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THE MICROLOCAL SMOOTHING EFFECT FOR SCHRÖDINGER TYPE OPERATORS IN GEVREY CLASSES

KUNIHIKO KAJITANI & GIOVANNI TAGLIALATELA (Un. Tsukuba)

Let $T>0$, we consider the Cauchy problem,

$$\begin{cases} Lu=0, & t\in[-T,T], \quad x\in\mathbb{R}^n, \\ u(0,x)=u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1}$$

where

$$Lu=\partial_t u-\sqrt{-1}\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}\Big(a_{jk}(x)\frac{\partial}{\partial x_k}u\Big)-\sum_{j=1}^{n}b_j(t,x)\frac{\partial}{\partial x_j}u-b_0(t,x)u,$$

$a_2(x,\xi)=\sum_{j,k=1}^{n}a_{jk}(x)\xi_j\xi_k$, is a real elliptic symbol with smooth and bounded coefficients.

We will consider the *smoothing effect* phenomenon: more the initial data decays at the infinity, more the solution of (1) is regular. In the case of microlocal smoothing effect, the decay of initial data is required only on a neighborhood of the backward bicharacteristic.

Recall that a function f belongs to the Gevrey class $\gamma^d(\Omega)$, if $f\in C^\infty(\Omega)$ and for any compact K of Ω there exist positive constants ρ and C such that

$$\sup_{x\in K}\big|D^\alpha f(x)\big|\le C\rho^{-|\alpha|}|\alpha|!^d,$$

for all $\alpha\in\mathbb{N}^n$.

A point $(y_0,\eta_0)\in T^*\mathbb{R}^n$ does not belong to the *Gevrey wave front set of order* d of a distribution u if there exists a function $\chi\in\gamma_0^d(\mathbb{R}^n)$ $\big(=\gamma^d(\mathbb{R}^n)\cap C_0^\infty(\mathbb{R}^n)\big)$, equal to 1 in a neighborhood of 0, such that if one sets $\chi_{y_0}(x)=\chi(x-y_0)$ and $\chi_{\eta_0}(\xi)=\Big(\frac{\xi}{|\xi|}-\frac{\eta_0}{|\eta_0|}\Big)$, then there exist positive constants C and ρ such that

$$\Big|D_x^\alpha\Big(\chi_{\eta_0}(D)\big(\chi_{y_0}(x)u\big)\Big)\Big|\le C\,\rho^{-|\alpha|}\alpha!^d,$$

for all $\alpha\in\mathbb{N}^n$. We will note $\mathrm{WF}_d\,u$ the wave front set of order d of u.

Assumption I.

1. $a_{jk}\in\gamma^d(\mathbb{R}^n)$, for some $d\ge 1$;
2. there exists a positive constant C such that
$$C^{-1}|\xi|^2\le a_2(x,\xi)\le C|\xi|^2,$$
for all $\xi\in\mathbb{R}^n$;
3. there exist positive constants δ, ρ and C such that
$$\big|D_x^\alpha a_{jk}(x)\big|\le C_\rho\,\rho^{-|\alpha|}\,\alpha!^d\,\langle x\rangle^{-|\alpha|-\delta},$$

for all $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n \setminus \{0\}$, for all $j, k = 1, \dots, n$;

4. there exists $\theta(x,\xi) \in \gamma^d(\mathbb{R}^n \times \mathbb{R}^n)$ such that
   (a) there exist positive constants ρ and C such that
$$\left|D_\xi^\alpha D_x^\beta \theta(x,\xi)\right| \le C_\rho \rho^{-|\alpha+\beta|} |\alpha|!|\beta|!^d \langle\xi\rangle^{1-|\alpha|} \langle x\rangle^{1-|\beta|},$$
   for all $x, \xi \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n$;
   (b) $H_{a_2}\theta(x,\xi) \ge C|\xi|^2$, for some positive constant C, where H_{a_2} is the *Hamiltonian flow* associated to a_2: $H_{a_2} = \sum_{j=1}^n \frac{\partial a_2}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial a_2}{\partial x_j}\frac{\partial}{\partial \xi_j}$.

Assumption II.

1. $b_j \in \mathcal{C}\big([0,T];\gamma^d(\mathbb{R}^n)\big)$, for $j = 0, 1, \dots, n$ and there exist positive constants ρ and C such that
$$\left|D_x^\alpha b_j(t,x)\right| \le C_\rho \rho^{-|\alpha|} \alpha!^d \langle x\rangle^{-|\alpha|},$$
   for all $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n \setminus \{0\}$, for all $j = 0, 1, \dots, n$;
2. $\operatorname{Im} b_j(t,x) = 0$, for $j = 1, \dots, n$.

For $(y_0, \eta_0) \in T^*\mathbb{R}^n$, we consider the bicharacteristic of $a_2(x,\xi)$ passing through (y_0,η_0):
$$\begin{cases} \dot X(s) = \dfrac{\partial a_2}{\partial \xi}(X(s), \Xi(s)) \\ \dot\Xi(s) = -\dfrac{\partial a_2}{\partial x}(X(s), \Xi(s)) \end{cases} \qquad (X(0), \Xi(0)) = (y_0, \eta_0).$$
The hypothesis on the principal symbol imply that $\lim_{s\to+\infty} \left|X(s; y_0, \eta_0)\right| = +\infty$.
We set
$$\Gamma_{\varepsilon_0}^{\pm}(y_0,\eta_0) = \bigcup_{s\le 0}\left\{x \in \mathbb{R}^n \;\middle|\; \left|x - X(\pm s; y_0, \eta_0)\right| \le \varepsilon\big(1+|s|\big)\right\}.$$
For $\varepsilon \in \mathbb{R}$ denote
$$\phi_\varepsilon(x,\xi) = x\cdot\xi - i\varepsilon \frac{\theta(x,\xi)}{\langle x\rangle^{1-\sigma}\langle\xi\rangle^{1-\delta}},$$
where $\sigma > 0$, $\delta \ge 0$ and $\sigma + \delta = \kappa$, $\kappa < 1$, and define
$$I_{\phi_\varepsilon}(x,D)u(x) = \int_{\mathbb{R}^n} e^{i\phi_\varepsilon(x,\xi)} \hat u(\xi) \bar d\xi,$$
where $\hat u(\xi)$ stands for a Fourier transform of u and $\bar d\xi = (2\pi)^{-n} d\xi$.

Theorem. *Assume that Assumptions I and II are satisfied for $d\kappa \le 1$, $\kappa < 1$ and $d \ge 1$. If $u_0 \in L^2(\mathbb{R}^n)$ and $I_{\phi_\varepsilon}(x,D)u_0 \in L^2\big(\Gamma_{\varepsilon_0}^-(y_0,\eta_0)\big)$, for $|\varepsilon| \le \varepsilon_0$, then there exists a solution u of* (1) *such that $(y_0,\eta_0) \notin \mathrm{WF}_{1/\kappa}\, u(t,\cdot)$, for all $t > 0$.*

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, 1-1-1 TENNŌDAI, 305 TSUKUBA IBARAKI,