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Hartogs' phenomena for microfunctions with holomorphic parameters

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0 Introduction

Hyperfunctions and microfunctions with holomorphic parameters of form $u(x', z'')$ have been considered in classical microlocalization and they play an important role in second microlocalization. It is standard to define them in a cohomological way: cf. in particular (1.1) and (1.2) below. The reason why they are not just defined as "holomorphic functions in the variables z'' with values in hyper- or microfunctions in the variables x'", is that spaces of hyperfunctions, respectively microfunctions, admit no natural topology. On the other hand, it is quite easy to associate a value $u(\cdot, z'')$ with u for every fixed z''. Since hyper- and microfunctions with holomorphic parameters share many properties in common with standard, complex-valued holomorphic functions, the following problem arises:

Problem 0.1. *Do the "values" $u(\cdot, \dot{z}'') = u|_{z''=\dot{z}''}$ of u with respect to the z''-variable determine the original u?*

It is a result of K. Kataoka and T. Oshima that the answer to this problem is affirmative in the hyperfunction case and we shall show in this note that the same is true also in the case of microfunctions: see Theorem 2.1 below. Actually, Kataoka and Oshima considered a slightly more general situation in which the parameter is assumed to be "real-analytic", rather than "holomorphic". For this reason we shall also consider a result similar to Theorem 2.1 for the case of real-analytic parameters: see Corollary 1.2 and Theorem 2.8.

Remark 0.2.. *K. Kataoka and T. Oshima did not publish their result themselves, but A. Kaneko included it (with proof) in his book [3] as Theorem 4.4.7'.*

1 Statement of the results

1. Let us fix the situation in which we work:

Let M' be a real analytic manifold with complexification X' and let X'' be a complex manifold. Local coordinates of M', X', and X'' are denoted by x', z', and z'' respectively. We consider the embedding $N := M' \times X'' \hookrightarrow X := X' \times X''$ and identify the conormal bundle $T^*_N X$ along N with $T^*_{M'}X' \times X''$. The canonical projection from $T^*_N X$ to N (resp. $T^*_{M'}X'$ to M') is denoted by π_N (resp. $\pi_{M'}$). We introduce the sheaf $\mathcal{CO}_N$ of microfunctions with holomorphic parameter z'' on $T^*_N X$ by

$$\mathcal{CO}_N := \mu_N(\mathcal{O}_X) \otimes or_{N/X}[\dim_{\mathbb{R}} M'] \tag{1.1}$$

and the sheaf $\mathcal{BO}_N$ of hyperfunctions with holomorphic parameter z'' on N by

$$\mathcal{BO}_N := \mathcal{CO}_N|_N. \tag{1.2}$$

Here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X, μ_N Sato's microlocalization functor along N, and $or_{N/X}$ the relative orientation sheaf. For any fixed point $\dot{z}'' \in X''$, we can define the restriction morphisms

$$\mathcal{BO}_N|_{\{z''=\dot{z}''\}} \to \mathcal{B}_{M'}, \quad u \mapsto u|_{z''=\dot{z}''}$$

and

$$\mathcal{CO}_N|_{\{z''=\dot{z}''\}} \to \mathcal{C}_{M'}, \quad u \mapsto u|_{z''=\dot{z}''}$$

under the identifications $M' \times \{\dot{z}''\} \simeq M'$ and $T^*_{M'}X' \times \{\dot{z}''\} \simeq T^*_{M'}X'$. Here we denote by $\mathcal{B}_{M'}$ the sheaf of hyperfunctions on M' and by $\mathcal{C}_{M'}$ that of microfunctions on $T^*_{M'}X'$.

Now we state our main theorem:

Theorem 1.1. *Let $\dot{q}' \in T^*_{M'}X'$ be a point, $U'' \subset X''$ an open subset and $u \in \mathcal{CO}_N(\{\dot{q}'\} \times U'')$ a microfuntion with holomorphic parameter z'' defined in a neighborhood of $\{\dot{q}'\} \times U''$. Assume that for any fixed $\dot{z}'' \in U''$, the restriction $u|_{z''=\dot{z}''}$ is 0 at $\dot{q}'$. Then $u = 0$ in a neighborhood of $\{\dot{q}'\} \times U''$.*

We give explicitly two corollaries of Theorem 1.1.

Corollary 1.2. *Let $\dot{x}' \in M'$ be a point, $U'' \subset X''$ an open subset and $u \in \mathcal{BO}_N(\{\dot{x}'\} \times U'')$ a hyperfunction with holomorphic parameter z'' defined in a neighborhood of $\{\dot{x}'\} \times U''$. Assume that for any $\dot{z}'' \in U''$, the restriction $u|_{z=\dot{z}''} = 0$ at $\dot{x}'$. Then $u = 0$ in a neighborhood of $\{\dot{x}'\} \times U''$.*

Corollary 1.3. *Let $\Omega' \subset M'$ and $U'' \subset X''$ be two open subsets and $u \in \mathcal{BO}_N(\Omega' \times U'')$ a hyperfunction with holomorphic parameter. Assume that for any $\dot{z}'' \in U''$, the restriction $u|_{z=\dot{z}''} \in \mathcal{B}_{M'}(\Omega')$ is real analytic. Then u itself is an analytic function on $\Omega' \times U''$, i.e., there exist a neighborhood $\tilde{U} \subset X$ of $\Omega' \times U''$ and a holomorphic function $f \in \mathcal{O}_X(\tilde{U})$ with $u = f|_{\Omega' \times U''}$.*

We remark that Corollary 1.2 can be obtained from a result due to Kataoka and Oshima given in Theorem 2.8, in which $u(x', x'')$ is assumed to have x'' as real analytic parameters.

Let us consider moreover the case where X'' is the complexification of some real analytic manifold M''. In this situation, we give a stronger result:

Theorem 1.4. *Let $\dot{q}' \in T^*_{M'}X'$ be a point, $U'' \subset X''$ a connected open subset and $u \in \mathcal{CO}_N(\{\dot{q}'\}\times U'')$ a microfunction with holomorphic parameters defined in a neighborhood of $\{\dot{q}'\}\times U''$. Assume that $M''\cap U''$ is non-empty and that $u|_{z''=\dot{x}''} = 0$ for any $\dot{x}'' \in M''\cap U''$. Then $u = 0$ in a neighborhood of $\{\dot{q}'\} \times U''$.*

Note that since we can argue locally, Theorem 1.1 will be a consequence of Theorem 1.4.

2. Let us next consider the sequence of embeddings

$$M := M' \times M'' \hookrightarrow N \hookrightarrow X,$$

which defines the sheaf

$$\mathcal{A}^2_\Sigma := \mathcal{CO}_N|_\Sigma$$

of second analytic functions defined on the real regular involutive submanifold

$$\Sigma := T^*_M X \times_{T^*X} T^*_N X.$$

It is again a consequence of Theorem 1.4 that the sections of $\mathcal{A}^2_\Sigma$ are determined pointwisely:

Corollary 1.5. *Let $\dot{q}' \in T^*_{M'}X'$ be a point, $\Omega'' \subset M''$ an open subset, and $u \in \mathcal{A}^2_\Sigma(\{\dot{q}'\} \times \Omega'')$ a second analytic function. Assume that $u|_{x''=\dot{x}''} = 0$ for any fixed $\dot{x}'' \in \Omega''$. Then $u = 0$.*

The following particular case of Theorem 1.4 will be the main intermediate step in the argument. We denote in it by $\dot{T}^*_{M'}X' = T^*_{M'}X' \setminus M'$ the conormal bundle to M' with the zero section removed and by $\dot{\pi}_{M'}$ the canonical projection from $\dot{T}^*_{M'}X'$ to M'.

Theorem 1.6. *Let $\Omega' \subset M'$ be an open subset, $U'' \subset X''$ a connected open subset with $U''\cap M'' \neq \emptyset$, and $Z \subset \dot{\pi}^{-1}_{M'}(\Omega')$ a closed conic subset such that for each base point $\dot{x}' \in \Omega'$ the intersection $Z \cap \dot{\pi}^{-1}_{M'}(\dot{x}')$ consists of only one direction. Also consider a fixed point $\dot{q}'$ in Z.*

Assume that a section $u \in \mathcal{CO}_N(\dot{\pi}^{-1}_{M'}(\Omega') \times U'')$ satisfies the condition

$$\mathrm{supp}(u) \subset Z \times U''$$

and that $u|_{z''=\dot{x}''} = 0$ at $\dot{q}'$ for any $\dot{x}'' \in U'' \cap M''$. Then we have $u = 0$ in a neighborhood of $\{\dot{q}'\} \times U''$.

2 Local forms of the main results

1. The theorems above in Section 1 are of a local nature. We may argue therefore in local coordinates and assume that $X' = \mathbb{C}^d$, $X'' = \mathbb{C}^{n-d}$ for some d and n. We shall identify $X' \times X''$ with $\mathbb{C}^n$ in a natural way: if $z = (z_1, \ldots, z_n)$ are the coordinates in $\mathbb{C}^n$, we write z' for $(z_1, \ldots, z_d)$ and z'' for $(z_{d+1}, \ldots, z_n)$. Thus, $z = (z', z'')$ and $X' = \{z \in \mathbb{C}^n; z'' = 0\}$, $X'' = \{z \in \mathbb{C}^n; z' = 0\}$. We denote $M = \{z \in X; \operatorname{Im} z = 0\} = \mathbb{R}^n$, regarded as a real analytic submanifold in $\mathbb{C}^n$ and consider its partial complexification $N = \{z \in X; \operatorname{Im} z' = 0\} = \mathbb{R}^d \times \mathbb{C}^{n-d}$. Variables in M shall be written as $x = (x', x'')$, $x' = (x_1, \ldots, x_d)$, $x'' = (x_{d+1}, \ldots, x_n)$. Coordinates of $T^*_N X = T^*_{\mathbb{R}^d}\mathbb{C}^d \times \mathbb{C}^{n-d}$ are denoted by $(x', z''; \xi' \cdot dx')$ or $(x', z''; \xi')$ under the identification $T^*_{\mathbb{R}^d}\mathbb{C}^d \simeq \sqrt{-1}T^*\mathbb{R}^d \simeq T^*\mathbb{R}^d$.

It is instructive to rewrite Theorem 1.1 in local variables and in terms of defining functions:

Theorem 2.1. *Assume that $h \in \mathcal{O}(\{z \in \mathbb{C}^n; |z'| < \varepsilon, \operatorname{Im} z' \in G', z'' \in \mathbb{C}^{n-d}, |z''| < \delta\})$ and denote by u the hyperfunction with holomorphic parameters on $\{(x', z''); |x'| < \varepsilon, |z''| < \delta\}$ associated with h. Also consider $\dot{\xi}' \in G'^{\perp}$ and assume that $(0, \dot{\xi}') \notin \mathrm{WF}_A\, u(\cdot, z'')$, for any z'' with $|z''| < \delta$. Then there are $\varepsilon' > 0$, $\delta' > 0$, open convex cones $G'_1, \ldots, G'_s$ in $\mathbb{R}^d$ so that $\dot{\xi}' \notin G'^{\perp}_j$ and holomorphic functions h_j defined on $\{z \in \mathbb{C}^n; |z'| < \varepsilon', y' \in G'_j, |z''| < \delta'\}$ so that u is for $|z''| < \delta'$ equal to $\sum_{j=1}^s b(h_j)$.*

Theorem 2.1 may be considered as a microlocal variant of a theorem of Malgrange-Zerner. For the classical situation, see H. Komatsu [6]. We also give a version of Theorem 1.6 in local coordinates:

Theorem 2.2. *Let $G' \subset \mathbb{R}^d$ be an open convex cone and h a holomorphic function defined on $\{z \in \mathbb{C}^n; |z'| < \varepsilon, y' \in G', |z''| < \varepsilon\}$. Assume that for every $\dot{x}'' \in \mathbb{R}^{n-d}$ with $|\dot{x}''| < \varepsilon$, the holomorphic function $h(\cdot, \dot{x}'') = h|_{z''=\dot{x}''}$ defined on $\{z' \in \mathbb{C}^d; |z'| < \varepsilon, y' \in G'\}$ extends holomorphically to a neighborhood of $0 \in \mathbb{C}^d$. Then h extends holomorphically to a neighborhood of the set $\{z \in \mathbb{C}^n; z' = 0, |z''| < \varepsilon\}$.*

Remark 2.3. *As a consequence of Corollary 1.2 it is possible to recast the definition of hyperfunctions (respectively microfunctions) with holomorphic parameters considered above as follows. Let again $U'' \subset \mathbb{C}^{n-d}$ be some open subset. A function $h : U'' \to \mathcal{B}_0$ ($\mathcal{B}_0$ denotes the set of germs of hyperfunctions at the point $0 \in \mathbb{R}^d$) is then a hyperfunction with holomorphic parameter z'' precisely if for any $\tilde{z}'' \in U''$ there is an open neighborhood $\tilde{U}'' = \tilde{U}_{\tilde{z}''}$ of $\tilde{z}''$, $\varepsilon > 0$, a finite collection of open convex cones $G'_j \subset \mathbb{R}^d$, $j = 1, \ldots, s$, and holomorphic functions $\tilde{h}_j \in \mathcal{O}(z; |z'| < \varepsilon, \operatorname{Im} z' \in G'_j, z'' \in \tilde{U}'')$, so that for any $z'' \in \tilde{U}''$, $h(z'')$ is equal to $\sum_{j=1}^s b(h_j(\cdot, z''))$.*

Likewise, a function $h : U'' \to \mathcal{C}_{(0,\dot{\xi}')}$ ($\mathcal{C}_{(0,\dot{\xi}')}$ denotes here the set of germs of microfunctions at the point $(0, \dot{\xi}')$) will be a microfunction with holomorphic parameter z'' precisely if for any $\tilde{z}'' \in U''$ there is an open neighborhood $\tilde{U}'' = \tilde{U}_{\tilde{z}''}$ of $\tilde{z}''$, $\varepsilon > 0$, an open cone $G' \subset \mathbb{R}^d$ which contains $\dot{\xi}'$ and a holomorphic function $\tilde{h} \in \mathcal{O}(z; |z'| < \varepsilon, \operatorname{Im} z' \in G', z'' \in \tilde{U}'')$, so that for any $\dot{z}'' \in \tilde{U}''$, $h(\dot{z}'')$ is the microfunction defined by the microfunctional boundary value of the holomorphic function $z' \to \tilde{h}(z', \dot{z}'')$, $|z'| < \varepsilon, \operatorname{Im} z' \in G'$. We shall call $\tilde{h}$ a local defining function for h (near $\tilde{z}''$). When we want to make the dependence of $\tilde{h}$ on $\tilde{z}''$ explicit, we shall occasionally write $\tilde{h}_{\tilde{z}''}$.

Remark 2.4. *It is a significant fact that the local defining functions $\tilde{h}_{\tilde{z}''}$ associated with the various $\tilde{z}''$ do not always admit a common holomorphic extension for all $z'' \in U''$. (I.e., in general there will exist no $f \in \mathcal{O}(z; |z'| < \varepsilon, \operatorname{Im} z' \in G', z'' \in U'')$ with $(0, \dot{\xi}') \notin \mathrm{WF}_A\, b'[f(\cdot, z'') - h(z'')]$, $\forall z'' \in U''$.)*

Remark 2.5. *Let us consider Corollary 1.3 again. This corollary says that the real analyticity of $u(x', z'')|_{z''=\dot{z}''}$ for each $\dot{z}''$ implies the real analyticity of u. It is important in this result that z'' is allowed to vary in an open set in $\mathbb{C}^n$. Indeed, there is no analogous result when we only have assumptions for z'' real. This is the content of the following*

Example 2.6. *Let u be the hyperfunction on $\mathbb{R}^2$ defined by*

$$u(x_1, x_2) = \frac{x_2}{x_1 + ix_2^2 + i0}.$$

Then

$$\mathrm{WF}_A(u) = \{(0, 0; 1, 0)\}.$$

In particular the restrictions of u with respect to the x_2 variable are well-defined. All these restrictions are real analytic (in one variable), but u itself is not real analytic.

We remark that a similar example was already obtained in A. Kaneko [2].

We also give the following

Example 2.7. *Consider a holomorphic function*

$$h(z_1, z_2) = \sum_{j=1}^{\infty} \frac{i(-iz_2)^j}{j^j(z_1 + i(z_2^2 + j^{-3j}))}$$

on $\{(z_1, z_2) \in \mathbb{C}^2; \operatorname{Im} z_1 > (\operatorname{Im} z_2)^2 - (\operatorname{Re} z_2)^2\}$. The boundary value $u(x_1, x_2)$ of h satisfies

(1) for any $\dot{x}_2$ and any $k = 0, 1, 2, \ldots$, $\partial_{x_2}^k u|_{x_2=\dot{x}_2}$ is well-defined and real analytic,

(2) $\mathrm{WF}_A(u) = \{(0, 0; 1, 0)\}$.

2. We give a remark. In view of Theorem 4.4.7' in A.Kaneko[3] mentioned in the introduction, we have the following result due to Kataoka and Oshima concerning hyperfunctions with holomorphic parameters:

Theorem 2.8 (K.Kataoka and T.Oshima). *Let Ω' and Ω'' be open subsets of M' and M'' respectively. Let $u(x',x'') \in \mathcal{BA}(\Omega' \times \Omega'')$ and assume that*

$$u(x',x'')\Big|_{x''=x''_0} = 0 \tag{2.1}$$

for any $x''_0 \in \Omega''$. Then it follows that $u = 0$.

Here we have used the notation

$$\mathcal{BA} := \mathcal{H}^d_M\left(\mathcal{O}_X\Big|_{X'\times M''}\right), \tag{2.2}$$

and recall that the sheaf $\mathcal{BA}$ was considered by M.Sato[10]. (We remark that M. Sato used the sheaf $\mathcal{BA}$ to discuss restriction of hyperfunctions in [10], before the notion of singular spectrum came into being.)

3. It follows from the above dicussion that all results mentioned so far will be reduced to Theorem 2.2. We shall therefore give the proof of this theorem in the next section.

The reduction of 1.6 to 2.2 will be done by using the characterization of real-analyticity of hyperfunctions with single defining functions and the unique continuation property along holomorphic parameters. Likewise, the reduction of Theorem 1.4 to Theorem 1.6 will be done by using two morphisms due to Kashiwara and which were used in the proof of the flabbiness of the sheaf of microfunctions. We omit these two steps in this note.

3 Proof of Theorem 2.2.

1. In this section, we give a brief sketch of the proof of Theorem 2.2. The argument will be based on several tools: a characterization of extendibility of holomorphic functions by duality, Baire's principle, a theorem of Hartogs' type, and the unique continuation property of singularities along holomorphic parameters.

First we shall give a very simple result on extendibility of holomorphic functions. Before we can state the result we need to introduce some additional notations and conventions. We shall in fact denote by $B'(\delta)$ the polydisc $\{z' \in \mathbb{C}^d; |z_j| < \delta, \forall j\}$ in $\mathbb{C}^d$ and shall use, for subsets $U', V' \subset \mathbb{C}^d$, the conventions:

$$H_{U'}(\zeta') := \sup_{z'\in U'} \mathrm{Re}(-i\langle z',\zeta'\rangle), \quad U'+V' := \{z'+\tilde{z}'; z' \in U', \tilde{z}' \in V'\}. \tag{3.1}$$

It is immediate that $H_{U'+B'(\delta)}(\zeta') = H_{U'}(\zeta') + \delta\sum_{j=1}^d |\zeta_j|$. Finally, if $U' \subset \mathbb{C}^d$ is an open set, we denote by $\mathcal{O}'_{\mathbb{C}^d}(U')$ the space of analytic functionals on $\mathcal{O}_{\mathbb{C}^d}(U')$. For simplicity we shall assume that U' is convex. It is well-known that analytic functionals v in $\mathcal{O}'_{\mathbb{C}^d}(U')$ are characterized by the fact their Fourier-Borel transform

$$\zeta' \to \hat{v}(\zeta') = \mathcal{F}(v)(\zeta') = v(\exp[-i\langle z',\zeta'\rangle])$$

satisfies an estimate of form

$$|\hat{v}(\zeta')| \le c \exp[H_Q(\zeta')]$$

for some constant C and some compact set $Q \subset U'$.

Theorem 3.1 (Holomorphic extensions and Duality). *Let $U' \subset \mathbb{C}^d$ be a bounded open convex domain and $h \in \mathcal{O}_{\mathbb{C}^d}(U')$ a holomorphic function defined on U'. Then h extends holomorphically to $U' + B'(\delta)$ if and only if for any δ' with $0 < \delta' < \delta$ there exists a constant $c_{\delta'}$ satisfying*

$$\forall v \in \mathcal{O}'_{\mathbb{C}^d}(U'), |\hat{v}(\zeta')| \le \exp\{H_{U'+B'(\delta')}(\zeta')\} \Longrightarrow |v(h)| \le c_{\delta'}.$$

We next prove a modified version of Hartogs' theorem. We only consider the case of convex sets.

Theorem 3.2 (Hartogs-type theorem). *Let $U' \subset \mathbb{C}^d$ and $U'' \subset \mathbb{C}^{n-d}$ be bounded open convex domains with $0 \in \partial U'$, $U'' \cap \mathbb{R}^{n-d} \ne \emptyset$ and $h \in \mathcal{O}_{\mathbb{C}^n}(U' \times U'')$ a holomorphic function defined on $U' \times U''$. Assume that for any $x'' \in U'' \cap \mathbb{R}^{n-d}$, there exists a positive number $\delta(x'') > 0$ for which the function $h(\cdot, x'') \in \mathcal{O}_{\mathbb{C}^d}(U')$ extends holomorphically to $B'(\delta(x''))$. Then we can find an open ball $B'' \subset U''$ centered at some point $\dot{x}'' \in U'' \cap \mathbb{R}^{n-d}$ and a constant $\delta > 0$ in such a way that h extends holomorphically to $B'(\delta) \times B''$.*

Before we enter the proof of Theorem 3.2, we recall a local variant of the Phragmén-Lindelöf principle.

Lemma 3.3. *Let B'' be the unit disc in $\mathbb{C}^{n-d}$ and let $\rho : B'' \to \mathbb{R}$ be a plurisubharmonic function on B''. Assume that $\rho(z'') \le 1$ on B'' and that $\rho(x'') \le 0$ for $x'' \in \mathbb{R}^{n-d} \cap B''$. Then there is a constant C independent of ρ satisfying $\rho(z'') \le C|\operatorname{Im} z''|$ for $|z''| \le 1/2$.*

For a proof of this result cf. e.g. Meise-Taylor-Vogt[9]. Note that the lemma implies in a trivial way the following remark:

Remark 3.4. *Assume that $\rho : B''(\varepsilon) \to \mathbb{R}$ is plurisubharmonic but assume now that $\rho(z'') \le c$ on $B''(\varepsilon)$ whereas $\rho(x'') \le c'$ for $x'' \in \mathbb{R}^{n-d} \cap B'(\varepsilon)$ for some constants c, c'. Then $\rho(z'') \le c' + cC|\operatorname{Im} z''|$ for $|z''| \le \varepsilon/2$. In particular it follows that if we fix $c'' > c'$ that there is ε' (which depends on c and C but not on ρ) so that $\rho(z'') \le c''$ if $|z''| < \varepsilon'$.*

Proof of Theorem 3.2. Take any compact convex set $K' \subset\subset U'$ whose interior $\operatorname{Int} K'$ is non-empty and denote by $\hat{K}'$ the convex hull of the set $\{0\} \cup K'$ in $\mathbb{C}^d$. Then we have:

- $0 \in \partial \hat{K}'$,
- h is holomorphic in $\operatorname{Int} \hat{K}' \times U''$,

- for any $x'' \in U'' \cap \mathbb{R}^{n-d}$, $h(\cdot, x'')$ extends holomorphically to a neighborhood of $\hat{K}'$.

Thus by shrinking U' to Int $\hat{K}'$ and by also shrinking $\delta(x'')$ suitably, we may assume, from the beginning, that for any $x'' \in U'' \cap \mathbb{R}^{n-d}$, the function $h(\cdot, x'')$ extends holomorphically to $U' + B'(\delta(x''))$.

Set

$$E''_j := \bigcap_{v \in \mathcal{O}'_{\mathbb{C}^d}(U'), |\hat{v}(\zeta')| \le \exp H_{U'+B'(1/j)}(\zeta')} \{x'' \in U'' \cap \mathbb{R}^{n-d}; |v(h(\cdot, x''))| \le j\}.$$

By Theorem 3.1, we have that $U'' \cap \mathbb{R}^{n-d} = \bigcup_j E''_j$. We can also see that every $E''_j \subset U'' \cap \mathbb{R}^{n-d}$ is closed. Thus from Baire's principle, some E''_{j_0} must include an open ball $E'' := \{x'' \in \mathbb{R}^{n-d}; |x'' - \dot{x}''| < \varepsilon\}$. By shrinking ε, we can assume that $\dot{x}'' + B''(\varepsilon) \subset\subset U''$. We define δ by $\delta = 1/(4j_0)$, take a point $\dot{z}' \in U'$ with $|\dot{z}'| < \delta$, and also take a positive constant δ' with $\dot{z}' + B'(\delta') \subset\subset U'$. From the considerations above, our function h satisfies the following two properties.

(P1) h is holomorphic in a neighborhood of the closure of $(\dot{z}' + B'(\delta')) \times (\dot{x}'' + B''(\varepsilon))$,

(P2) each $h(\cdot, x'')$ satisfies $|v(h(\cdot, x''))| \le j_0$ for any $v \in \mathcal{O}'_{\mathbb{C}^d}(\dot{z}' + B'(\delta))$ with $|\hat{v}(\zeta')| \le \exp H_{\dot{z}'+B'(3\delta)}(\zeta')$.

Take the Taylor expansion of h in the variables z' around $\dot{z}'$:

$$h(z) = \sum_\alpha a_\alpha(z'')(z' - \dot{z}')^\alpha. \tag{3.2}$$

Now we will estimate the functions $z'' \to |a_\alpha(z'')|$ in two ways. Our aim is to show that $a_\alpha(z'')$ satisfy estimates which are good enough to ensure that the function h is analytic on a larger domain than its initial domain of definition.

First we apply Cauchy's integral formula to the functions $h(\cdot, z'')$ in the variables z' for all $z'' \in \dot{x}'' + B''(\varepsilon)$. Using the property (P1), we obtain:

$$|a_\alpha(z'')| \le C_1 \delta'^{-|\alpha|} \quad \text{for any } z'' \in \dot{x}'' + B''(\varepsilon) \text{ and any } \alpha, \tag{3.3}$$

where $C_1 := \sup_{z \in (\dot{z}'+B'(\delta')) \times (\dot{x}''+B''(\varepsilon))} |h(z)|$.

On the other hand, the expression:

$$a_\alpha(z'') = v_\alpha(h(\cdot, z'')),$$

where v_α is the analytic functional v_α: $f \mapsto v_\alpha(f) := (1/\alpha!)((\partial/\partial z')^\alpha f)|_{z'=\dot{z}'}$, and the property (P2) give us the following estimate:

$$|a_\alpha(x'')| \le C_2 (3\delta)^{-|\alpha|} \quad \text{for any } x'' \in B''(\varepsilon) \cap \mathbb{R}^{n-d} \text{ and any } \alpha, \tag{3.4}$$

where $C_2 := j_0 e^{-d}$.

In this situation we apply Remark 3.4 to the plurisubharmonic functions

$$z'' \to \frac{1}{|\alpha|}(\log |a_\alpha(z'' + \dot{x}'')| - \log \max(C_1, C_2)),$$

From the estimates (3.3) and (3.4) we obtain

$$|a_\alpha(z'')| \le C(2\delta)^{-|\alpha|} \quad \text{for any } z'' \in \dot{x}'' + B''(\varepsilon'),$$

with some constant ε'. This estimate shows that for any $z'' \in \dot{x}'' + B''(\varepsilon')$, the Taylor series (3.2) converges at least on $\dot{z}' + B'(2\delta)$ and that our function h extends holomorphically to the domain $(\dot{z}' + B'(2\delta)) \times (\dot{x}'' + B''(\varepsilon'))$, which includes $B'(\delta) \times (\dot{x}'' + B''(\varepsilon'))$. Thus we have the desired result if we take $\dot{x}'' + B''(\varepsilon')$ for B''. □

2. Now we give a proof of Theorem 2.2.

Proof of Theorem 2.2. Let us assume that h is a holomorphic function satisfying the assumption of Theorem 2.2. For each $\dot{x}'' \in \mathbb{R}^{n-d}$ with $|\dot{x}''| < \varepsilon$, the restriction $h(\cdot, \dot{x}'')$ extends holomorphically to a set of type $\{z'; |z'| < \delta(\dot{x}'')\}$ with some positive number $\delta(\dot{x}'')$.

Then from Theorem 3.2, we can take a positive constant δ and an open ball $B'' \subset \{z'' \in \mathbb{C}^{n-d}; |z''| < \varepsilon\}$ centered at some point $\dot{x}'' \in \mathbb{R}^{n-d}$ such that h extends holomorphically to $B'(\delta) \times B''$. Let us consider the boundary value $u = \dot{b}(h) \in \mathcal{BO}(\{(x', z''); |x'| < \varepsilon, |z''| < \varepsilon\})$. From the domain of holomorphy of h, we can see that u is real analytic on the domain $\{(x', z''); |x'| < \delta, z'' \in B''\}$. Then from the unique continuation property for the analytic wave front sets along holomorphic parameters, we can conclude that u is real analytic on the domain $\{(x', z''); |x'| < \delta, |z''| < \varepsilon\}$. Since there is only one defining function h in the boundary value representation $u = \dot{b}(h)$, the real analyticity of u asserts that h extends holomorphically to the domain $\{(x', z''); |x'| < \delta, |z''| < \varepsilon\}$. This completes the proof. □

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