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Asymptotic Expansions for Solutions to Semilinear Fuchsian Equations

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Let X be a C^∞ manifold with boundary, ∂X. We consider the semilinear elliptic equation

$$Au=F(x,B_1u,\ldots,B_Ku)\ \text{ on } X^\circ=X\setminus\partial X \tag{1}$$

for the unknown $u=u(x)$, where $A\in\mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$, $B_J\in\mathrm{Diff}^{\mu_J}_{\mathrm{Fuchs}}(X)$ for $1\le J\le K$, $\mu>\max_{1\le J\le K}\mu_J$, and $F\in C^\infty_R(X\times\mathbb{R}^K)\ (=C^\infty(\mathbb{R}^K;C^\infty_R(X)))$. Here R is a certain asymptotic type (see below) and $C^\infty_R(X)$ is the space of C^∞ functions on X° admitting an asymptotic expansion as $x\to\partial X$ the data of which are prescribed by R.

$A\in\mathrm{Diff}^{\mu}_{\mathrm{Fuchs}}(X)$ means that $A\in\mathrm{Diff}^{\mu}(X^\circ)$ and in a collar neighborhood $\mathcal{U}$ of ∂X, $\mathcal{U}\xrightarrow{\cong}[0,1)\times Y$, $x\mapsto(t,y)$ (so that $Y\approx\partial X$), A takes the form

$$A=t^{-\mu}\sum_{j=0}^{\mu}a_j(t,y,D_y)\,(t\partial_t)^j\,, \tag{2}$$

where $a_j(t)=a_j(t,y,D_y)\in C^\infty([0,1);\mathrm{Diff}^{\mu-j}(Y))$ for $1\le j\le\mu$. The ellipticity of A then means that A is elliptic on X° and, in addition,

$$\sum_{j=0}^{\mu}\sigma^{\mu-j}(a_j(t))(y,\eta)(i\tilde\tau)^j\neq0\ \text{ for all } (t,y,\tilde\tau,\eta)\in\tilde T^*([0,1)\times Y)\setminus0,$$

where $\sigma^{\mu-j}(a_j(t))$ denotes the principal symbol of $a_j(t)$, $\tilde T^*([0,1)\times Y)$ is the compressed cotangent bundle of $[0,1)\times Y$, and $\tilde\tau=t\tau$, τ being the covariable to t.

A typical result reads as follows:

Theorem 1. *Let A be elliptic in the above sense. Further let u be a solution to (1) in the class of extendible distributions such that $B_Ju\in L^\infty(X)$ for all $1\le J\le K$. Then, under some natural technical assumptions, there exists an asymptotic type P such that $u\in C^\infty_P(X)$. Furthermore, P can be expressed in terms of A, $B_1,\ldots,B_K$, R, and the conormal order (= order of flatness, in an L^2 sense) of u close to ∂X.*

Asymptotic expansions for solutions $u=u(x)$ to (1) are of the form

$$u(x)\sim\sum_{j=0}^{\infty}\sum_{k=0}^{m_j}t^{-p_j}\log^k t\,c_{jk}(y)\ \text{ as } t\to+0,$$

where $p_j \in \mathbb{C}$, $\mathrm{Re}\, p_j \to -\infty$ as $j \to \infty$, $m_j \in \mathbf{N}$, and the coefficients $c_{jk}(y)$ vary in a finite-dimensional subspace $L_j \subset C^\infty(Y)$. In the notion of asymptotic type introduced by B.-W. Schulze [3, 4], the sequence $\{(p_j, m_j, L_j)\}_{j=0}^\infty$ is regarded as constituting an asymptotic type.

Results of the sort of Theorem 1 rely on a *refined notion of asymptotic type* in which linear relations between the various coefficients $c_{jk} \in L_j$, even for different j, are allowed. Whereas in the former notion of asymptotic type only the aspect of the "production" of asymptotics is emphasized, now also the aspect of their "annihilation" is underscored.

The basic technical question to answer is *what kind of linear relation* between the coefficients c_{jk} is admissible [2, 6]. The resulting notion of asymptotic type both turns out to be coordinate-invariant [2] and seems to admit adequate pseudodifferential calculi [1, 5].

These results were jointly obtained with Liu Xiaochun, Wuhan University, China.

References

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