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Mild Solutions to the discrete Boltzmann equation with linear and quadratic terms for the initial data with locally finite entropy

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1. Introduction

We study the discrete velocity models of the Boltzmann equation in one space dimension. These models describe the motion of particles in a rarefied gas. To observe the evolution of particles in a thin infinite tube, we take into account both collisions between particles and reflection over the inner wall of tube, which are represented by quadratic terms and a linear terms respectively. The discrete velocity models consist in discretizing the velocity $v\in\mathbf{R}^{3}$ and then the velocity of particles in the models can be taken only in a finite set of $\{C_{i}\in\mathbf{R}^{3};i\in I\}$. Let the variable x be the direction of the axis of tube and the variables y and z be transversal to the axis. The thinness of the tube enables us to suppose that the behavior of the particles is homogeneous and uniform with respect to the variables y and z. The distribution of the particles with velocity C_{i} is then represented by a function $u_{i}(x, t)$. Denoting the $x-$component of C_{i} by $c_{i}\in\mathbf{R}$, we have the system of the hyperbolic partial differential equations which describes our models :

$$(1.1)\qquad \left\{\begin{array}{ll}\dfrac{\partial u_{i}}{\partial t}+c_{i}\dfrac{\partial u_{i}}{\partial x} & =Q_{i}(u) ,\\ u_{i}|_{t=0} & =u_{i}^{0}(x) , \quad i\in I,\end{array}\right.$$

where $Q_{i}(u)$ and $L_{i}(u)$ represent the terms of binary collisions and the ones of linear reflection respectively. These terms are in the form of :

$$(1.2)\qquad Q_{i}(u)=\sum_{j,k,\ell\in I}(A_{ij}^{k\ell}u_{k}u_{\ell}-A_{k\ell}^{ij}u_{i}u_{j}) ,$$

$$(1.3)\qquad L_{i}(u)=\sum_{k\in I}(\alpha_{i}^{k}u_{k}-\alpha_{k}^{i}u_{i}) ,$$

where the constants satisfy

$$A_{k\ell}^{ij} = A_{k\ell}^{ji} = A_{\ell k}^{ij} \geq 0, \quad A_{k\ell}^{ii} = 0 \quad \alpha_k^i \geq 0. \tag{1.4}$$

In this talk, we prove the time global existence and the uniqueness of the solutions $(u_i)_{i \in I}$ to the system (1.1) for initial data which are not necessary bounded but with locally finite entropy, provided that the distinct velocity assumption :

$$i \neq j \Rightarrow c_i \neq c_j \ , \tag{dv}$$

and the weak microreversibility condition :

$$\sum_{k,\ell \in I} A_{k\ell}^{ij} = \sum_{k,\ell \in I} A_{ij}^{k\ell} \quad \text{for} \quad \forall i,j \in I. \tag{μr}$$

The condition (μr) is weaker than the usual microreversibility condition :

$$A_{ij}^{k\ell} = A_{k\ell}^{ij} \quad \text{for} \quad \forall i,j,k,\ell \in I. \tag{1.5}$$

It is worthy to remark that, in the mesonic process $h_\nu + P \longrightarrow N + \psi^+$, the condition (1.5) is violated but the condition (μr) is satisfied ([3,4]).

For the bounded data, we [7] obtained more precise results without condition (dv), which show the existence of solution [resp. locally] bounded and global in time for data positive and [resp. locally] bounded. We [7,8] have moreover the explicit estimates of solutions for bounded data. For generalized Broadwell models, we [11] derive a more precise concrete estimates for bounded and summable data. Nevertheless, for merely summable data, it is necessary suppose the condition (dv) in order to define a solution to the initial problem in some sense which is weaker than the distribution sense, as we will see in Proposition 2.1.

In the case that the right hand side includes only the quadratic terms, Toscani [6] showed the global existence of solutions for the data bounded, summable with weight $(1+|x|)^\alpha (\alpha > 0)$ and with 'globally' finite entropy.

2. Bounded data

To consider the solutions to the initial value problem (1.1), we introduce a Banach space $\mathcal{B}(\mathbf{R} \times [0,T])$ and a Fréchet space $\mathcal{B}_{\ell oc}(\mathbf{R} \times [0,T])$ ($T < \infty$ fixed), the former being introduced by Toscani [6].

Definition.— *We denote by $\mathcal{B}(\mathbf{R} \times [0,T])$ [resp. $\mathcal{B}_{\ell oc}(\mathbf{R} \times [0,T])$] the Banach [resp. Fréchet] space of classes of measurable functions $u = (u_i(x,t))_{i \in I}$ defined on $\mathbf{R} \times [0,T]$ such that the following norm [resp. semi-norm] is finite :*

$$\|u\|_{\mathcal{B}} = \sum_{i \in I} \int_{\mathbf{R}} \operatorname*{ess\,sup}_{t \in [0,T]} \left| u_i^\sharp(x,t) \right| dx \ , u_i^\sharp(x,t) \stackrel{def}{=} u_i(x + c_i t, t), \tag{2.1}$$

$$[resp. \|u\|_{\mathcal{B},K} = \sum_{i\in I} \int_K \operatorname*{ess\,sup}_{t\in[0,T]} \left|u_i^\sharp(x,t)\right| dx \ , \forall K \text{ compact subset of } \mathbf{R}\ ].$$

Proposition 2.1.— *We assume the condition (dv). For $u \in \mathcal{B}(\mathbf{R}\times[0,T])$ [resp. $u \in \mathcal{B}_{\ell oc}(\mathbf{R}\times[0,T])$], we have $Q_i(u)$, $L_i(u) \in L^1(\mathbf{R}\times[0,T])$ [resp. $L^1_{\ell oc}(\mathbf{R}\times[0,T])$]. We denote $(Ku)_i(x,t) = \int_0^t (Q_i(u) + L_i(u))\,(x - c_i(t-s), s)\, ds$. Then we obtain*

$$\|Ku\|_{\mathcal{B}} \leq C^{st}\left(\|u\|_{\mathcal{B}}{}^2 + T\|u\|_{\mathcal{B}}\right) \ , \tag{2.2}$$

$$\|Ku - Kv\|_{\mathcal{B}} \leq C^{st}\|u-v\|_{\mathcal{B}}\,(\|u\|_{\mathcal{B}} + \|v\|_{\mathcal{B}} + T) \ . \tag{2.3}$$

Proof. It is crucial to suppose that the condition (dv) is verified.
We give a proof only for the case that $u \in \mathcal{B}(\mathbf{R}\times[0,T])$. Another case can be proved similary. We take $u \in \mathcal{B}(\mathbf{R}\times[0,T])$ and denote $U_i(x) = \operatorname{ess\,sup}_{t\in[0,T]}\left|u_i^\sharp(x,t)\right|$. Then, using $U_i \in L^1(\mathbf{R})$ and $\left|u_i^\sharp(x,t)\right| \leq U_i(x)$ for any $t \in [0,T]$, we have

$$\int_{\mathbf{R}}\int_0^T Q_i(u)dt\ dx \leq C^{st}\sum_{k\neq\ell}\|U_k\|_{L^1}\|U_\ell\|_{L^1} \leq C^{st}\,\|u\|^2_{\mathcal{B}} < \infty \ , \tag{2.4}$$

$$\int_{\mathbf{R}}\int_0^T L_i(u)dt\ dx \leq C^{st}T\,\|u\|_{\mathcal{B}} < \infty \ . \tag{2.5}$$

Therefore we obtain

$$\sum_i \int_{\mathbf{R}} \operatorname*{ess\,sup}_{t\in[0,T]} \left|(Ku)_i^\sharp(x,t)\right| dx \leq C^{st}\left(\|u\|_{\mathcal{B}}{}^2 + T\|u\|_{\mathcal{B}}\right) \ . \tag{2.6}$$

Similarly we have

$$\begin{aligned}&\sum_i \int_{\mathbf{R}} \operatorname*{ess\,sup}_{t\in[0,T]} \left|(Ku)_i^\sharp(x,t) - (Kv)_i^\sharp(x,t)\right| \\ &\leq C^{st}\|u-v\|_{\mathcal{B}}\,(\|u\|_{\mathcal{B}} + \|v\|_{\mathcal{B}} + T) \ . \ \blacksquare\end{aligned} \tag{2.7}$$

In order to prove the global existence for data with locally finite entropy, we define weak solution, so called mild solutions :

Definition.— *Let be $u_i^0 \in L^1_{\ell oc}(\mathbf{R})$ and $u \in \mathcal{B}_{\ell oc}(\mathbf{R}\times[0,T])$. We say that u is a mild solution of the initial value problem (1.1) if*

$$u_i^\sharp(x,t) = u_i^0(x) + \int_0^t \{Q_i^\sharp(u)(x,s) + L_i^\sharp(u)(x,s)\}ds \ . \tag{2.8}$$

Remark : For bounded functions $u = (u_i)$, there is an equivalence between the notion of mild solutions and one of solutions in the distribution sense.

We obtain, in a classical way, the local results for bounded data :

Proposition 2.2.— *We suppose that the data are bounded.*
i) (local existence) There exists a unique bounded solution at least up to the time $T_0 = C/\left(1 + \|u^0\|_{L^\infty}\right)$. *Furthermore, for* $t \le T_0$, *we have* $\|u(\cdot,t)\|_{L^\infty} \le C\|u^0\|_{L^\infty}$ *where a constant* C *depend only on the system.*
ii) (uniqueness) If we have two bounded solutions on the same interval $[0,T]$ *for the same initial data, then they coincide on this interval. We can then define the existence time* T^* *as supremum of* T *such that the solution exists and is bounded up to the time* T .
iii) (positivity) If the data are positive, the solution is also positive up to the time T^* .
iv) (conservation of the mass) If the data are positive and summable, we have

$$\int_{\mathbf{R}} \sum_{i\in I} u_i(x,t)dx = \int_{\mathbf{R}} \sum_{i\in I} u_i^0(x)dx \ , \tag{2.9}$$

for $\forall t \in [0,T^*[$.
v) (finite velocity propagation) If two data u^0 *and* v^0 *coincide in the interval* $[a,b]$, *then the solutions* u *and* v *coincide in the triangle or the trapezoid* $\{(x,t) \ : t \in [0,T_1[, T_1 \le T^*, a+\gamma t \le x \le b-\gamma t\}$ *where* $\gamma = \max_{i\in I}|c_i|$.
vi) (entropy with controlled increase) If we assume the condition (μr *), then, for positive data* u^0 *supported in* $[-R,R]$ *and with its finite entropy i.e.* $\sum_{i\in I}\int_{\mathbf{R}} u_i^0 \log u_i^0(x)dx < \infty$, *we have* $H(t) \le H(0) + C_* t$ *for* $t \in [0,T_1[$, $T_1 \le T^*$ *where* C_* *depends only on the system and* $H(t) = \sum_{i\in I}\int_{\mathbf{R}} u_i \log u_i(x,t)dx$.

Proof. The classical iteration method enables us to show i)-v). For details, we refer to [8].

On the increase of the entropy $H(t)$, noting that the support of the solutions $u(\cdot,t)$ with respect to x is contained in $[-R',R']$ with $R' = R+\gamma T_1$, we see that the quantity $H(t) = \sum_i \int_{\mathbf{R}} u_i \log u_i(x,t)dx$ is well-defined. It follows from the system that

$$\sum_i \left(\frac{\partial}{\partial t} + c_i \frac{\partial}{\partial x}\right) u_i \log u_i \ \le -\sum_{ik} \alpha_i^k u_k \log \frac{u_k}{u_i}. \tag{2.10}$$

Applying the Jensen's inequality to the convex function $x\log x$, we have

$$\begin{aligned} \int_{\mathbf{R}} u_k \log \frac{u_k}{u_i} \ dx \ &= \int_{\mathbf{R}} \frac{u_k}{u_i} \log \frac{u_k}{u_i} \cdot u_i \ dx \\ &\ge \int_{\mathbf{R}} u_k \ dx \cdot \log \int_{\mathbf{R}} u_k \ dx. \end{aligned} \tag{2.11}$$

Integrating the inequality (2.10) on $\mathbf{R}$ in x then between 0 and t in t, we have

$$\begin{aligned} H(t) - H(0) \ &\le -\sum_{ik} \alpha_i^k \int_0^t d\tau \int_{\mathbf{R}} u_k \ dx \cdot \log \int_{\mathbf{R}} u_k \ dx \\ &\le \frac{\sum_{ik} \alpha_i^k}{e} t \end{aligned} \tag{2.12}$$

because we have $x \log x \geq -\frac{1}{e}$. ∎

3. Local existence

We show the time local existence for data with small mass :

Theorem 3.1.— *We assume the condition (dv) . Then there exists a $\delta > 0$ such that, for $\|u^0\|_{L^1} \leq \delta$, there exists a unique solution u in $\mathcal{B}(\mathbf{R} \times [0,\delta])$, and we have $\|u\|_{\mathcal{B}} \leq 2\delta$. Furthermore, the mapping $G(t) \; : u^0 \mapsto u(\cdot,t)$ is continuous from the ball with radius δ of L^1 into the ball with radius 2δ of L^1 for $t \leq \delta$.*

Proof. We write the system in the form $u - Ku = f$, where $f_i = u_i^0(x - c_i t)$. We put $u_{\nu+1} = Ku_\nu + f$, $u_0 = 0$. By virtue of Proposition 2.1, for a sufficiently small δ, we have $u_\nu \in \mathcal{B}(\mathbf{R} \times [0,\delta])$ and

$$(3.1) \qquad \left\{ \begin{array}{l} \|u_\nu\|_{\mathcal{B}} \leq 2\delta, \; \|Ku_\nu\|_{\mathcal{B}} \leq \dfrac{1}{2}\|u_\nu\|_{\mathcal{B}} , \\ \|Ku_{\nu+1} - Ku_\nu\|_{\mathcal{B}} \leq \dfrac{1}{2}\|u_{\nu+1} - u_\nu\|_{\mathcal{B}} . \end{array} \right.$$

The fixed point theorem permits us to conclude that u_ν converges to a solution $u \in \mathcal{B}(\mathbf{R} \times [0,\delta])$ and we have $\|u\|_{\mathcal{B}} \leq 2\delta$.

We suppose that v is a solution for data v^0 and we put $g_i = v_i^0(x - c_i t)$. Then we have $u - v = Ku - Kv + f - g$. By virtue of the inequality (3.1), we have $\|u - v\|_{\mathcal{B}} \leq \frac{1}{2}\|u - v\|_{\mathcal{B}} + \|f - g\|_{\mathcal{B}}$ therefore $\|u - v\|_{\mathcal{B}} \leq 2\|u_0 - v_0\|_{L^1}$. It implies the continuity of the mapping $G(t)$. ∎

Corollary 3.2.— *We suppose the condition (dv) and $\|u^0\|_{L^1} \leq \delta$, where δ is brought by Theorem 3.1.*
a) (finite velocity propagation) The value $u(x,t)$ depends only on $\{u^0(y) \; : y \in [x - \gamma t, x + \gamma t]\}$ where $\gamma = \max_{i \in I} |c_i|$. In particular, if the data are supported in $[a,b]$, then the support of $u(\cdot,t)$ is included in $[a - \gamma t, b + \gamma t]$.
b) (conservation of positivity) If the data are positive, then the solution is also positive up to the time δ .
c) If u^0 is bounded, then u is bounded up to the time δ and we have $\|u(\cdot,t)\|_{L^\infty} \leq 2\left(1 + \|u^0\|_{L^\infty}\right)$ for $t \in [0,\delta]$.

Proof. a) If $u_\nu(x,t)$ is determined by $\{u^0(y) \; : y \in [x - \gamma t, x + \gamma t]\}$, then $u_{\nu+1} = Ku_\nu + f$ does also. As the solution u is a limit of the sequence u_ν , $u(x,t)$ depends only on $\{u^0(y) \; : y \in [x - \gamma t, x + \gamma t]\}$.
b) We approximate the data by the $u_n^0 = \inf(u^0, n)$. Then the solution u_n which corresponds to the data u_n^0 exists in $\mathcal{B}(\mathbf{R} \times [0,\delta])$ and the sequence $u_n(\cdot,t)$ converges to $u(\cdot,t)$ by the continuity of $G(t)$. By virtue of Proposition 2.2, the u_n are positive, then u is positive.

c) Let $[0,T[$ be with $T \leq \delta$ the supremum of T such that $\sup_{i,x} |u_i(x,\cdot)|$ is bounded up to the time T. Then, by Proposition 2.2, we have $T > 0$. For $\varepsilon > 0$, we put $M_\varepsilon = \sup_{t\in[0,T-\varepsilon]} \sup_{i,x} |u_i(x,t)| < \infty$. By the system, we have

$$\text{(3.2)} \qquad \begin{aligned} \sup_x |u_i^\sharp(x,t)| &\leq \left\|u^0\right\|_{L^\infty} + C^{st} M_\varepsilon \|u\|_{\mathcal{B}} + C^{st}\|u\|_{\mathcal{B}} + M(T-\varepsilon) \\ &\leq C^{st}\delta M_\varepsilon + C^{st}\delta + \left\|u^0\right\|_{L^\infty} \ , \text{ for } t \in [0, T-\varepsilon], \end{aligned}$$

where constants depend only on the system. For $\delta < (2C^{st})^{-1}$, we obtain

$$\text{(3.3)} \qquad M_\varepsilon = \sup_{t\in[0,T-\varepsilon]} \sup_{i,x} |u_i(x,t)| \leq 2\left(1 + \left\|u^0\right\|_{L^\infty}\right) .$$

The right hand side being independent of ε, we have

$$\text{(3.4)} \qquad M = \sup_{t\in[0,T]} \sup_{i,x} |u_i(x,t)| \leq 2\left(1 + \left\|u^0\right\|_{L^\infty}\right) .$$

This bound depends only on the initial data. Let T^* be the existence time of bounded solution which is associated by Proposition 2.2. If we had $T < \delta$, taking as initial data the $u_i(x, T-\varepsilon')$ with $\varepsilon' < T^*$, we would obtain a contradiction. ∎

Corollary 3.3.— *We assume the condition (dv) . Let u^0 be positive data in L^1 and h a real number such that $\sum_{i\in I}\int_a^{a+h} u_i^0(x)dx \leq \delta$ for any $a \in \mathbf{R}$.*
a) Then there exists a unique solution u in $\mathcal{B}(\mathbf{R}\times[0,\theta])$ with $\theta = \min\{\delta, h/\gamma\}$ and $\gamma = \max_{i\in I}|c_i|$. Furthermore the value $u(x,t)$ depends only on $\{u^0(y) : y \in [x-\gamma t, x+\gamma t]\}$.
b) Assume the condition (μr) and we put $H(t) = \sum_{i\in I}\int_{\mathbf{R}} u_i \log u_i(x,t)dx$. If the data are supported in $[-R,R]$ and verify $\sum_{i\in I}\int_{\mathbf{R}} u_i^0 \log u_i^0(x)dx < \infty$, we have, for $t \in [0,\theta]$, $H(t) \leq H(0) + C_* t$ with C_* which depends only on the system.

Proof. a) If we restrict the initial data in $[a, a+h]$, extending them by 0 outside, there exists a solution by Theorem 3.1. The restrictions of these solutions in small triangles with base $[a, a+h]$ and with height $\min\{\delta, h/\gamma\}$ can be sticked together by virtue of the finite velocity propagation.
b) For the calculus on the increase of $H(t)$, we approximate the data by $u_n^0 = \inf(u^0, n)$. Then we have $\sum_i \int_{\mathbf{R}} u_{n,i}^0 \log u_{n,i}^0(x)dx \leq \sum_i \int_{\mathbf{R}} u_i^0 \log u_i^0(x)dx < \infty$. Then the solutions u_n which correspond to data u_n^0 exist up to the time θ . Furthermore, they are bounded and positive up to the time θ . Moreover, the solutions have their support in x included in $[-R', R']$ with $R' = R + \gamma\theta$. Therefore the quantity $H_n(t) = \sum_i \int_{\mathbf{R}} u_{n,i} \log u_{n,i}(x,t)dx$ is well-defined. By virtue of Proposition 2.2, we have $H_n(t) \leq H_n(0) + C_* t \leq H(0) + C_* t$.
Theorem 3.1. enables us to conclude that the $u_n(\cdot,t)$ converge to the solution $u(\cdot,t)$ in L^1 for each $t \in [0,\theta]$. By extracting a sub-sequence if necessary, the $u_n(\cdot,t)$ converges to $u(\cdot,t)$ almost everywhere. By the fact that $u_{n,i}\log u_{n,i}(\cdot,t)$

are estimated from below by $-1/e$ and that they are supported in a fixed compact set, the Fatou's lemma implies that

$$
\begin{aligned}
H(t) &= \sum_i \int_{\mathbf{R}} u_i \log u_i(x,t)\, dx \\
&\leq \liminf \sum_i \int_{\mathbf{R}} u_{n,i} \log u_{n,i}(x,t)\, dx \qquad (3.5) \\
&= \liminf H_n(t) \leq H(0) + C_* t \ . \ \blacksquare
\end{aligned}
$$

4. Global existence

In this section, we show the time global existence for the initial data with locally finite entropy.

Proposition 4.1.— *We assume the condition* (dv) .
a) We suppose that there exist two solutions u *and* v *in* $\mathcal{B}(\mathbf{R} \times [0,T])$ *corresponding to the summable and positive initial data which coincide in a interval* $[a,b]$. *Then the solutions coincide in the triangle or the trapezoid* $\{(x,t)\ ; t \in [0,T], a + \gamma t \leq x \leq b - \gamma t\}$.
b) Let the initial data be supported in $[-R,R]$ *, summable and positive. We suppose that there exists a solution* u *in* $\mathcal{B}(\mathbf{R} \times [0,T])$. *Then the support of* $u(\cdot,t)$ *is included in* $[-R-Ct, R+Ct]$.

Proof. a) Let t_0 be the infimum of t such that $u(\cdot,t) \neq v(\cdot,t)$. We have then $u(\cdot,t_0) = v(\cdot,t_0) \in L^1$. As u and v are in L^1 , there exists a q such that $\int_{\{x\ : u_i(x,t_0) \geq q\}} u_i(x,t_0)dx < \delta/2$ for any $i \in I$. Taking $h = \delta/(2q)$, we have, for any $a \in \mathbf{R}$,

$$
\int_a^{a+h} u_i(x,t_0)dx \leq \frac{\delta}{2} + hq \leq \delta \ , \ i \in I. \qquad (4.1)
$$

Using Corollary 3.3, u are v coincide in small triangles with base $[a, a+h] \times \{t = t_0\}$ and with height θ and we are led to a contradiction.
b) It is sufficient to apply the previous result to u^0 and 0 as two initial data. ∎

Lemma 4.2.— *We assume the conditions* (vd) *and* (μr) . *Let* $u(x,t)$ *be a positive solution defined in* $\mathbf{R} \times [0,T]$ *with its support in* $[-R,R]$. *We suppose that* $\sum_{i\in I} \int_{\mathbf{R}} u_i \log u_i(x,t)\ dx$ *are estimated from above for any* $t \in [0,T]$ *by a constant* C *which does not depend on* t . *Then, for any* $\delta > 0$ *, there exists a* h, *which depends only on* R *,* C *and* δ *, such that* $\sum_{i\in I} \int_a^{a+h} u_i(x,t)\ dx \leq \delta$ *for any* $a \in \mathbf{R}$ *and any* $t \in [0,T]$.

Proof. If not, for any $h > 0$, there would exist a $a_* \in \mathbf{R}$ and a $t_* \in [0,T]$ such that $\sum_i \int_{a_*}^{a_*+h} u_i(x,t_*)dx > \delta$. We use the argument owing to Toscani [6] and Tartar-Crandall [5]. We put, for $m \geq 1$,

$$
B_{1,i} = \{x \in [a_*, a_* + h]\ : u_i(x,t_*) \geq e^m\} \qquad (4.2)
$$

$$B_{2,i} = [a_*, a_* + h] \setminus B_{1,i} \ . \tag{4.3}$$

Then we would have

$$\begin{aligned}
\int_{a_*}^{a_*+h} u_i(x,t_*)dx &\le \frac{1}{m}\int_{B_{1,i}} u_i\log^+ u_i(y,t_*)dy + he^m \\
&\le \frac{1}{m}\int_{a_*}^{a_*+h} u_i\log^+ u_i(y,t_*)dy + he^m \ .
\end{aligned} \tag{4.4}$$

On the other hand, we would have

$$\begin{aligned}
C &\ge \sum_i \int_{\mathbf{R}} u_i \log u_i(y,t_*)dy \\
&\ge \sum_i \int_{-R}^{R} \left(u_i\log^+ u_i(y,t_*) - 1\right) dy \\
&= \sum_i \int_{-R}^{R} u_i\log^+ u_i(y,t_*)dy - 2pR \text{ where } p = \sharp I.
\end{aligned} \tag{4.5}$$

Therefore we would obtain

$$\begin{aligned}
\delta &< \sum_i \int_{a_*}^{a_*+h} u_i(y,t_*)dy \le \frac{1}{m}\sum_i \int_{a_*}^{a_*+h} u_i\log^+ u_i(y,t_*)dy + phe^m \\
&\le \frac{1}{m}(C + 2pR) + phe^m \ .
\end{aligned} \tag{4.6}$$

Choosing a m such that $\frac{1}{m}(C+2pR) < \frac{\delta}{4}$, then a h such that $phe^m < \frac{\delta}{4}$, we would have $\delta < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$, which is a contradiction. ∎

Corollary 4.3.— *We assume the conditions (vd) and (μr) . We suppose that the initial data are supported in $[-R,R]$, positive and with finite entropy and that there exists a solution in $\mathcal{B}(\mathbf{R}\times[0,T])$. Then we have*

$$H(t) \le H(0) + C_* t \quad \text{for} \quad \forall t \in [0,T] \tag{4.7}$$

where the constant C_* depends only on the system and where J depends only on the system, on R and on T .

Proof. The support of $u(\cdot,t)$ in x is contained in $[-R',R']$ with $R' = R + \gamma T$. Let t_0 be the infimum of t such that the estimates does not hold at the time t . Taking a small $\varepsilon > 0$, we have $H(t_0-\varepsilon) \le H(0) + C_*(t_0-\varepsilon) \le H(0) + C_* T$. Owing to the fact that $u(\cdot, t_0-\varepsilon)$ is positive and supported in $[-R',R']$, the previous lemma shows that there exists a h independent of ε such that we have, for any $a \in \mathbf{R}$, $\sum_i \int_a^{a+h} u_i(x,t_0-\varepsilon)dx \le \delta$. By virtue of Corollary 3.3, we have

$$\begin{aligned}
H(t_0-\varepsilon+\theta) &\le H(t_0-\varepsilon) + C_*\theta \\
&\le H(0) + C_*(t_0-\varepsilon) + C_*\theta \ .
\end{aligned} \tag{4.8}$$

The estimate is then verified up to the time $t_0 - \varepsilon + \theta$ with $\theta > 0$ independent of ε , which is a contradiction. ∎

We state our main result :

Theorem 4.4.— *We assume the conditions* (dv) *and* (μr) . *For the initial data positive and with locally finite entropy, there exists a unique mild solution to* (1.1) *defined on* $\mathbf{R} \times [0, \infty[$.

Proof. By virtue of Proposition 4.1, we are led to the case that the initial data are supported in $[-R, R]$ and with finite entropy. Let T^* be the existence time for these data. Suppose that T^* is finite. By Corollary 4.3, the entropy is bounded for $t < T^*$: $H(t) \leq H_T < \infty$. Owing to Lemma 4.2, the solution $u(\cdot, t)$ being supported in $[-R - CT^*, R + CT^*]$, there exists a $h > 0$ such that $\sum_i \int_a^{a+h} u_i(x,t)dx \leq \delta$ for any $a \in \mathbf{R}$ and $t < T^*$. For $t < T^*$, by virtue of Corollary 3.3, applied to the initial data $u_i(\cdot, t)$, there exists a $\theta > 0$ independent of t such that the solution can be extended in $\mathbf{R} \times [0, t + \theta]$. It is sufficient to choose $t > T^* - \theta$ for reaching to a contradiction. ∎

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