数理解析研究所講究録 1211 巻 2001 年 34-43

Edge Sobolev Spaces, Weakly Hyperbolic Equations, and Branching of Singularities

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Summary

Edge Sobolev spaces are proposed as a main new tool for the investigation of weakly hyperbolic equations. The well-posedness of the linear and the semilinear Cauchy problem in the class of such edge Sobolev spaces is proved. Applications to the propagation of singularities for solutions to semilinear problems are considered.

1 Introduction

We consider the two semilinear Cauchy problems

$$Lu = f(u), \qquad (\partial_t^j u)(0,x) = u_j(x), \quad j = 0,1, \tag{1.1}$$

$$Lu = f(u, \partial_t u, t^{l_*}\nabla_x u), \qquad (\partial_t^j u)(0,x) = u_j(x), \quad j = 0,1, \tag{1.2}$$

where L is the weakly hyperbolic operator

$$\begin{aligned} L = \partial_t^2 + 2\sum_{j=1}^{n} \lambda(t) c_j(t) \partial_t \partial_{x_j} - \sum_{i,j=1}^{n} \lambda(t)^2 a_{ij}(t) \partial_{x_i} \partial_{x_j} \\ + \sum_{j=1}^{n} \lambda'(t) b_j(t) \partial_{x_j} + c_0(t) \partial_t \end{aligned} \tag{1.3}$$

with coefficients a_{ij}, b_j, c_j belonging to $C^\infty([-T_0, T_0], \mathbb{R})$ and $\lambda(t) = t^{l_*}$ with some $l_* \in \mathbb{N}_+ = \{1, 2, 3, \ldots\}$.

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The variables t and x satisfy $(t,x) \in [0,T_0] \times \mathbb{R}^n$; in the end of this paper we will also consider the case $(t,x) \in [-T_0,T_0] \times \mathbb{R}^n$. The operator L is supposed to be weakly hyperbolic with degeneracy for $t=0$ only, i.e.,

$$\left(\sum_{j=1}^{n} c_j(t)\xi_j\right)^2 + \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j \geq \alpha_0|\xi|^2, \quad \alpha_0 > 0, \quad \forall (t,\xi).$$

The choice of the exponents of t in (1.3) reflects so-called Levi conditions which are necessary and sufficient conditions for the C^∞ well-posedness of the *linear* Cauchy problem, see [8], [10]. If, for instance, the t-exponent of the coefficient of ∂_{x_j} were less than l_*-1, the linear Cauchy problem for that L would be well-posed only in certain Gevrey spaces, see [14].
We list some known results. The Cauchy problems (1.1), (1.2) are locally well-posed in $C^k([0,T],H^s(\mathbb{R}^n))$ for s large enough ([9], [10]) and $C^k([0,T],C^\infty(\mathbb{R}^n))$ ([1], [2]).
Furthermore, singularities of the initial data may propagate in an astonishing way: in [11], it has been shown that the solution $v=v(t,x)$ of

$$Lv = v_{tt} - t^2 v_{xx} - (4m+1)v_x = 0, \quad m \in \mathbb{N}, \tag{1.4}$$

with initial data $v(0,x)=u_0(x)$, $v_t(0,x)=0$ is given by

$$v(t,x) = \sum_{j=0}^{m} C_{jm} t^{2j} (\partial_x^j u_0)(x+t^2/2), \quad C_{jm} \neq 0. \tag{1.5}$$

This shows that singularities of u_0 propagate only to the left.
Taniguchi and Tozaki discovered branching phenomena for similar operators in [15]. They have studied the Cauchy problem

$$v_{tt} - t^{2l_*} v_{xx} - b l_* t^{l_*-1} v_x = 0, \quad (\partial_t^j v)(-1,x) = u_j(x), \quad j=0,1,$$

and assumed that the initial data have a singularity at some point x_0. Since the equation is strictly hyperbolic for $t<0$, this singularity propagates, in general, along each of the two characteristic curves starting at $(-1,x_0)$. When these characteristic curves cross the line $t=0$, they split, and the singularities then propagate along four characteristics for $t>0$. However, in certain cases, determined by a discrete set of values for b, one or two of these four characteristic curves do not carry any singularities.
The function spaces $C^k([0,T],H^s(\mathbb{R}^n))$ and $C^k([0,T],C^\infty(\mathbb{R}^n))$, for which local well-posedness could be proved, have the disadvantage that their elements have different smoothness with respect to t and x. We do not know

any previous result concerning the weakly hyperbolic Cauchy problem stating that solutions belong to a function space that embeds into the Sobolev spaces $H^s_{\text{loc}}((0,T) \times \mathbf{R}^n)$, for some $s \in \mathbf{R}$, under the assumption that the initial data and the right-hand side belong to appropriate function spaces of the same kind.

In this paper, solutions to (1.1) and (1.2) are sought in edge Sobolev spaces, a concept which has been initially invented in the analysis of elliptic pseudodifferential equations near edges, see [7], [13].

The operator L can be written as $L = t^{-\mu}P(t, t\partial_t, \Lambda(t)\partial_x)$, where $\Lambda(t) = \int_0^t \lambda(t')\, dt'$ and $P(t, \tau, \xi)$ is a polynomial in τ, ξ of degree $\mu = 2$ with coefficients depending on t smoothly up to $t = 0$. Operators with such a structure arise in the investigation of edge pseudodifferential problems on manifolds with cuspidal edges, where cusps are described by means of the function $\lambda(t)$. The singularity of the manifold requires the use of adapted classes of Sobolev spaces, so–called *edge Sobolev spaces.*

We shall define edge Sobolev spaces $H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n)$, where $s \geq 0$ denotes the Sobolev smoothness with respect to (t, x) for $t > 0$ and $\delta \in \mathbf{R}$ is an additional parameter. More precisely, we have continuous embeddings

$$\begin{aligned} H^s_{\text{comp}}(\mathbf{R}_+ \times \mathbf{R}^n)\big|_{(0,T)\times\mathbf{R}^n} \subset H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n) \\ \subset H^s_{\text{loc}}(\mathbf{R}_+ \times \mathbf{R}^n)\big|_{(0,T)\times\mathbf{R}^n}. \end{aligned}$$

The elements of the spaces $H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n)$ have different Sobolev smoothness at $t = 0$ in the following sense: There are traces τ_j, $\tau_j u(x) = (\partial_t^j u)(0, x)$, with continuous mappings

$$\tau_j : H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n) \to H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbf{R}^n), \quad \beta = \frac{1}{l_* + 1}$$

for all $j \in \mathbf{N}$, $j < s - 1/2$. This reflects the loss of Sobolev regularity observed when passing from the Cauchy data at $t = 0$ to the solution. Namely, (1.5) shows that $u_0 \in H^{s+m}(\mathbf{R})$ implies $v(t, .) \in H^s(\mathbf{R})$ only, since $C_{mm} \neq 0$.

This phenomenon has consequences for the investigation of the nonlinear problems (1.1), (1.2). The usual iteration procedure giving the existence of solutions for small times cannot be applied in the case of the standard function space $C([0,T], H^s(\mathbf{R}^n))$, since we have no longer a mapping which maps this Banach space into itself.

However, it turns out, that the iteration approach is applicable if we employ the specially chosen edge Sobolev spaces $H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n)$. Roughly speaking, the iteration algorithm does not feel the loss of regularity, because it has

been absorbed in the function spaces. The idea to choose a special function space adapted to the weakly hyperbolic operator has also been used in [3], [4], and [12].
Our results are the following. We claim the $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ well–posedness of (1.1) and (1.2). In Section 4, we consider the hyperbolic equation from (1.1), but with data prescribed at $t=-T_0$, and show that the strongest singularities of the solution u propagate in the same way as the singularities of the solution v solving $Lv=0$ and having the same initial data as u for $t=-T$. The propagation of the singularities of v was discussed in [15]. The proofs of the results mentioned here can be found in [5] and [6].

2 Edge Sobolev Spaces

Details on the abstract approach to edge Sobolev spaces can be found, e.g., in [7], [13]. Proofs of the results listed here are given in [5].

2.1 Weighted Sobolev Spaces on $\mathbb{R}_+$

We say that $u=u(t)\in\mathcal{H}^{s,\delta}(\mathbb{R}_+)$, $s\in\mathbb{N}$, $\delta\in\mathbb{R}$, if

$$\|u\|^2_{\mathcal{H}^{s,\delta}(\mathbb{R}_+)}=\sum_{k=0}^{s}\int_0^\infty\left|t^{-\delta}(t\partial_t)^k u(t)\right|^2\,dt<\infty.$$

For arbitrary $s,\delta\in\mathbb{R}$, this Mellin Sobolev space $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$ can be defined by means of interpolation and duality, or by the requirement that

$$\|u\|^2_{\mathcal{H}^{s,\delta}(\mathbb{R}_+)}=\frac{1}{2\pi i}\int_{\operatorname{Re}z=1/2-\delta}\langle z\rangle^{2s}|Mu(z)|^2\,dz<\infty,$$

where $Mu(z)=\int_0^\infty t^{z-1}u(t)\,dt$ denotes the Mellin transform. (Both norms coincide if $s\in\mathbb{N}$.) Furthermore, the space $C^\infty_{\text{comp}}(\mathbb{R}_+)$ is dense in $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$.
We introduce the notations

$$H^s(\mathbb{R}_+\times\mathbb{R}^n)=\left\{v\big|_{\mathbb{R}_+\times\mathbb{R}^n}:\ v\in H^s(\mathbb{R}^{1+n})\right\},\quad n\geq 0,$$
$$H^s_0(\overline{\mathbb{R}}_+\times\mathbb{R}^n)=\left\{v\in H^s(\mathbb{R}^{1+n}):\ \operatorname{supp}v\subseteq\overline{\mathbb{R}}_+\times\mathbb{R}^n\right\},\quad n\geq 0,$$
$$\mathcal{S}(\overline{\mathbb{R}}_+\times\mathbb{R}^n)=\left\{v\big|_{\mathbb{R}_+\times\mathbb{R}^n}:\ v\in\mathcal{S}(\mathbb{R}^{1+n}\right\},\quad n\geq 0.$$

Example 2.1. For $s\geq 0$, $H^s_0(\overline{\mathbb{R}}_+)=\mathcal{H}^{0,0}(\mathbb{R}_+)\cap\mathcal{H}^{s,s}(\mathbb{R}_+)$.

Definition 2.2. Let $s\geq 0$, $\delta\in\mathbb{R}$ and $\omega\in C^\infty(\overline{\mathbb{R}}_+)$ be a cut–off function close to $t=0$, i.e., $\operatorname{supp}\omega$ is bounded and $\omega(t)=1$ for t close to 0. Then the *cone Sobolev spaces* $H^{s,\delta;\lambda}(\mathbb{R}_+)$, $H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+)$ are defined by

$$H^{s,\delta;\lambda}(\mathbb{R}_+)=\{\omega u_0+(1-\omega)u_1:\ u_0\in H^s(\mathbb{R}_+),\ u_1\in\mathcal{H}^{s,\delta;\lambda}_{\sharp}(\mathbb{R}_+)\},$$

$$H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+) = \{\omega u_0 + (1-\omega)u_1\colon\ u_0 \in H_0^s(\overline{\mathbb{R}}_+),\ u_1 \in \mathcal{H}_\sharp^{s,\delta;\lambda}(\mathbb{R}_+)\},$$

where $\mathcal{H}_\sharp^{s,\delta;\lambda}(\mathbb{R}_+) = \mathcal{H}^{0,\delta l_*}(\mathbb{R}_+) \cap \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)$. The space $H^{s,\delta;\lambda}(\mathbb{R}_+)$ is equipped with the norm

$$\|u\|^2_{H^{s,\delta;\lambda}(\mathbb{R}_+)} = \|\omega u_0\|^2_{H^s(\mathbb{R}_+)} + \|(1-\omega)u_1\|^2_{\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\cap\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)}\,.$$

2.2 The Spaces $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$

Definition 2.3. Let E be a Hilbert space and $\{\kappa_\nu\}_{\nu>0}$ be a strongly continuous group of isomorphisms acting on E with $\kappa_\nu\kappa_{\nu'} = \kappa_{\nu\nu'}$ for ν, $\nu' > 0$ and $\kappa_1 = \mathrm{id}_E$. For $s \in \mathbb{R}$, the abstract edge Sobolev space $\mathcal{W}^s\big(\mathbb{R}^n; (E, \{\kappa_\nu\}_{\nu>0})\big)$ consists of all $u \in \mathcal{S}'(\mathbb{R}^n; E)$ such that $\hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^n; E)$ and

$$\|u\|^2_{\mathcal{W}^s\left(\mathbb{R}^n;(E,\{\kappa_\nu\}_{\nu>0})\right)} = \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \|\kappa^{-1}_{\langle\xi\rangle}\hat{u}(\xi)\|^2_E\, d\xi < \infty.$$

Definition 2.4. Let $s \geq 0$, $\delta \in \mathbb{R}$. Then we define the group $\{\kappa_\nu^{(\delta)}\}_{\nu>0}$ by

$$\kappa_\nu^{(\delta)} w(t) = \nu^{\beta/2-\beta\delta l_*} w(\nu^\beta t),\quad \nu > 0,$$

where $\beta = 1/(l_* + 1)$, and set

$$H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) = \mathcal{W}^s\big(\mathbb{R}^n; (H^{s,\delta;\lambda}(\mathbb{R}_+), \{\kappa_\nu^{(\delta)}\}_{\nu>0})\big),$$
$$H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) = \mathcal{W}^s\big(\mathbb{R}^n; (H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+), \{\kappa_\nu^{(\delta)}\}_{\nu>0})\big).$$

Proposition 2.5. **(a)** *$\mathcal{S}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ is dense in $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$.*

(b) *For every fixed $\delta \in \mathbb{R}$, $\{H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)\colon\ s \geq 0\}$ forms an interpolation scale with respect to the complex interpolation method.*

(c) *If $l_* = 0$, then $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) = H^s(\mathbb{R}_+ \times \mathbb{R}^n)$.*

(d) *We have the continuous embeddings*

$$H^s_{\mathrm{comp}}(\mathbb{R}_+ \times \mathbb{R}^n) \subset H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \subset H^s_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^n).$$

The spaces $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ admit traces at $t = 0$ in the following sense.

Proposition 2.6. *Let $s \geq 0$, $\delta \in \mathbb{R}$. Then, for each $j \in \mathbb{N}$, $j < s - 1/2$, the map $\mathcal{S}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $u \mapsto (\partial_t^j u)(0, x)$, extends by continuity to a map*

$$\tau_j\colon H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n).$$

Furthermore, we have a surjective map

$$H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to \prod_{j<s-1/2} H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n),\quad u \mapsto \{\tau_j u\}_{j<s-1/2}\,.$$

Proposition 2.7. *For $s \geq 0$, $\delta \in \mathbf{R}$, the following maps are continuous:*
(a) $\partial_t \colon H^{s+1,\delta;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n) \to H^{s,\delta+1;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n)$;
(b) $t^l \colon H^{s,\delta;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n) \to H^{s,\delta+l/l_*;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n)$ *for* $l = 0, 1, \ldots, l_*$;
(c) $\partial_{x_j} \colon H^{s+1,\delta;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n) \to H^{s,\delta;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n)$ *for* $1 \leq j \leq n$;
(d) $\varphi \colon H^{s,\delta;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n) \to H^{s,\delta;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n)$ *for each* $\varphi = \varphi(t) \in \mathcal{S}(\overline{\mathbf{R}}_+)$.
Here t^l means the operator of multiplication by t^l. Similarly for φ.

2.3 The Spaces $H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$

For $T > 0$, we set $H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n) = H^{s,\delta;\lambda}(\mathbf{R}_+ \times \mathbf{R}^n)\big|_{(0,T)\times\mathbf{R}^n}$ and equip this space with its infimum norm. There is an alternative description of this space provided that $s \in \mathbf{N}$.

Lemma 2.8. *Let $s \in \mathbf{N}$, $\delta \in \mathbf{R}$ and $T > 0$. Then the infimum norm of the space $H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n)$ is equivalent to the norm $\|.\|_{s,\delta;T}$, where*

$$\|u\|^2_{s,\delta;T} = \sum_{l=0}^{s} T^{2l-1} \int_0^T \int_{\mathbf{R}^n_\xi} \vartheta_l(t,\xi)^2 \left|\partial_t^l \hat{u}(t,\xi)\right|^2 d\xi\, dt,$$

$$\vartheta_l(t,\xi) = \begin{cases} \langle \xi \rangle^{s-l} \lambda(t_\xi)^{-\delta-l} & : 0 \leq t \leq t_\xi, \\ \langle \xi \rangle^{s-l} \lambda(t)^{-\delta-l} & : t_\xi \leq t \leq T. \end{cases}$$

Here we have introduced the notation $t_\xi = \langle \xi \rangle^{-\beta}$, $\beta = 1/(l_* + 1)$.

Lemma 2.9. *For s, $s' \geq 0$, δ, $\delta' \in \mathbf{R}$, and $T > 0$,*

$$H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n) \subseteq H^{s',\delta';\lambda}((0,T) \times \mathbf{R}^n)$$

if and only if $s \geq s'$, $\quad s + \beta\delta l_* \geq s' + \beta\delta' l_*$.

The two conditions on s are related to the fact that the elements of the edge Sobolev spaces have different smoothness for $t > 0$ and $t = 0$, respectively. The following result provides a criterion when the superposition operators defined by the right–hand sides of the hyperbolic equations in (1.1) and (1.2) map an edge Sobolev space into itself. This result is related to the fact that the usual Sobolev spaces are Banach algebras for sufficiently high smoothness.

Proposition 2.10. *Let $f = f(u)$ be an entire function with $f(0) = 0$, i.e., $f(u) = \sum_{j=1}^\infty f_j u^j$ for all $u \in \mathbf{R}$. Assume that $\lfloor s \rfloor + \delta \geq 0$ and $\min\{\lfloor s \rfloor, \lfloor s \rfloor + \beta\delta l_*\} > (n+2)/2$. Then there is, for each $R > 0$, a constant $C_1(R)$ with the property that*

$$\|f(u)\|_{s,\delta;T} \leq C_1(R) \|u\|_{s,\delta;T}, \quad \|f(u) - f(v)\|_{s,\delta;T} \leq C_1(R) \|u - v\|_{s,\delta;T}$$

provided that $u, v \in H^{s,\delta;\lambda}((0,T) \times \mathbf{R}^n)$ and $\|u\|_{s,\delta;T} \leq R$, $\|v\|_{s,\delta;T} \leq R$.

3 Linear and Semilinear Cauchy Problems

Our considerations start with the linear Cauchy problem

$$Lw(t,x) = f(t,x), \qquad (\partial_t^j w)(0,x) = w_j(x), \quad j = 0, 1. \tag{3.1}$$

We introduce the number

$$Q_0 = -\frac{1}{2} + \sup_\xi \frac{|\sum_j(-b_j(0)\xi_j + c_j(0)\xi_j)|}{2\sqrt{(\sum_j c_j(0)\xi_j)^2 + \sum_{i,j} a_{ij}(0)\xi_i\xi_j}}, \tag{3.2}$$

and fix $A_0 = Q_0 l_*/(l_* + 1) = \beta Q_0 l_*$.

Theorem 3.1. *Let s, $Q \in \mathbf{R}$, $s \geq 1$, $Q \geq Q_0$. Further let $w_0 \in H^{s+A}(\mathbf{R}^n)$, $w_1 \in H^{s+A-\beta}(\mathbf{R}^n)$, and $f \in H^{s-1,Q+1;\lambda}((0,T) \times \mathbf{R}^n)$, where $A = \beta Q l_*$. Then there is a solution $w \in H^{s,Q;\lambda}((0,T) \times \mathbf{R}^n)$ to* (3.1). *Moreover, the solution w is unique in the space $H^{s,Q_0;\lambda}((0,T) \times \mathbf{R}^n)$.*

Remark 3.2. The parameter A_0 describes the loss of regularity. The explicit representations of the solutions for special model operators in [11] and [15] show that the statement of the Theorem becomes false if $A < A_0$.

Theorem 3.3. *Let $s \in \mathbf{N}$ and assume that* $\min\{s, s + \beta Q_0 l_*\} > (n+2)/2$, *where Q_0 be the number from* (3.2). *Suppose that $f = f(u)$ is an entire function with $f(0) = 0$. Let $Q \geq Q_0$ and $A = \beta Q l_*$. Then, for $u_0 \in H^{s+A}(\mathbf{R}^n)$, $u_1 \in H^{s+A-\beta}(\mathbf{R}^n)$, there is a number $T > 0$ with the property that a solution $u \in H^{s,Q;\lambda}((0,T) \times \mathbf{R}^n)$ to the Cauchy problem* (1.1) *exists. This solution u is unique in the space $H^{s,Q_0;\lambda}((0,T) \times \mathbf{R}^n)$.*

Theorem 3.4. *Let $s \in \mathbf{N}$ and assume that $s - 1 > (n+2)/2$. Suppose that $f = f(u, v, v_1, \dots, v_n)$ is entire with $f(0, \dots, 0) = 0$. Let $Q \geq Q_0$ and $A = \beta Q l_*$. Then, for $u_0 \in H^{s+A}(\mathbf{R}^n)$, $u_1 \in H^{s+A-\beta}(\mathbf{R}^n)$, there is a number $T > 0$ with the property that a solution $u \in H^{s,Q;\lambda}((0,T) \times \mathbf{R}^n)$ to the Cauchy problem* (1.2) *exists. This solution u is unique in the space $H^{s,Q_0;\lambda}((0,T) \times \mathbf{R}^n)$.*

Eventually, we state a result concerning the propagation of mild singularities.

Theorem 3.5. *Let s satisfy the assumptions of* Theorem 3.3. *Assume $u_0 \in H^{s+\beta Q_0 l_*}(\mathbf{R}^n)$, $u_1 \in H^{s+\beta Q_0 l_* - \beta}(\mathbf{R}^n)$, where Q_0 is given by* (3.2). *Let v be the solution to*

$$Lv = 0, \quad (\partial_t^j v)(0,x) = u_j(x), \quad j = 0, 1. \tag{3.3}$$

Then the solutions $u, v \in H^{s,Q_0;\lambda}((0,T) \times \mathbf{R}^n)$ to (1.1) *and* (3.3) *satisfy*

$$u - v \in H^{s+\beta,Q_0;\lambda}((0,T) \times \mathbf{R}^n).$$

Example 3.6. Consider Qi Min–You's operator L from (1.4). Then $l_* = 1$, $\beta = 1/2$, and $Q_0 = 2m$. Theorems 3.1, 3.3, and 3.5 state that the solutions u, v to (1.1), (3.3) satisfy

$$u,\ v \in H^{s,2m;\lambda}((0,T)\times\mathbb{R}), \qquad u-v \in H^{s+1/2,2m;\lambda}((0,T)\times\mathbb{R})$$

if $u_0 \in H^{s+m}(\mathbb{R})$, $u_1 \in H^{s+m-1/2}(\mathbb{R})$. Proposition 2.5 then implies

$$u,\ v \in H^s_{\mathrm{loc}}((0,T)\times\mathbb{R}), \qquad u-v \in H^{s+1/2}_{\mathrm{loc}}((0,T)\times\mathbb{R}).$$

We find that the strongest singularities of u coincide with the singularities of v. The latter can be looked up in (1.5) in case $u_1 \equiv 0$.

4 Branching Phenomena for Solutions to Semilinear Equations

In this section, we consider the Cauchy problems

$$Lu = f(u), \qquad (\partial_t^j u)(-T_0,x) = \varepsilon w_j(x), \quad j=0,1, \tag{4.1}$$
$$Lv = 0, \qquad (\partial_t^j v)(-T_0,x) = \varepsilon w_j(x), \quad j=0,1, \tag{4.2}$$

with L from (1.3), and we are interested in branching phenomena for singularities of the solution u. Our main result is Theorem 4.2.
We know, e.g., from the example of Qi Min–You that we have to expect a loss of regularity when we pass from the Cauchy data at $\{t=0\}$ to the solution at $\{t\neq 0\}$. However, we *also* will observe a loss of smoothness if we prescribe Cauchy data at, say, $t=-T_0$ and look at the solution for $t=0$.

Definition 4.1. Let $s\geq 0$, $\delta\in\mathbb{R}$. We say that $u\in H^{s,\delta;\lambda}((-T,T)\times\mathbb{R}^n)$ if $u(t,x)\in H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$, $u(-t,x)\in H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$, and $u(t,x)-u(-t,x)\in H_0^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$.
Let $s_-, s_+ \geq 0$, $\delta_-,\delta_+\in\mathbb{R}$ and suppose that $s_- + \beta\delta_- l_* = s_+ + \beta\delta_+ l_*$, $s_+ \leq s_-$. We say that $u\in H^{s_-,s_+,\delta_-,\delta_+;\lambda}((-T,T)\times\mathbb{R}^n)$ if $u\in H^{s_+,\delta_+;\lambda}((-T,T)\times\mathbb{R}^n)$ and $u(-t,x)\in H^{s_-,\delta_-;\lambda}((0,T)\times\mathbb{R}^n)$. We define the norm by

$$\begin{aligned}&\|u(t,x)\|_{H^{s_-,s_+,\delta_-,\delta_+;\lambda}((-T,T)\times\mathbb{R}^n)}\\&\quad=\|u(t,x)\|_{H^{s_+,\delta_+;\lambda}((0,T)\times\mathbb{R}^n)}+\|u(-t,x)\|_{H^{s_-,\delta_-;\lambda}((0,T)\times\mathbb{R}^n)}\,.\end{aligned}$$

This choice of the norm is possible, since $H^{s_-,\delta_-;\lambda}((0,T)\times\mathbb{R}^n) \subset H^{s_+,\delta_+;\lambda}((0,T)\times\mathbb{R}^n)$, compare Lemma 2.9.

The next theorem relates branching phenomena for the semilinear problem (4.1) with branching phenomena for the linear reference problem (4.2). This relation between a semilinear Cauchy problem and an associated linear reference problem has already been discussed in Example 3.6. The explicit representations of solutions in [15] show that the statements about the smoothness of solutions in the following theorem are optimal.

Theorem 4.2. *Let L be the operator from* (1.3), *Q_0 be the number from* (3.2), *and suppose that* $\min\{\lfloor s_\pm \rfloor, \lfloor s_\pm \rfloor + \beta Q_\pm l_*\} > (n+2)/2$, $\lfloor s_\pm \rfloor + Q_\pm \geq 0$, $s_\pm \geq 1$, *where* $Q_+ = Q_0$, $Q_- = -1 - Q_0$, *and* $s_+ = s_- + \beta Q_- l_* - \beta Q_+ l_*$.
Assume that $w_0 \in H^{s_-}(\mathbf{R}^n)$, $w_1 \in H^{s_- -1}(\mathbf{R}^n)$, *and that* $f = f(u)$ *is an entire function with* $f(0) = f'(0) = 0$.
Then there is an $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon \leq \varepsilon_0$ *there are unique solutions* $u, v \in H^{s_-, s_+, Q_-, Q_+; \lambda}((-T_0, T_0) \times \mathbf{R}^n)$ *to* (4.1) *and* (4.2), *respectively, which, in addition, satisfy*

$$u - v \in H^{s_- + \beta, s_+ + \beta, Q_-, Q_+; \lambda}((-T_0, T_0) \times \mathbf{R}^n). \tag{4.3}$$

Remark 4.3. Due to (3.2), $Q_0 \geq -1/2$, which is equivalent to $s_+ \leq s_-$. If $s_+ = s_-$, no loss of regularity occurs when we cross the line of degeneracy. The case of a linear hyperbolic operator with this property and countably many points of degeneracy (or singularity) accumulating at $t = 0$ has been discussed in [16].

References

[1] F. Colombini, E. Jannelli, and S. Spagnolo. Well–posedness in the Gevrey classes of the Cauchy problem for a non–strictly hyperbolic equation with coefficients depending on time. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 10:291–312, 1983.

[2] F. Colombini and S. Spagnolo. An example of a weakly hyperbolic Cauchy problem not well posed in C^∞. *Acta Math.*, 148:243–253, 1982.

[3] M. Dreher. Weakly hyperbolic equations, Sobolev spaces of variable order, and propagation of singularities. *submitted.*

[4] M. Dreher and M. Reissig. Propagation of mild singularities for semilinear weakly hyperbolic differential equations. *J. Analyse Math.*, 82:233-266, 2000.

[5] M. Dreher and I. Witt. Edge Sobolev spaces and weakly hyperbolic equations. *submitted to Ann. Mat. Pura Appl. (4).*

[6] M. Dreher and I. Witt. Edge Sobolev spaces, weakly hyperbolic equations, and branching of singularities. *to appear in the Proceedings of "Equations aux derivees partielles et physique mathematique", Paris 2000*

[7] Y. Egorov and B.-W. Schulze. *Pseudo-Differential Operators, Singularities, Applications*, volume 93 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1997.

[8] V. Ivrii and V. Petkov. Necessary conditions for the Cauchy problem for non–strictly hyperbolic equations to be well–posed. *Russian Math. Surveys*, 29(5):1–70, 1974.

[9] K. Kajitani and K. Yagdjian. Quasilinear hyperbolic operators with the characteristics of variable multiplicity. *Tsukuba J. Math.*, 22(1):49–85, 1998.

[10] O. Oleinik. On the Cauchy problem for weakly hyperbolic equations. *Comm. Pure Appl. Math.*, 23:569–586, 1970.

[11] M.-Y. Qi. On the Cauchy problem for a class of hyperbolic equations with initial data on the parabolic degenerating line. *Acta Math. Sinica*, 8:521–529, 1958.

[12] M. Reissig and K. Yagdjian. Weakly hyperbolic equations with fast oscillating coefficients. *Osaka J. Math.*, 36(2):437–464, 1999.

[13] B.-W. Schulze. *Boundary Value Problems and Singular Pseudo-differential Operators.* Wiley Ser. Pure Appl. Math. J. Wiley, Chichester, 1998.

[14] K. Shinkai. Stokes multipliers and a weakly hyperbolic operator. *Comm. Partial Differential Equations*, 16(4,5):667–682, 1991.

[15] K. Taniguchi and Y. Tozaki. A hyperbolic equation with double characteristics which has a solution with branching singularities. *Math. Japon.*, 25(3):279–300, 1980.

[16] T. Yamazaki. Unique existence of evolution equations of hyperbolic type with countably many singular or degenerate points. *J. Differential Equations*, 77:38–72, 1989.