, z = \frac{n+1}{2} - \gamma$, $n = \dim X$, for some $s \in \mathbb{R}$.*

Note that condition (i) can be formulated invariantly as non–vanishing of $\sigma_{\psi,\mathfrak{f}}(A)$ globally on $T^*_\mathfrak{f}\mathbb{B} \setminus 0$, where $T^*_\mathfrak{f}\mathbb{B}$ is the compressed cotangent bundle. Moreover, if (ii) is satisfied for an $s = s_0 \in \mathbb{R}$, then it holds for all $s \in \mathbb{R}$.

Let $\mathcal{H}^{s,\gamma}(\mathbb{B})$ denote the weighted Sobolev space of smoothness $s \in \mathbb{R}$ and weight $\gamma \in \mathbb{R}$, cf. [25] or [28]. Recall that when $\omega(r)$ is any cut–off function (i.e., $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\omega(r) = 1$ in a neighbourhood of $r = 0$) we set

$$\mathcal{H}^{s,\gamma}(\mathbb{B}) = \omega\mathcal{H}^{s,\gamma}(X^\wedge) + (1-\omega)H^s_{\mathrm{loc}}(\operatorname{int}\mathbb{B}),$$

for $X^\wedge = \mathbb{R}_+ \times X$. Here, $\overline{\mathbb{R}}_+ \times X$ is identified with a collar neighbourhood of $\partial\mathbb{B}$ in $\mathbb{B}$ such that $\overline{\mathbb{R}}_+$ corresponds to the inner normal (with respect to a Riemannian metric on $\mathbb{B}$), and $\mathcal{H}^{s,\gamma}(X^\wedge)$ is the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\left\{\frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\operatorname{Im} z)(Mu)(z)\|^2_{L^2(X)}\,dz\right\}^{\frac{1}{2}}$$

for an arbitrary family $R^\mu(\tau)$ of classical parameter–dependent elliptic operators on X with parameter $\tau \in \mathbb{R}$ that induces isomorphisms $R^\mu(\tau) : H^s(X) \to H^{s-\mu}(X)$ for all $s \in \mathbb{R}$ and $\tau \in \mathbb{R}$; further

$$\Gamma_\beta := \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}.$$

Theorem 2 *Let $\mathbb{B}$ be compact, and let A be an operator of Fuchs type on $\mathbb{B}$ of order μ. Then the following conditions are equivalent:*

(i) *A is elliptic with respect to the weight γ,*

(ii) *$A : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{B})$ is a Fredholm operator for any $s \in \mathbb{R}$.*

For the iteration of pseudo–differential calculi on spaces with higher singularities (edges, corners, etc.) it is essential to consider pseudo–differential operators on (stretched) infinite cones $X^\wedge = \mathbb{R}_+ \times X$. The configuration in this case has an exit to infinity that causes specific precautions at infinity, cf. also [11].

The weighted Sobolev spaces in this case are denoted $\mathcal{K}^{s,\gamma}(X^\wedge)$, where $\omega\mathcal{K}^{s,\gamma}(X^\wedge) = \omega\mathcal{H}^{s,\gamma}(X^\wedge)$ for any cut–off function $\omega(r)$, and $(1-\omega)\mathcal{K}^{s,\gamma}(X^\wedge)$ equals the standard Sobolev space in the sense that for $X = S^n$ (the unit sphere in $\mathbb{R}^{n+1}$) and $X^\wedge = \mathbb{R}^{n+1} \setminus \{0\}$ we have $(1-\omega)\mathcal{K}^{s,\gamma}(X^\wedge) = (1-\omega)H^s(\mathbb{R}^{n+1})$, where on the right hand side of the latter relation $r = |\tilde{x}|$, $\tilde{x} \in \mathbb{R}^{n+1}$. The definition for general X can be reduced to this case.

Remark 3 *Setting $(\kappa_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}}u(\lambda r, x)$ for $u(r,x) \in \mathcal{K}^{s,\gamma}(X^\wedge)$, $\lambda \in \mathbb{R}_+$, we have a strongly continuous group of isomorphisms*

$$\kappa_\lambda : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

for every $s, \gamma \in \mathbb{R}$.

3 Abstract edge spaces

Let E be a Hilbert space and $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ be a strongly continuous group of isomorphisms $\kappa_\lambda : E \to E$, $\lambda \in \mathbb{R}$. Then the abstract edge Sobolev space of smoothness $s \in \mathbb{R}$ (with model space E) is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, E)$ (the Schwartz space of E–valued functions) in the norm

$$\left\{ \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle}\hat{u}(\eta)\|_E^2 \, d\eta \right\}^{\frac{1}{2}}.$$

Here $\hat{u}(\eta)$ is the Fourier transform with respect to $y \in \mathbb{R}^q$.

Example 4 *Set* $E = H^s(\mathbb{R}^{n+1})$, $(\kappa_\lambda u)(\tilde{x}) := \lambda^{\frac{n+1}{2}} u(\lambda\tilde{x})$, $\lambda \in \mathbb{R}_+$. *Then we have* $\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^{n+1})) = H^s(\mathbb{R}^{n+1+q})$.

Another example is $E = H^s(\mathbb{R}_+)$ $(= H^s(\mathbb{R})|_{\mathbb{R}_+})$, $(\kappa_\lambda u)(t) = \lambda^{\frac{1}{2}} u(\lambda t)$, $\lambda \in \mathbb{R}_+$, where

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}_+)) = H^s(\mathbb{R}^q \times \mathbb{R}_+) = H^s(\mathbb{R}^q \times \mathbb{R})|_{\mathbb{R}^q \times \mathbb{R}_+}.$$

Parallel to the edge spaces we have spaces of operator–valued symbols

$$S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}),$$

$U \subseteq \mathbb{R}^p$ open (subscript "(cl)" means classical or non–classical), associated with E, $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\widetilde{E}$, $\{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ that are Hilbert spaces with given strongly continuous groups of isomorphisms. $S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ is defiend to be the subspace of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that

$$\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,\widetilde{E})} \leq c\langle\eta\rangle^{\mu-|\beta|}$$

for all $y \in K$ for arbitrary $K \subset\subset U$, $\eta \in \mathbb{R}^q$, $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, with constants $c = (\alpha, \beta, K) > 0$.

Classical symbols are based on "twisted homogeneity", that is a relation of the form

$$f(y, \lambda\eta) = \lambda^\nu \widetilde{\kappa}_\lambda f(y,\eta)\kappa_\lambda^{-1}, \tag{4}$$

$\lambda \in \mathbb{R}_+$, $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$, for a function $f(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E,\widetilde{E}))$. A symbol $a(y,\eta)$ is said to be classical of order ν, if there are elements $a_{(\mu-j)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E,\widetilde{E}))$ that are homogeneous of order $\mu - j$ in the above–mentioned sense, $j \in \mathbb{N}$, such that for any excision function $\chi(\eta)$

$$a(y,\eta) - \chi(\eta)\sum_{j=0}^{N} a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; E, \widetilde{E})$$

for all $N \in \mathbb{N}$.

Abstract edge Sobolev spaces based on strongly continuous groups of isomorphisms on a parameter space E have been introduced in [24] in connection with edge pseudo–differential operators, cf. also Section 4 below.

4 The edge algebra

Let us set

$$\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)). \tag{5}$$

It can be proved that then

$$H^s_{\mathrm{comp}}(X^\wedge \times \mathbb{R}^q) \subset \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \subset H^s_{\mathrm{loc}}(X^\wedge \times \mathbb{R}^q)$$

for all $s, \gamma \in \mathbb{R}$.

The spaces of type (5) can be globally defined on a manifold W with edges $Y \subset W$. By this we understand a topological space (locally compact, paracompact) such that $W \setminus Y$ and Y are C^∞ manifolds of dimensions $n+1+q$ and q, respectively, and every $y \in Y$ has a neighbourhood V in W such that there is a homeomorphism $\overline{\alpha} : V \to X^\Delta \times \Omega$ for a closed compact C^∞ manifold X and open $\Omega \subseteq \mathbb{R}^q$, where $\overline{\alpha} : V \to X^\Delta \times \Omega$ induces diffeomorphisms $\overline{\alpha}|_{V\setminus Y} =: \alpha : V \setminus Y \to X^\wedge \times \Omega$ and $\overline{\alpha}|_{V\cap Y} =: \alpha' : V \cap Y \to \Omega$ (such an $\overline{\alpha}$ is called a singular chart), and the transition maps $\beta\alpha^{-1}$ for different $\alpha : V\setminus Y \to X^\wedge \times \Omega$, $\beta : V \setminus Y \to X^\wedge \times \widetilde{\Omega}$ extend to diffeomorphisms

$$\overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \widetilde{\Omega}.$$

For simplicity, let us assume here that Y has a neighbourhood in W corresponding to $X^\Delta \times Y$; then the transition maps $(r, x, y) \to (\widetilde{r}, \widetilde{x}, \widetilde{y})$ may (and will) be chosen to be independent of r and x for small r. Instead of W we mainly look at the stretched manifold $\mathbb{W}$.

By definition $\mathbb{W}$ is a smooth manifold with boundary, where the local structure near $\partial\mathbb{W}$ is given by $\overline{\mathbb{R}}_+ \times X \times \Omega$ and (under our assumption) $\partial\mathbb{W} \cong X \times Y$. There is then a canonical projection $\mathbb{W} \to W$ that is a diffeomorphism $\operatorname{int}\mathbb{W} \to W \setminus Y$ and locally near $\partial\mathbb{W}$ induced by $\overline{\mathbb{R}}_+ \times X \times \Omega \to X^\Delta \times \Omega$. Locally near $\partial\mathbb{W}$ we employ the splitting of variables into (r, x, y) with covariables (ϱ, ξ, η). Local symbols are assumed to be edge–degenerate, i.e., of the form $r^{-\mu} p(r, x, y, \varrho, \xi, \eta)$, where

$$p(r, x, y, \varrho, \xi, \eta) = \widetilde{p}(r, x, y, r\varrho, \xi, r\eta)$$

for a classical symbol $\widetilde{p}(r, x, y, \widetilde{\varrho}, \xi, \widetilde{\eta})$ of order $\mu \in \mathbb{R}$ that is smooth up to $r = 0$.

Assume, for simplicity, $\mathbb{W}$ to be compact. Then, using the local spaces (5) we can invariantly define weighted Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathbb{W}, E)$ of sections in vector bundles E on $\mathbb{W}$. The bundles are assumed to be pull–backs of bundles on W under the projection $\mathbb{W} \to W$. There is then a restriction E' of the former bundle to Y. Analogously, we employ the notation with prime for other bundles on $\mathbb{W}$ of the described kind.

The pseudo–differential algebra on $\mathbb{W}$ consists of block matrix operators of the form

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ H^s(Y, J^-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \\ \oplus \\ H^{s-\mu}(Y, J^+) \end{array},$$

where $E, F \in \operatorname{Vect}(\mathbb{W})$, $J^-, J^+ \in \operatorname{Vect}(Y)$. The upper left corner u. l. c. $\mathcal{A} = A$ is a pseudo–differential operator with edge–degenerate symbol, plus certain Mellin and Green operators. The principal symbol structure consists of a pair

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\wedge(\mathcal{A})),$$

where $\sigma_\psi(\mathcal{A})$ is the principal interior symbol, $\sigma_\wedge(\mathcal{A})$ the principal boundary symbol. Let us illustrate the nature of the symbols for the case of edge–degenerate differential operators. Locally near Y in stretched coordinates $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ they are assumed to be of the form

$$A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y) \left(-r \frac{\partial}{\partial r} \right)^j (r D_y)^\alpha$$

where $a_{j\alpha}(r, y) \in \mathrm{Diff}^{\mu-(j+|\alpha|)}(X; E'_y, F'_y)$ with smooth dependence on $(r, y) \in \overline{\mathbb{R}}_+ \times \Omega$.

$\sigma_\psi(A)(r, x, y, \varrho, \xi, \eta)$ (in local coordinates x on X with covariables ξ) is the standard homogeneous principal symbol, where

$$\sigma_{\psi,\mathfrak{f}}(A)(r, x, y, \varrho, \xi, \eta) := r^\mu \sigma_\psi(A)(r, x, y, r^{-1}\varrho, \xi, r^{-1}\eta)$$

is smooth up to $r = 0$. Globally, $\sigma_{\psi,\mathfrak{f}}(A)$ has an invariant meaning as a homomorphism

$$\sigma_{\psi,\mathfrak{f}}(A) : \pi^*_{\mathbb{W},\mathfrak{f}} E \to \pi^*_{\mathbb{W},\mathfrak{f}} F, \tag{6}$$

$\pi_{\mathbb{W},\mathfrak{f}} : T^*_\mathfrak{f}\mathbb{W} \setminus 0 \to \mathbb{W}$, where $T^*_\mathfrak{f}\mathbb{W}$ is the compressed cotangent bundle of $\mathbb{W}$. Moreover, the homogeneous principal edge symbol of A is given by the expression

$$\sigma_\wedge(A)(y, \eta) = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y) \left(-r \frac{\partial}{\partial r} \right)^j (r\eta)^\alpha.$$

It represents a homomorphism

$$\sigma_\wedge(A) : \pi^*_Y \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E' \to \pi^*_Y \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F', \tag{7}$$

$\pi_Y : T^*Y \setminus 0 \to 0$. The operator A is said to be σ_ψ–elliptic, if (6) is an isomorphism (note that this is required up to $\partial\mathbb{W}$).

Theorem 5 *Let A be σ_ψ–elliptic. Then for every $y \in Y$ there exists a countable set $D(y) \subset \mathbb{C}$ such that $D(y) \cap \{z : \ c \leq \mathrm{Re}\, z \leq c'\}$ is finite for every $c \leq c'$, such that*

$$\sigma_\wedge(A)(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E'_y \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F'_y \tag{8}$$

is a Fredholm operator for all $\gamma \in \mathbb{R}$ such that $\Gamma_{\frac{n+1}{2}-\gamma} \cap D(y) = \emptyset$, for all $s \in \mathbb{R}$.

Remark 6 *$D(y)$ is the set of all non–bijectivity points of the principal conormal symbol of $\sigma_\wedge(A)(y, \eta)$, regarded as an operator of Fuchs type on the cone $X^\wedge$, namely*

$$\begin{aligned}\sigma_M(A)(y, z) &:= \sigma_M(\sigma_\wedge(A))(y, z) \\ &:= \sum_{j=0}^{\mu} a_{j0}(0, y) z^j : H^s(X) \otimes E'_y \to H^{s-\mu}(X) \otimes F'_y.\end{aligned}$$

We now assume that there is a $\gamma \in \mathbb{R}$ such that $\Gamma_{\frac{n+1}{2}-\gamma} \cap D(y) = \emptyset$ for all $y \in Y$. Similarly to methods in pseudo–differential boundary value problems we fill up the Fredholm family (8) to a family of isomorphisms

$$\sigma_\wedge(\mathcal{A})(y, \eta) = \begin{pmatrix} \sigma_\wedge(A) & \sigma_\wedge(K) \\ \sigma_\wedge(T) & \sigma_\wedge(Q) \end{pmatrix} (y, \eta) : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge) \otimes E'_y \\ \oplus \\ J^-_y \end{matrix} \to \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \otimes F'_y \\ \oplus \\ J^+_y \end{matrix},$$

homogeneous in the sense

$$\sigma_\wedge(\mathcal{A})(y,\lambda\eta) = \lambda^\mu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \sigma_\wedge(\mathcal{A})(y,\eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tag{10}$$

for all $(y,\eta) \in T^*Y \setminus 0$, $\lambda \in \mathbb{R}_+$. Analogous considerations are possible for edge–degenerate pseudo–differential operators A. Implicitly we assume that a certain topological condition on $\sigma_\psi(A)$ is fulfilled, i.e., that bundles $J^-, J^+ \in \mathrm{Vect}(Y)$ can be chosen such that (9) are isomorphisms. In general, if that condition is violated, we can only choose the bundles $J^\pm$ to be given on $T^*Y \setminus 0$, whereas in (9) they are given on Y itself, cf. the remarks at the end of Section 5 below.

Consider the $(1,2)$, $(2,1)$ and $(2,2)$–entries of (9) in local coordinates $\Omega \times \mathbb{R}^q$, $\Omega \subseteq \mathbb{R}^q$ open. Then, multiplying them by an excision function $\chi(\eta)$ in $\mathbb{R}^q$ we get corresponding entries of a 2×2–matrix $g(y,\eta)$ classical operator–valued symbols $g(y,\eta) = (g_{ij}(y,\eta))_{i,j=1,2}$,

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma} \otimes \mathbb{C}^e) \oplus \mathbb{C}^{j_-}, (\mathcal{K}^{\infty,\gamma-\mu+\varepsilon} \otimes \mathbb{C}^f) \oplus \mathbb{C}^{j_+}), \tag{11}$$

$s \in \mathbb{R}$, where, for simplicity, $X^\wedge$ is omitted in the spaces e, f and $j_\pm$ are the fibre dimensions of E, F and $J^\pm$, respectively, and $\varepsilon > 0$. Associated pseudo–differential operators $\mathrm{Op}(g)u(y) = \iint e^{i(y-y')\eta} g(y,\eta)u(y')\,dy' đ\eta$ give us (after a globalisation by means of a partition of unity on $\mathbb{W}$ and $\partial\mathbb{W}$) continuous maps

$$\mathcal{A} = \begin{pmatrix} A & K \\ T & Q \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(\mathbb{W},E) \\ \oplus \\ H^s(Y,J^-) \end{matrix} \to \begin{matrix} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W},F) \\ \oplus \\ H^{s-\mu}(Y,J^+) \end{matrix} \tag{12}$$

for all s (the upper left corner is the given edge–degenerate operator A itself).

Remark 7 *In general, e.g., in compositions of operators of the form* (12) *or in parametrices (of elliptic operators) we have to replace A in the upper left corner by $A + M + G$, where M is a "smoothing" Mellin operator and (locally) $G = \mathrm{Op}(g_{11})$, where $g_{11}(y,\eta)$ is a so–called Green symbol that appears as an upper left corner of a symbol of the form* (11).

The details of the calculus of operators of the kind (12) show that it is reasonable to further specify the nature of the entries of (11) by requiring symbols with values that map to subspaces of $\mathcal{K}^{\infty,\gamma-\mu}(X^\wedge) \oplus \mathbb{C}^{j_+}$ with specific asymptotics in the first component and that the pointwise formal adjoints have analogous properties.

An operator

$$\mathcal{A} = \begin{pmatrix} A + M + G & K \\ T & Q \end{pmatrix} \tag{13}$$

of the described kind, where, in general, A is a (classical) pseudo–differential operator with edge degenerate symbols (in local coordinates near $\partial\mathbb{W}$) is said to belong to the space $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, where $\boldsymbol{g} = (\gamma, \gamma - \mu)$ are the involved weight data and $\boldsymbol{v} = (E, F; J^-, J^+)$ the bundles. (Weight strips Θ in $\boldsymbol{g}$ as they are used in other expositions on edge degenerate operators are omitted, here; we simply assume that those strips equal $(-\infty, 0]$, i.e., they are then superfluous. Operators of this form constitute what we call the edge algebra (that is, the union of all $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ over $\mu \in \mathbb{R}$ and weight and bundle data, where algebra operations are admitted when these data fit together. To each $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ we have a pair of principal symbols, namely

$$\sigma(\mathcal{A}) = (\sigma_{\psi}(\mathcal{A}), \sigma_{\wedge}(\mathcal{A})),$$

where $\sigma_{\psi}(\mathcal{A}) := \sigma_{\psi}(\mathcal{A}_{11})$ is the homogeneous principal pseudo-differential symbol of order μ of the upper left corner; it has a Fuchs type variant $\sigma_{\psi,\mathbf{f}}(\mathcal{A})$, cf. formula (6), and $\sigma_{\wedge}(\mathcal{A})$ is the "κ_{λ}-homogeneous" principal edge symbol of order μ that is a homomorphism of the form (9) satisfying relation (10) (the choice of s is unessential).

Theorem 8 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\boldsymbol{v} = (E, F; J^-, J^+)$ and $\sigma(\mathcal{A}) = 0$ implies that* (12) *is compact for every $s \in \mathbb{R}$.*

Theorem 9 *$\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, $\mathcal{B} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{h}; \boldsymbol{w})$ for $\boldsymbol{g} = (\gamma-\nu, \gamma-(\mu+\nu))$, $\boldsymbol{v} = (E_0, F; J_0, J^+)$, $\boldsymbol{h} = (\gamma, \gamma - \nu)$, $\boldsymbol{w} = (E, E_0; J^-, J_0)$ implies $\mathcal{AB} \in \mathcal{Y}^{\mu+\nu}(\mathbb{W}; \boldsymbol{g} \circ \boldsymbol{h}; \boldsymbol{v} \circ \boldsymbol{w})$, where $\boldsymbol{g} \circ \boldsymbol{h} = (\gamma, \gamma - (\mu + \nu))$, $\boldsymbol{v} \circ \boldsymbol{w} = (E, F; J^-, J^+)$, and we have*

$$\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

(with componentwise multiplication).

Remark 10 *The edge algebra is also closed under formal adjoints $\mathcal{A}^*$ with respect to the $\mathcal{W}^{0,0}$- and H^0-scalar products and we have a corresponding rule $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})^*$.*

The basics on edge operators are elaborated in [25], [6] or [28]. A new version of edge symbol calculus is given in [10].

Definition 11 *An operator $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ is said to be elliptic if both $\sigma_{\psi,\mathbf{f}}(\mathcal{A})$ and $\sigma_{\wedge}(\mathcal{A})$ are isomorphisms.*

The condition concerning $\sigma_{\wedge}(\mathcal{A})$ is an analogue of the Shapiro–Lopatinskij condition in elliptic boundary value problems.

Remark 12 *Smooth manifolds with boundary may be regarded as manifolds with edges, where the boundary is interpreted as an edge and the inner normal (with respect to a Riemannian metric) as the model cone of the local wedge. Classical symbols that are smooth up to the boundary may be viewed (modulo smoothing symbols) as particular edge-degenerate ones. So the calculus of boundary value problems for pseudo-differential operators with and without the transmission property can be regarded as a subcalculus of the edge algebra in general.*

Theorem 13 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ where $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\boldsymbol{v} = (E, F; J^-, J^+)$. Then the following conditions are equivalent,*

(i) *$\mathcal{A}$ is elliptic,*

(ii) *$\mathcal{A}$ is a Fredholm operator* (12) *for some $s \in \mathbb{R}$.*

Remark 14 *If* (12) *is Fredholm for $s = s_0 \in \mathbb{R}$ then so is for all $s \in \mathbb{R}$.*

Remark 15 *If $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ is elliptic, there is a parametrix $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}$ where $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma)$, $\boldsymbol{v}^{-1} = (F, E; J^+, J^-)$, in the sense that $\mathcal{I} - \mathcal{AP}$ and $\mathcal{I} - \mathcal{PA}$ are of order $-\infty$ in the respective classes (and as such compact in our weighted edge Sobolev spaces).*

The assertion (i) $\to$ (ii) of Theorem 13 as well as Remarks 14, 15 may be found in [28]. The converse (ii) $\to$ (i) has been proved in [2].

Remark 16 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ be elliptic, and let*

$$u \in \mathcal{W}^{-\infty,\gamma}(\mathbb{W}, E) \oplus H^{-\infty}(Y, J^-).$$

Then $\mathcal{A}u = f \in \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \oplus H^{s-\mu}(Y, J^+)$ implies

$$u \in \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \oplus H^{s}(Y, J^-).$$

An analogous result holds for subspaces with asymptotics, i.e., when the first component of f has (say, continuous) edge asymptotics near Y then the first component of u has asymptotics, too, of some type that is determined by f and the operator, cf. [25].

5 Higher algebras on manifolds with geometric singularities

As noted in the beginning (pseudo–) manifolds with geometric singularities such as conical points, edges, boundaries, corners, etc. play a role in many applications. Let us first give a list of examples of "regular" singular spaces that directly occur in models of mechanics, theoretical physics, etc., and let us describe the principal symbol hierarchies in corresponding algebras. If we talk about degenerate symbols we refer to stretched coordinates near the singularities.

(i) Closed manifolds B with conical singularities. Symbols of operators A in the "cone algebra" are

$$\sigma(A) = (\sigma_{\psi}(A), \sigma_M(A)) \tag{14}$$

where $\sigma_{\psi}(A)$ is the Fuchs type homogeneous principal symbol outside the singularity and $\sigma_M(A)$ is the principal conormal symbol (the latter is operator–valued and takes values in pseudo–differential operators on the base X of the cone), cf. [23], [25], [27], [40], [17].

(ii) Manifolds B with boundary and conical singularities. Symbols of operators A in the cone algebra of boundary value problems are

$$\sigma(A) = (\sigma_{\psi}(A), \sigma_{\partial}(A), \sigma_M(A)), \tag{15}$$

where $\sigma_{\psi}(A)$, $\sigma_M(A)$ are analogues of the corresponding components in (14) while $\sigma_{\partial}(A)$ is the boundary symbol (which is operator–valued and acts in Sobolev spaces in normal direction to the boundary, if we talk about the transmission property at the smooth part of the boundary), cf. [21], [22], [5].

(iii) Closed manifolds W with edges. Operators constitute the "edge algebra"; a particular case is the algebra of boundary value problems without the transmission property. Symbols of operators A in the edge algebra are

$$\sigma(A) = (\sigma_{\psi}(A), \sigma_{\wedge}(A));$$

here $\sigma_\psi(A)$ is the edge–degenerate homogeneous principal symbol and $\sigma_\wedge(A)$ the homogeneous principal edge symbol. It takes values in operators in weighted Sobolev spaces on the infinite model cone, cf. [25], [6], [28], [27].

(iv) Manifolds W with boundary and edges. Symbols of operators A in the edge algebra of boundary value problems are

$$\sigma(A) = (\sigma_\psi(A), \sigma_\partial(A), \sigma_\wedge(A)), \tag{16}$$

where subscripts indicate the meaning of components as before, cf. [12], [31], [9].

(v) Manifolds K with corners (that are locally cones where the base manifolds have also conical singularities. There are then one–dimensional edges connecting the corner singularities.) Let A be an element in the corresponding "corner algebra". The corner induces a corner conormal symbol component $\sigma_c(A)$ that takes values in the cone algebra of the local base of the corner, while the other symbol components have a similar meaning as before, and we have

$$\sigma(A) = (\sigma_\psi(A), \sigma_\wedge(A), \sigma_c(A)) \tag{17}$$

for the boundaryless case,

$$\sigma(A) = (\sigma_\psi(A), \sigma_\partial(A), \sigma_\wedge(A), \sigma_c(A)) \tag{18}$$

where K has a boundary, where, in particular, the homogeneous principal symbol $\sigma_\psi(A)$ is corner degenerate in both cases (and, for boundary value problems, has the transmission property with respect to the smooth part of the boundary), cf. [26], [31], [30], [18], [36].

In general, if M is a stratified space in the sense of notation in Section 1, there is a general method of building up higher analogues of the edge and corner algebras of pseudo–differential operators by an iterative method. Details are published [30]. The idea consists of repeatedly applying a machinery, called "conification" and "edgification" of an already achieved calculus. Conification yields a calculus on a cone, starting from an already constructed calculus on a (compact) base space with singularities. Edgification yields a calculus on a wedge, starting from an already constructed calculus on a cone. Invariance considerations and patching together such local calculi gives us operator algebras globally on manifolds with singularities that are locally modelled by corresponding local cones and wedges. Then the procedure can start again, and we reach in this way the full hierarchy of operator algebras on stratified spaces. The construction in [30] assumes a certain regularity of the singularities in the sense of some transversality of intersections of faces near lower–dimensional strata. The cuspidal case should be possible as well. Special such theories for cuspidal cones, wedges or corners have been elaborated in [37], [34].

A result of [30] is that each conification step contributes an extra "conormal symbol" and an extra weight in the Sobolev spaces. Each edgification step contributes additional edge conditions along the new arising edge and a new symbol that encodes a higher analogue of the Shapiro–Lopatinskij condition in the elliptic case. Ellipticity of edge conditions in such a sense requires that a certain topological obstruction vanishes. This is analogous to the Atiyah–Bott condition in boundary value problems, cf. Atiyah and Bott [1], Boutet de Monvel [3], Rempel and Schulze [19], Seeley [39], or the author's joint paper with Sternin and Shatalov [35]. The methods developed in Schulze [29] may also be applied to edge singularities, cf. the author's joint paper with Seiler [32].

Let us finally note that ellipticity in operator algebras with symbol hierarchies gives rise to many interesting new elements of the index theory, and it is a very ambitious program to express the Fredholm index in terms of the symbol structure. This is a wide field with contributions by many authors, cf. the papers [7], [33], [8], [9], [20], [16] and the references there.

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