数理解析研究所講究録 1211 巻 2001 年 112-119

Existence of singular solutions with bounds of linear partial differential equations in the complex domain

大内 忠 (上智大学)

Sunao Ōuchi (Sophia Univ.)

§0 Introduction

In this paper we consider a linear partial differential equation in the complex domain in $\mathbb{C}^{d+1}$, $L(z,\partial)u(z)=f(z)$. $L(z,\partial_z)$ is an m-th linear partial differential operator with coefficients are holomorphic in a neighborhood U of $z=0$ in $\mathbb{C}^{d+1}$. The inhomogeneous term $f(z)$ has singularities on a complex hypersurface K. The author reported the results concerning the growth properties and the asymptotic behaviors of solutions, and those concerning the existence of solutions with asymptotic expansion, when $f(z)$ has an asymptotic expansion, at the conference held here, RIMS of Kyoto Univ.(see [5],[6],[7] and [8]). In the present paper our concern is the existence of solutions, when $f(z)$ has not necessary asymptotic expansion, but the singularities are tempered, that is, singularities are of the fractional order. The details will be given elsewhere.

§1 Notations and Definitions

In order to state our problem and results more precisely, let us introduce notations, function spaces and characteristic polygon.

1.1. Notations. $z=(z_0,z_1,\cdots,z_d)=(z_0,z')\in\mathbb{C}\times\mathbb{C}^d$. $|z|=\max\{|z_i|;\ 0\le i\le d\}$ and $|z'|=\max\{|z_i|;\ 1\le i\le d\}$. Its dual variables are $\xi=(\xi_0,\xi')=(\xi_0,\xi_1,\cdots,\xi_d)$. $\partial_i=\partial/\partial z_i$, and $\partial=(\partial_0,\partial_1,\cdots,\partial_d)=(\partial_0,\partial')$. $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ is the set of all nonnegative integers. For a multi-index $\alpha=(\alpha_0,\alpha')\in\mathbb{N}\times\mathbb{N}^d$, $|\alpha|=\alpha_0+|\alpha'|=\sum_{i=0}^d\alpha_i$. For a polydisk $U=U_0\times U'$ in $\mathbb{C}^{d+1}$, where $U_0=\{z_0\in\mathbb{C};|z_0|<R_0\}$ and $U'=\{z\in\mathbb{C};|z'|<R\}$, set $U_0(\theta)=\{z_0\in U_0-\{0\};\ |\arg z_0|<\theta\}$ and $U(\theta)=U_0(\theta)\times U'$. $U(\theta)$ is a sectorial region with respect to z_0. K is a complex hypersurface through $z=0$ in U. We choose the coordinate so that $\{z\in U;z_0=0\}$

---

e-mail s_ohuti@hoffman.cc.sophia.ac.jp

KEY WORDS: complex partial differential equations, asymptotic expansion, existence of singular solutions. 1991 Mathematical Subject Classification(s): Primary 35A20; Secondary 35A07, 35G20.

1.2. Function spaces. For a region U in $\mathbb{C}^n$, $\mathcal{O}(U)$ is the set of all holomorphic functions on U.

Definition 1.1. (i). *$\mathcal{O}_{temp,c}(U(\theta))$ is the set of all $u(z) \in \mathcal{O}(U(\theta))$ such that for any θ' with $0 < \theta' < \theta$*

$$|u(z)| \leq C|z_0|^c \quad z \in \Omega(\theta') \tag{1.1}$$

holds for a constant $C = C(\theta')$.
(ii). $\mathcal{O}_{temp}(U(\theta)) = \bigcup_{c\in\mathbb{R}} \mathcal{O}_{temp,c}(U(\theta))$. *We say that $u(z) \in \mathcal{O}_{temp}(U(\theta))$ is tempered singular, or regular singular, on K in $U(\theta)$.*

Definition 1.2. $\mathcal{A}_{\{\kappa\}}(U(\theta))$ $(0 < \kappa \leq +\infty)$ *is the set of all $u(z) \in \mathcal{O}(U(\theta))$ such that for any θ' with $0 < \theta' < \theta$*

$$|\partial_0^N u(z)| \leq AB^N \Gamma(N(1+\frac{1}{\kappa})+1) \quad \textit{for } z \in U(\theta') \tag{1.2}$$

holds for all $n \in \mathbb{N}$ and for some constants $A = A(\theta')$ and $B = B(\theta')$.

$u(z) \in \mathcal{A}_{\{+\infty\}}(U(\theta))$ means that $u(z)$ is holomorphic at $z = 0$. $\mathcal{A}_{\{\kappa\}}(U(\theta))$ is coincident with $Asy_{\{\kappa\}}(U(\theta))$ in the preceding papers [5],[7] and [8], which consists of all $u(z) \in \mathcal{O}(U(\theta))$ with asymptotic expansion of Gevrey type, that is, for any θ' with $0 < \theta' < \theta$

$$|u(z) - \sum_{n=0}^{N-1} u_n(z')z_0^n| \leq AB^N|z_0|^N\Gamma(\frac{N}{\kappa}+1) \quad \textit{for } z \in U(\theta'), \tag{1.3}$$

where $u_n(z') \in \mathcal{O}(U')$ $(n \in \mathbb{N})$, $A = A(\theta')$ and $B = B(\theta')$. The notation $u(z) \sim 0$ in $A_{\{\kappa\}}(U(\theta))$ means that $u_n(z') \equiv 0$ for all n in (1.3).

1.3. Characteristic polygon. Let $L(z, \partial)$ be an m-th order linear partial differential operator with holomorphic coefficients in a neighborhood of $z = 0$,

$$L(z,\partial) = \sum_{|\alpha|\leq m} a_\alpha(z)\partial^\alpha. \tag{1.4}$$

We introduce the characteristic polygon of $L(z, \partial)$ with respect to hypersurface $K = \{z_0 = 0\}$, which is indispensable for our purpose, to study the existence of solutions with bounds. Let us introduce a notation $\lrcorner(a, b) :=$ $\{(x, y) \in \mathbb{R}^2; x \leq a, y \geq b\}$, which means an infinite rectangle. Let j_α be the valuation of $a_\alpha(z)$ with respect to z_0, that is, if $a_\alpha(z) \not\equiv 0$, $a_\alpha(z) = z_0^{j_\alpha} b_\alpha(z)$ with $b_\alpha(0, z') \not\equiv 0$ and set $j_\alpha = \infty$ for $a_\alpha(z) \equiv 0$. Define

$$e_{L,\alpha} = j_\alpha - \alpha_0, \tag{1.5}$$

where $e_{L,\alpha} = +\infty$ if $a_\alpha(z) \equiv 0$.

The characteristic polygon of Σ is defined by

$$\Sigma := \textit{the convex hull of } \bigcup_\alpha \rfloor(|\alpha|, e_{L,\alpha}).$$

The boundary of Σ consists of a vertical half line $\Sigma(0)$ and a horizontal half line $\Sigma(p^*)$ and $p^* - 1$ segments $\Sigma(i)$ $(1 \leq i \leq p^* - 1)$ with slope γ_i, $0 = \gamma_{p^*} < \gamma_{p^*-1} < \cdots < \gamma_1 < \gamma_0 = +\infty$.

Let $\{(m_i, e(i)) \in \mathbb{R}^2; 0 \leq i \leq p^* - 1\}$ be vertices of Σ, where $0 \leq m_{p^*-1} < \cdots < m_i < m_{i-1} < \cdots < m_0 = m$. So the endpoints of $\Sigma(i)$ $(1 \leq i \leq p^* - 1)$ are $(m_{i-1}, e(i-1))$ and $(m_i, e(i))$. We call the slope γ_i of $\Sigma(i)$ the *i-th characteristic index* of $L(z, \partial)$ with respect to $K = \{z_0 = 0\}$.

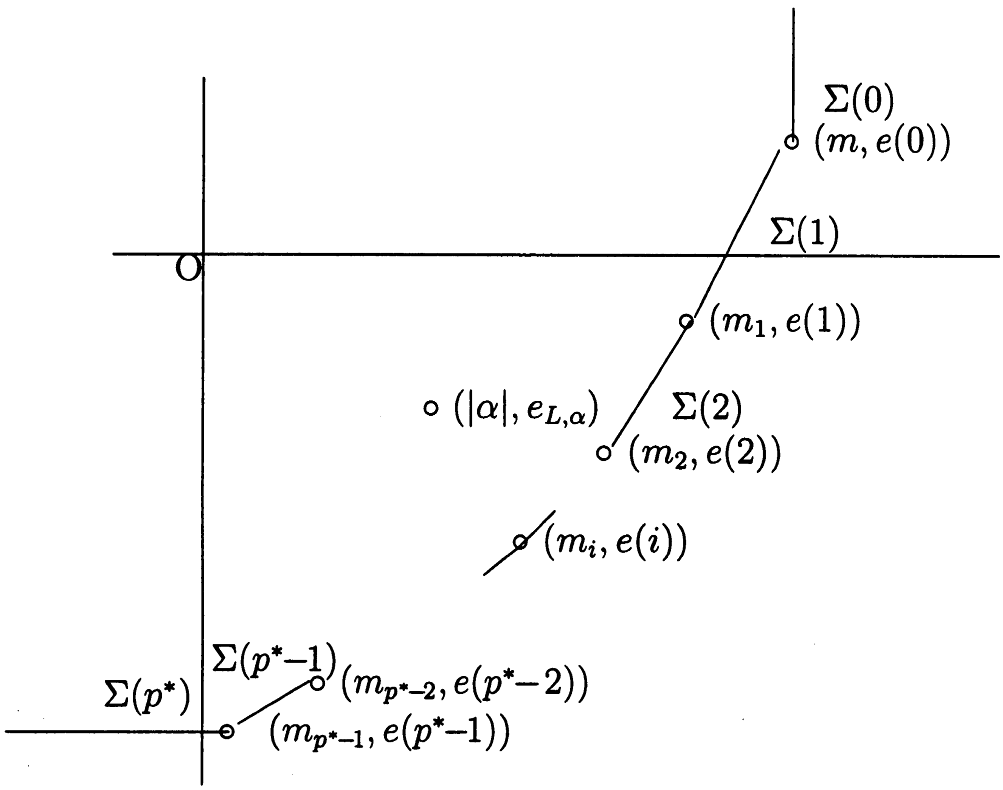

Figure 1: Characteristic polygon

Let $\Delta(i)$ be a subset of multi-indices and $l_i \in \mathbb{N}$ $(0 \le i \le p^* - 1)$ defined by

$$
(1.6) \qquad \begin{cases} \Delta(i) := \{\alpha \in \mathbb{N}^{d+1}; |\alpha| = m_i,\ e_{L,\alpha} = e(i)\}, \\ l_i := \max\{|\alpha'| :\ \alpha \in \Delta(i)\}. \end{cases}
$$

Define a subset $\Delta_0(i)$ of $\Delta(i)$ and a polynomial $\chi_{L,i}(z', \xi')$ in ξ' $(0 \le i \le p^* - 1)$ by

$$
(1.7) \qquad \begin{cases} \Delta_0(i) = \{\alpha \in \Delta(i); |\alpha'| = l_i\}, \\ \chi_{L,i}(z', \xi') = \displaystyle\sum_{\alpha \in \Delta_0(i)} b_\alpha(0, z')\xi^{\alpha'}. \end{cases}
$$

$\chi_{L,i}(z', \xi')$ is homogeneous in ξ' with degree l_i.

§2 Existence of singular solutions

Let us return to the equation

$$
(Eq) \qquad L(z, \partial)u(z) = f(z) \in \mathcal{O}(U(\theta)).
$$

The existence of singular solutions are studied by [2], [4], [10] and other papers referred in these papers. More generally we have

Theorem 2.1. *Suppose that $\chi_{L,0}(0, \xi') \not\equiv 0$. Then there is a solution $u(z) \in \mathcal{O}(V(\theta))$ of (Eq) for some $V \subset U$.*

In this paper we consider the case $f(z)$ has tempered singularities on K. We have a solution $u(z)$ by Theorem 2.1, but $u(z)$ has not always tempered singularities. We can generally show $|u(z)| \le A\exp(c|z_0|^{-\gamma_1})$ for $z \in V(\theta')$ $(0 < \theta' < \theta)$. So our interest is to find a solution $u(z) \in \mathcal{O}_{temp,c'}(V(\theta'))$ of the equation

$$
L(z, \partial)u(z) = f(z) \in \mathcal{O}_{temp,c}(U(\theta)))
$$

for some polydisc $V \subset U$ and constants c' and $0 < \theta' < \theta$.

Let us give conditions (C_i) $(0 \le i \le p^* - 1)$. For fixed i, $0 \le i \le p^* - 2$

(C_i) $\quad j_\alpha = 0$ *for* $\alpha \in \Delta_0(i)$ *and* $\chi_{L,i}(0, \xi') \not\equiv 0$.

For $i = p^* - 1$

(C_{p^*-1}) $\quad |\alpha'| \le l_{p^*-1}$ *for* $\alpha \in \{\alpha \in \mathbb{N}^{d+1};\ e_{L,\alpha} = e(p^*-1)\}$ *and* $\chi_{L,p^*-1}(0, \xi') \not\equiv 0$.

Our main existence theorem is

Theorem 2.2. *Suppose* $p^* \ge 2$ *and* (C_i) *hold for all* $0 \le i \le p^* - 1$. *Let* $f(z) \in \mathcal{O}_{temp,c}(U(\theta))$ *and* θ' *be a constant with* $0 < \theta' < \min\{\theta, \pi/2\gamma_1\}$. *Then there is a solution* $u(z) \in \mathcal{O}_{temp,c'}(V(\theta'))$ *of* (Eq) *for some polydisc* V *and a constant* c'.

We note that the opening angle θ' of sectorial region is restricted by γ_1. We need two theorems in order to show Theorem 2.2. One is

Theorem 2.3. *Suppose* $p^* \ge 2$ *and* $L(z, \partial)$ *satisfies* (C_{p^*-1}). *Let* $f(z) \in \mathcal{O}_{temp,c}(U(\theta))$ *and* θ' *be a constant with* $0 < \theta' < \min\{\theta, \pi/2\gamma_{p^*-1}\}$. *Then there is a* $v(z) \in \mathcal{O}_{temp,c'}(V(\theta'))$ *for some polydisc* V *and a constant* c' *such that* $(Rf)(z) := (L(z, \partial)v(z) - f(z)) \sim 0$ *in* $\mathcal{A}_{\{\gamma_{p^*-1}\}}(V(\theta'))$.

The other is

Theorem 2.4. *Suppose* $p^* \ge 2$ *and* $L(z, \partial)$ *satisfies* (C_i) *for* $i = 0, 1, \cdots p^* - 2$ *and let* $f(z) \in \mathcal{A}_{\{\gamma_{p^*-1}\}}(U(\theta))$. *Then for any* $0 < \theta' < \min\{\theta, \pi/2\gamma_1\}$ *there is* $u(z) \in \mathcal{A}_{\{\gamma_{p^*-1}\}}(V(\theta'))$ *satisfying* $L(z, \partial)u(z) = f(z)$ *in* $V(\theta')$ *for some polydisc* V.

Theorem 2.4 is given in [8] and [9], where we considered the existence of solutions with asymptotic expansion under the condition that $f(z)$ in (Eq) has an asymptotic expansion. We exclude $p^* = 1$ in the preceding theorems, however, we have from results in [4]

Theorem 2.5. *Suppose* $p^* = 1$ *and* (C_0) *holds. Let* $f(z) \in \mathcal{O}_{temp,c}(U(\theta))$. *Then there is a solution* $u(z) \in \mathcal{O}_{temp,c'}(V(\theta))$ *of* (Eq) *for some polydisc* V *and a constant* c'

The operators of Fuchsian type (see [1]) satisfy the conditions of Theorem 2.5.

Example. Let

$$L(z, \partial) = \partial_1^5 + A_1(z)\partial_1^3\partial_0 + A_2(z)\partial_0^2, \quad z = (z_0, z_1) \in \mathbb{C}^2, \tag{2.1}$$

where $A_i(z) = z_0^{j_i} B_i(z)$, $j_i \in \mathbb{N}$, $B_i(0) \neq 0$ for $i = 1, 2$. According to the values of j_1 and j_2, several cases occur. However the conditions in Theorem 2.2 or the conditions in Theorem 2.5 hold for any case. So $L(z, \partial)u(z) = f(z)$ has always a solution $u(z)$ with tempered singularities in a sectorial region for $f(z)$ with tempered singularities.

§3 Outline of the proof of Theroem 2.3.

In order to find $v(z)$ in Theorem 2.3 we construct a parametrix of $L(z, \partial_z)$. The method of construction of the parametrix is a modification of that in [6]. We may assume $e(p^* - 1) = 0$ and θ_0 be a constant with $0 < \theta_0 < \pi/2\gamma_{p^*-1}$. $v(z) = (Gf)(z)$ is constructed of the form

$$(3.1) \qquad (Gf)(z) := \int_S G(z, w) f(w) dw, \quad w = (w_0, w_1, \dots, w_d) = (w_0, w'),$$

where S is a chain in $V(\theta_0)$. The kernel $G(z, w)$ has the form

$$(3.2) \qquad G(z, w) = \frac{1}{2\pi i} \int_{\lambda_0}^{\infty} z_0^{\lambda} w_0^{-\lambda - 1} K(z, w', \lambda) d\lambda.$$

We can find $K(z, w', \lambda)$ with the following:

1. $K(z, w', \lambda)$ is holomorphic $\{z_0; 0 < |z_0| < r_0, |\arg z_0| < \theta_0\} \times \{(z', w'); |z_j| r_1 < r_2 < |w_j| < r_3,\ 1 \leq j \leq d\}$ and holomorphic in λ in some infinite region.

2. $K(z, w', \lambda)$ has an asymptotic expansion
$$K(z, w', \lambda) \sim \widehat{K}(z, w', \lambda) = \sum_{n=0}^{\infty} k_n(z, w', \lambda) z_0^n,$$
where $\widehat{K}(z, w', \lambda)$ is a formal power series of z_0.

3. $\widehat{K}(z, w', \lambda)$ satisfies formally
$$L(z, \partial)(z_0^{\lambda} \widehat{K}(z, w', \lambda)) = \frac{z_0^{\lambda}}{(2\pi i)^d} \prod_{j=1}^{d} \frac{1}{(w_j - z_j)}.$$

As for $G(z,w)$ we have

$$L(z,\partial)G(z,w)=\delta(z,w)+R(z,w), \tag{3.3}$$

where

$$\begin{cases} \delta(z,w)=\dfrac{1}{(2\pi i)^{d+1}}\Big(\displaystyle\int_{\lambda_0}^{\infty} z_0^{\lambda}w_0^{-\lambda-1}d\lambda\Big)\prod_{j=1}^{d}\dfrac{1}{(w_j-z_j)} \\ \\ |R(z,w)|\le C\exp(-c|z_0|^{-\gamma_{p^*-1}}). \end{cases} \tag{3.4}$$

It follows from (3.3) and (3.4) that $(Rf)(z)=L(z,\partial)v(z)-f(z)$ satisfies the conclusions of Theorem 2.3.

References

[1] Baouendi, M.S. and Goulaouic, C., Cauchy problems with characteristic initial hypersurface, *Comm. Pure Appl. Math.,* **26** (1973), 455-475.

[2] Hamada, Y., Leray, J. et Wagschal, C., Système d'équation aux derivées partielles à caractéristic multiples; problème de Cauchy ramifié; hyperbolicité partielle, *J. Math. Pures Appl.,* **55** (1976), 297-352.

[3] Ōuchi, S., Characteristic Cauchy problems and solutions of formal power series , *Ann. L'institut Fourier,* **33** (1983), 131-176.

[4] Ōuchi, S., Existence of singular solutions and null solutions for linear partial differential operators, *J. Fac. Sci. Univ. Tokyo,* **32** (1985), 457-498.

[5] Ōuchi, S., Singular solutions with asymptotic expansion of linear partial differential equations in the complex domain, *Publ. RIMS Kyoto Univ.,* **34** (1998), 291-311.

[6] Ōuchi, S., Growth property and slowly increasing behavior of singular solutions of linear partial differential equations in the complex domain, *J. Math. Soc. Japan.* **52** (2000), 767-792

[7] Ōuchi, S., Asymptotic expansion of singular solutions and the characteristic polygon of linear partial differential equations in the complex domain, *Publ. RIMS Kyoto Univ.,* **36** (2000), 457-482.

[8] Ōuchi, S., Existence of weakly singular solutions to linear partial differential equations with holomorphic coefficients, *Kokyûroku of RIMS Kyoto Univ., Complex Analysis and Microloca Analysis,* **1158** (2000) 66-72.

[9] Ōuchi, S. Existence of solutions with asymptotic expansion of linear partial differential equations in the complex domain, (*preprint*)

[10] Persson, J., Singular holomorphic solutions of linear partial differential equations with holomorphic coefficients and nonanalytic solutions with analytic coefficients, *Asterisque 89-90, analytic solutions of partial differential equations* (*Trento 1981*), Soc Math. France, 233-247.