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Explicit formulas for the reproducing kernels of the space of harmonic polynomials in the case of real rank 1

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Introduction.

Let $H_n(\mathbf{C}^p)$ be the space of homogeneous harmonic polynomials on $\mathbf{C}^p$ of degree n ($p \in \mathbf{N}$, $p \geq 2$). It is well known that the restriction mapping $f \longrightarrow f|_{S^{p-1}}$ is a bijection from $H_n(\mathbf{C}^p)$ onto $H_n(S^{p-1})$, where $H_n(S^{p-1})$ is the space of spherical harmonics of degree n in dimension p. This fact can be extended to the following form:

Theorem 0.1 (cf. [2], [5], [7], [11]). *Let $\mathcal{O}(\neq \{0\})$ be any $SO(p)$-orbit in $\mathbf{C}^p$. Then, the restriction mapping $r_{\mathcal{O}} : f \longrightarrow f|_{\mathcal{O}}$ is a bijection from $H_n(\mathbf{C}^p)$ onto $H_n(\mathbf{C}^p)|_{\mathcal{O}}$.*

In addition, we can express the inverse formula of this map $r_{\mathcal{O}}$ explicitly as an integral on the orbit $\mathcal{O}$, by using the Legendre poynomials (for details, see [2], [5], [7], [11]).

On the other hand, according to the formulation in [4], classical harmonic polynomials on $\mathbf{C}^p$ can be canonically identified with the harmonic polynomials on the space $\mathfrak{p}$, where $\mathfrak{so}(p,1) = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$ is the Cartan decomposition of the Lie algebra $\mathfrak{so}(p,1)$ and $\mathfrak{p}$ is the complexification of $\mathfrak{p}_{\mathbf{R}}$. In this situation, any $SO(p)$-orbit in $\mathbf{C}^p$ corresponds to a $K_{\mathbf{R}}$-orbit in $\mathfrak{p}$, where $K_{\mathbf{R}} = \exp \operatorname{ad} \mathfrak{k}_{\mathbf{R}}$. Therefore, Theorem 0.1 can be reformulated in this Lie algebraic form, and we can express the inverse formula $r_{\mathcal{O}}^{-1}(f)$ explicitly from this standpoint (Theorem 1.2).

In this note we shall give explicit reproducing formulas of harmonic polynomials on each single $K_{\mathbf{R}}$-orbit $\mathcal{O}$ for two remaining classical real rank 1 cases $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$ and $\mathfrak{sp}(p,1)$ ($p \geq 1$), and show that the similar results as in Theorem 0.1 hold for these cases. This is an extension of our previous note [10], where we expressed the inverse formula as an integral on every nilpotent $K_{\mathbf{R}}$-orbit in $\mathfrak{p}$.

§ 1. Harmonic polynomials on $\mathfrak{p}$.

In this section we fix notations which we use in this note and recall the definitions and the results on harmonic polynomials on $\mathfrak{p}$.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{g}_{\mathbf{R}}$ be a noncompact real form of $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the complexification of the Cartan decomposition $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$. We put G

$= \exp \operatorname{ad} \mathfrak{g}$ and $K_\theta = \{g \in G\,;\, \theta g = g\theta\}$, where $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ is defined by $\theta = 1$ on $\mathfrak{k}$ and $\theta = -1$ on $\mathfrak{p}$. Let K be the identity component of K_θ. Then we have $K = \exp \operatorname{ad} \mathfrak{k}$.

Now we define harmonic polynomials on $\mathfrak{p}$. S and S_n denote the spaces of polynomials on $\mathfrak{p}$ and homogeneous polynomials on $\mathfrak{p}$ of degree n, respectively. For $f \in S$ and $g \in K_\theta$, gf is defined by $(gf)(X) = f(g^{-1}X)$ $(X \in \mathfrak{p})$. J denotes the ring of K-invariant polynomials on $\mathfrak{p}$ and we put $J_+ = \{f \in J\,;\, f(0) = 0\}$. It is known that J is also K_θ-invariant. According to the definition in [4], $f \in S$ is harmonic if and only if $(\partial P)f = 0$ for any $P \in J_+$. $\mathcal{H}_n$ denotes the space of homogeneous harmonic polynomials on $\mathfrak{p}$ of degree n. We put $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$. The following results are known:

Theorem 1.1 (cf. [1], [4]). (i) *For any $n \in \mathbf{Z}_+$ we have*

$$S_n = (J_+S)_n \oplus \mathcal{H}_n,$$

where $(J_+S)_n = S_n \cap J_+S$.

(ii) *We put $\mathcal{N} = \{X \in \mathfrak{p}\,;\, P(X) = 0$ for any $P \in J_+\}$ and $h(X,Y) = \operatorname{Tr}({}^tX\overline{Y})$ for $X, Y \in \mathfrak{p}$. Then $\mathcal{H}_n$ is generated by $\{h(\ ,Z)^n\,;\, Z \in \mathcal{N}\}$.*

(iii) *Let Γ be a maximal dimensional K_θ-orbit in $\mathfrak{p}$. Then the restriction mapping $f \longrightarrow f|_\Gamma$ is a bijection from $\mathcal{H}_n$ onto $\mathcal{H}_n|_\Gamma$.*

For harmonic polynomials on $\mathfrak{p}$ for general semisimple Lie algebras $\mathfrak{g}$, see [4].

From now we consider the case where $\mathfrak{g}_{\mathbf{R}}$ is classical real rank 1, i.e., $\mathfrak{g}_{\mathbf{R}} = \mathfrak{so}(p,1)$, $\mathfrak{su}(p,1)$ or $\mathfrak{sp}(p,1)$. In this note we assume that $p \in \mathbf{N}$, $p \geq 2$, unless otherwise stated. Let $K_{\mathbf{R}}$ be the adjoint group of $\mathfrak{k}_{\mathbf{R}}$. We consider $K_{\mathbf{R}} \subset K$ and $\mathfrak{p}_{\mathbf{R}} \subset \mathfrak{p}$. Let $B(X,Y)$ $(X, Y \in \mathfrak{g})$ be the Killing form of $\mathfrak{g}$. Then, J is generated by $B(X,X)$ $(X \in \mathfrak{p})$. We put $\Sigma = \{X \in \mathfrak{p}\,;\, B(X,X) = 1\}$ and $\Sigma_{\mathbf{R}} = \Sigma \cap \mathfrak{p}_{\mathbf{R}}$. In this case we see that $\Sigma_{\mathbf{R}}$ consists of one $K_{\mathbf{R}}$-orbit and that $\mathcal{H}_n \simeq \mathcal{H}_n|_{\Sigma_{\mathbf{R}}}$ (see [5], [8], [9]).

Now we recall the results in the case of $\mathfrak{so}(p,1)$. When $\mathfrak{g}_{\mathbf{R}} = \mathfrak{so}(p,1)$, J is generated by $P(X) = \operatorname{Tr}({}^tXX)$ $(X \in \mathfrak{p})$. We put

$$Q_{n,p}(X,Y) = 2^{-n} P_{n,p}\left(\frac{h(X,Y)}{(P(X)P(\overline{Y}))^{1/2}}\right)(P(X)P(\overline{Y}))^{n/2} \quad (X, Y \in \mathfrak{p}),$$

where $P_{n,p}$ is the Legendre polynomial of degree n and dimension p. Then the following results hold:

Theorem 1.2 (cf. [2], [5], [7], [11]). *Assume that $\mathfrak{g}_{\mathbf{R}} = \mathfrak{so}(p,1)$ $(p \geq 2)$.*

(i) *Let $\mathcal{O} (\neq \{0\})$ be a $K_{\mathbf{R}}$-orbit in $\mathfrak{p}$. Then the restriction mapping $r_{\mathcal{O}} : f \longrightarrow f|_{\mathcal{O}}$ is a bijection from $\mathcal{H}_n$ onto $\mathcal{H}_n|_{\mathcal{O}}$.*

(ii) *We have for $f \in \mathcal{H}_n|_{\mathcal{O}}$*

$$r_{\mathcal{O}}^{-1} f(X) = \dim \mathcal{H}_n \int_{\mathcal{O}} f(Y) Q_{n,p}(X,Y) d\mu_{\mathcal{O}}(Y) \quad (X \in \mathfrak{p}),$$

where $d\mu_{\mathcal{O}}$ is the normalized $K_{\mathbf{R}}$-invarint measure on $\mathcal{O}$.

§ 2. **Integral formulas of harmonic polynomials: The case of $\mathfrak{su}(p,1)$.**

In this section we give the reproducing kernel of $\mathcal{H}_n$ on any $K_{\mathbf{R}}$-orbit in $\mathfrak{p}$ in the case of $\mathfrak{g} = \mathfrak{sl}(p+1, \mathbf{C})$ and $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$ ($p \in \mathbf{N}$, $p \geq 2$).

In this case, we have

$$\mathfrak{k}_{\mathbf{R}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} ;\ A \in \mathfrak{u}(p),\ \alpha \in \mathfrak{u}(1),\ \mathrm{Tr}\, A + \alpha = 0 \right\},$$
$$\mathfrak{p}_{\mathbf{R}} = \left\{ \begin{pmatrix} 0 & x \\ {}^t\overline{x} & 0 \end{pmatrix} ;\ x \in \mathbf{C}^p \right\},$$
$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} ;\ A \in M(p, \mathbf{C}),\ \mathrm{Tr}\, A + \alpha = 0 \right\},$$
$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix} ;\ x, y \in \mathbf{C}^p \right\},$$

and $K_{\mathbf{R}} = \mathrm{Ad}\, S(U(p) \times U(1)) = \{\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} ;\ A \in U(p)\}$. For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix} \in \mathfrak{p}$, $P(X) = \frac{1}{2}\mathrm{Tr}\,(X^2) = {}^t y x$ gives a generator of J. We put $\mathcal{N} = \{X \in \mathfrak{p};\ P(X) = 0\}$, $\Sigma = \{X \in \mathfrak{p};\ P(X) = 1\}$ and $\Sigma_{\mathbf{R}} = \Sigma \cap \mathfrak{p}_{\mathbf{R}}$. $\mathcal{H}_n = \{f \in S_n;\ \sum_{j=1}^{p} \frac{\partial^2}{\partial x_j \partial y_j} f = 0\}$ is the space of homogeneous harmonic polynomials on $\mathfrak{p}$ of degree n. For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix} \in \mathfrak{p}$ we define the bijection $\Psi : \mathfrak{p} \longrightarrow \mathbf{C}^{2p}$ by $\Psi(X) = \frac{1}{2}\begin{pmatrix} x + y \\ i(y - x) \end{pmatrix}$. Then $f \in \mathcal{H}_n$ if and only if $f \circ \Psi^{-1} \in H_n(\mathbf{C}^{2p})$ and we have $\dim \mathcal{H}_n = \dim H_n(\mathbf{C}^{2p}) = \frac{2(n+p-1)(n+2p-3)!}{n!(2p-2)!}$.

Remark that the mapping $\Psi : \Sigma_{\mathbf{R}} \longrightarrow S^{2p-1}$ is bijective and $\widetilde{H}_n(X, Y) = Q_{n,2p}(\Psi(X), \Psi(Y))$ is the reproducing kernel of $\mathcal{H}_n$ on $\Sigma_{\mathbf{R}}$ (see [8] Proposition 2.1).

For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix} \in \mathfrak{p}$ and $g = \mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in K_{\mathbf{R}}$ ($A \in U(p)$) we have $gX = \begin{pmatrix} 0 & Ax \\ {}^t(\overline{A} y) & 0 \end{pmatrix}$. We put

$$\widetilde{E}_r = \begin{pmatrix} 0 & re_1 \\ \sqrt{1-r^2}\, {}^t e_2 & 0 \end{pmatrix} \in \mathcal{N} \qquad (0 \leq r \leq 1),$$
$$E_1 = \begin{pmatrix} 0 & e_1 \\ {}^t e_1 & 0 \end{pmatrix} \in \Sigma_{\mathbf{R}},$$
$$\widetilde{E}_{r,q} = \begin{pmatrix} 0 & re_1 \\ {}^t((1/r)e_1 + qe_2) & 0 \end{pmatrix} \in \Sigma \quad (r > 0, q \geq 0),$$

where $e_1 = {}^t(1\,0 \cdots 0)$, and $e_2 = {}^t(0\,1 \cdots 0)$. Then we have

$$K_{\mathbf{R}} E_1 = \Sigma_{\mathbf{R}} \quad \text{and} \quad \mathfrak{p} = \mathcal{N} \cup \bigcup_{\lambda \in \mathbf{C} \setminus \{0\}} \lambda \Sigma.$$

Remark that

$$\Sigma = \bigcup_{q\geq 0, r>0} K_{\mathbf{R}}\widetilde{E}_{r,q} \quad \text{and} \quad \mathcal{N} = \bigcup_{\rho\geq 0, 0\leq r\leq 1} K_{\mathbf{R}}(\rho\widetilde{E}_r)$$

give the $K_{\mathbf{R}}$-orbit decompositions of Σ and $\mathcal{N}$, respectively.

We put $\Lambda = \{(n,k)\,;\, n \in \mathbf{Z}_+, 0 \leq k \leq n\}$. For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & a \\ {}^t b & 0 \end{pmatrix} \in \mathfrak{p}$ we put $\widetilde{K}_{n,k}(X,Y) = (x \cdot \bar{a})^k (y \cdot \bar{b})^{n-k}$ $((n,k) \in \Lambda)$, where $z \cdot w = {}^t z w$ for $z, w \in \mathbf{C}^p$. It is clear that $\widetilde{K}_{n,k}(\ ,Y) \in \mathcal{H}_n$ $(Y \in \mathcal{N})$. Let $\mathcal{H}_{n,k}$ be the space which is spanned by the elements $\widetilde{K}_{n,k}(\ ,Y)$ $(Y \in \mathcal{N})$. The equality $\widetilde{K}_{n,k}(gX, gY) = \widetilde{K}_{n,k}(X,Y)$ holds for any $g \in K_{\mathbf{R}}$, $X, Y \in \mathfrak{p}$. From [6] Theorem 14.4 we can easily see that $\mathcal{H}_n = \bigoplus_{k=0}^n \mathcal{H}_{n,k}$ gives the $K_{\mathbf{R}}$-irreducible decomposition of $\mathcal{H}_n$ and $\dim \mathcal{H}_{n,k} = \frac{(p+n-1)(k+p-2)!(n-k+p-2)!}{(p-1)!(p-2)!k!(n-k)!}$. We put $E_0 = \begin{pmatrix} 0 & e_1 \\ {}^t e_2 & 0 \end{pmatrix}$. Then we have the following proposition.

Proposition 2.1 (cf. [10]). (i) *For any $f \in \mathcal{H}_{n,k}$ and $X \in \mathfrak{p}$ we have*

$$\delta_{n,m}\delta_{k,l} f(X) = \dim \mathcal{H}_{n,k} \int_{K_{\mathbf{R}}} f(gE_0)\widetilde{K}_{m,l}(X, gE_0)dg. \tag{2.1}$$

(ii) *For any $f \in \mathcal{H}_{n,k}$ and $h \in \mathcal{H}_{m,l}$ we have*

$$\int_{K_{\mathbf{R}}} f(gE_1)\overline{h(gE_1)}dg = \delta_{n,m}\delta_{k,l}\binom{k+p-2}{k}\binom{n+p-2}{k}^{-1} \int_{K_{\mathbf{R}}} f(gE_0)\overline{h(gE_0)}dg. \tag{2.2}$$

Now we define $\widetilde{H}_{n,k}(X,Z)$ $(X, Z \in \mathfrak{p})$ by

$$\widetilde{H}_{n,k}(X,Z) = \dim \mathcal{H}_{n,k}\binom{n+p-2}{k}\binom{k+p-2}{k}^{-1} \int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(X, gE_0)\widetilde{K}_{n,k}(gE_0, Z)dg.$$

Clearly we have $\widetilde{H}_{n,k}(\ ,Z) \in \mathcal{H}_n$ $(Z \in \mathfrak{p})$ and

$$\begin{aligned} \widetilde{H}_{n,k}(X,Z) &= \overline{\widetilde{H}_{n,k}(Z,X)}, \\ \widetilde{H}_{n,k}(gX,gZ) &= \widetilde{H}_{n,k}(X,Z) \quad (g \in K_{\mathbf{R}}). \end{aligned}$$

We shall show that the reproducing kernel of $\mathcal{H}_n$ on each $K_{\mathbf{R}}$-orbit can be expressed in terms of $\widetilde{H}_{n,k}(\ ,Z)$ $(Z \in \mathfrak{p}$, $(n,k) \in \Lambda)$. Our main theorem in this section is the following

Theorem 2.2. *Let $\mathcal{O} = K_{\mathbf{R}}X_0$, $X_0 \in \mathfrak{p}$ and $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ $({}^\forall(n,k) \in \Lambda)$.*
(i) *The restriction mapping $r_{\mathcal{O}} : f \longrightarrow f|_{\mathcal{O}}$ is a bijection from $\mathcal{H}_n$ onto $\mathcal{H}_n|_{\mathcal{O}}$.*
(ii) *For $f \in \mathcal{H}_n|_{\mathcal{O}}$ we have*

$$r_{\mathcal{O}}^{-1}(f)(X) = \sum_{k=0}^{n} \frac{\dim \mathcal{H}_{n,k}}{\widetilde{H}_{n,k}(X_0, X_0)} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{H}_{n,k}(X, gX_0)dg. \tag{2.3}$$

To prove Theorem 2.2 we need the following lemma.

Lemma 2.3. *Let K_0 be the isotropy group of E_1 in $K_{\mathbf{R}}$ and $\mathcal{H}'_{n,k} = \{f \in \mathcal{H}_{n,k}\,;\, gf = f$ for any $g \in K_0\}$. If $f \in \mathcal{H}'_{n,k}$, we have*

$$f = f(E_1)\widetilde{H}_{n,k}(\ ,E_1). \tag{2.4}$$

Sketch of Proof. From [8] Lemmas 2.4 and 2.5 and the definition of $\mathcal{H}_{n,k}$ we can prove (2.4). q.e.d.

Proof of Theorem 2.2. From (2.1)-(2.4) we have for any $X_0 \in \mathfrak{p}$

$$\dim \mathcal{H}_{n,k} \int_{K_{\mathbf{R}}} \widetilde{H}_{m,l}(gX_0, E_1)\widetilde{H}_{n,k}(X, gX_0)dg = \delta_{n,m}\delta_{k,l}\widetilde{H}_{n,k}(X_0, X_0)\widetilde{H}_{n,k}(X, E_1). \tag{2.5}$$

(2.5) implies that for any $f \in \mathcal{H}_{m,l}$ and any $X_0 \in \mathfrak{p}$

$$\dim \mathcal{H}_{n,k} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{H}_{n,k}(X, gX_0)dg = \delta_{n,m}\delta_{k,l}\widetilde{H}_{n,k}(X_0, X_0)f(X) \tag{2.6}$$

because $\widetilde{H}_{n,k}(\ ,E_1)$ is a generator of $\mathcal{H}_{n,k}$. (2.6) gives Theorem 2.2. q.e.d.

Remark 2.4. From the definition it is valid that

$$\widetilde{H}_{n,k}(X_0, X_0) = 0 \quad \text{iff} \quad \int_{K_{\mathbf{R}}} |\widetilde{K}_{n,k}(gX_0, E_0)|^2 dg \quad (X_0 \in \mathfrak{p}).$$

Therefore the following two conditions (2.7) and (2.8) are equivalent.

$$\widetilde{H}_{n,k}(X_0, X_0) = 0, \tag{2.7}$$

$$\mathcal{H}_{n,k}|_{K_{\mathbf{R}}X_0} = \{0\}. \tag{2.8}$$

This implies that $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ if and only if $X_0 \notin \lambda K_{\mathbf{R}}\widetilde{E}_1$ and $X_0 \notin \lambda K_{\mathbf{R}}\widetilde{E}_0$ $(\lambda \in \mathbf{C})$.

§ 3. Integral formulas of harmonic polynomials: The case of $\mathfrak{sp}(p,1)$.

In this section we consider the case $\mathfrak{sp}(p,1)$ $(p \in \mathbf{N},\ p \geq 2)$. From now we put $\mathfrak{g} = \mathfrak{sp}(p+1, \mathbf{C})$, $\mathfrak{g}_{\mathbf{R}} = \mathfrak{sp}(p,1)$,

$$\mathfrak{k}_{\mathbf{R}} = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & a & 0 & b \\ -\overline{B} & 0 & \overline{A} & 0 \\ 0 & -\overline{b} & 0 & \overline{a} \end{pmatrix} ;\ \begin{array}{l} A \in \mathfrak{u}(p),\ a \in \mathfrak{u}(1),\ b \in \mathbf{C} \\ B \text{ is } p \times p \text{ symmetric} \end{array} \right\},$$

$$\mathfrak{p}_{\mathbf{R}} = \left\{ \begin{pmatrix} 0 & x & 0 & y \\ {}^t\overline{x} & 0 & {}^ty & 0 \\ 0 & \overline{y} & 0 & -\overline{x} \\ {}^t\overline{y} & 0 & -{}^tx & 0 \end{pmatrix} ;\ x, y \in \mathbf{C}^p \right\}.$$

Then we have

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ C & 0 & -{}^tA & 0 \\ 0 & \gamma & 0 & -\alpha \end{pmatrix} ; \begin{array}{c} A, B, C \in M(p, \mathbf{C}) \\ {}^tB = B,\ {}^tC = C \\ \alpha, \beta, \gamma \in \mathbf{C} \end{array} \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x & 0 & w \\ {}^ty & 0 & {}^tw & 0 \\ 0 & z & 0 & -y \\ {}^tz & 0 & -{}^tx & 0 \end{pmatrix} ; x, y, z, w \in \mathbf{C}^p \right\},$$

and

$$K_{\mathbf{R}} = \left\{ \mathrm{Ad} \begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ -\overline{B} & 0 & \overline{A} & 0 \\ 0 & -\overline{\beta} & 0 & \overline{\alpha} \end{pmatrix} \in \mathrm{Ad}\, U(2p+2) ; \begin{array}{c} {}^tA\overline{A} + {}^t\overline{B}B = I_p, \\ {}^tA\overline{B} = {}^t\overline{B}A, \\ \alpha\overline{\alpha} + \beta\overline{\beta} = 1 \end{array} \right\}.$$

For $X = \begin{pmatrix} 0 & x & 0 & w \\ {}^ty & 0 & {}^tw & 0 \\ 0 & z & 0 & -y \\ {}^tz & 0 & -{}^tx & 0 \end{pmatrix} \in \mathfrak{p}$, $P(X) = \frac{1}{4}\mathrm{Tr}\, X^2 = {}^txy + {}^tzw$ gives a generator of J and $\mathcal{H}_n = \{f \in S_n ; \sum_{j=1}^p \left(\frac{\partial^2}{\partial x_j \partial y_j} + \frac{\partial^2}{\partial z_j \partial w_j} \right) f = 0\}$.

For $X \in \mathfrak{p}$ we define the mapping $\Psi : \mathfrak{p} \longrightarrow \mathbf{C}^{4p}$ by $\Psi(X) = \frac{1}{2}\begin{pmatrix} x+y \\ z+w \\ i(y-x) \\ i(w-z) \end{pmatrix}$. We can see that $f \in \mathcal{H}_n$ if and only if $f \circ \Psi^{-1} \in H_n(\mathbf{C}^{4p})$ and from this fact, we have $\dim \mathcal{H}_n = \dim H_n(\mathbf{C}^{4p}) = \frac{2(n+2p-1)(n+4p-3)!}{n!(4p-2)!}$. We put $\mathcal{N} = \{X \in \mathfrak{p} ; P(X) = 0\}$, $\Sigma = \{X \in \mathfrak{p} ; P(X) = 1\}$ and $\Sigma_{\mathbf{R}} = \Sigma \cap \mathfrak{p}_{\mathbf{R}}$. Remark that $\Psi : \Sigma_{\mathbf{R}} \simeq S^{4p-1}$ and $\widetilde{H}_n(X, Y) = Q_{n,4p}(\Psi(X), \Psi(Y))$ is the reproducing kernel on $\Sigma_{\mathbf{R}}$ (see [9] Theorem 2.2).

Let $g = \mathrm{Ad}\begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ -\overline{B} & 0 & \overline{A} & 0 \\ 0 & -\overline{\beta} & 0 & \overline{\alpha} \end{pmatrix} \in K_{\mathbf{R}}$. If we put $\Phi(X) = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbf{C}^{4p}$, we have

$$\Phi(gX) = \begin{pmatrix} A(\overline{\alpha}x + \overline{\beta}w) + B(\overline{\alpha}z - \overline{\beta}y) \\ \overline{B}(-\beta x + \alpha w) + \overline{A}(\alpha y + \beta z) \\ -\overline{B}(\overline{\alpha}x + \overline{\beta}w) + \overline{A}(\overline{\alpha}z - \overline{\beta}y) \\ A(-\beta x + \alpha w) - B(\alpha y + \beta z) \end{pmatrix}.$$

We put $\widetilde{E}_r = \Phi^{-1}\begin{pmatrix} re_1 \\ 0 \\ 0 \\ \sqrt{1-r^2}\, e_2 \end{pmatrix} \in \mathcal{N}$ $(0 \le r \le 1)$, $\widetilde{E}_{r,q} = \Phi^{-1}\begin{pmatrix} re_1 \\ (1/r)e_1 + qe_2 \\ 0 \\ 0 \end{pmatrix} \in \Sigma$ $(r > 0,$

It is clear that $\mathfrak{p} = \mathcal{N} \cup \bigcup_{\lambda\in\mathbf{C}\setminus\{0\}} \lambda\Sigma$. Remark that

$$\mathcal{N} = \bigcup_{q\geq 0, 1/\sqrt{2}\leq r\leq 1} K_{\mathbf{R}}(q\widetilde{E}_r) \quad \text{and} \quad \Sigma = \bigcup_{q\geq 0, r>0} K_{\mathbf{R}}\widetilde{E}_{r,q}$$

give the orbit decompositions of $\mathcal{N}$ and Σ, respectively. We put $E_1 = \widetilde{E}_{1,0} \in \Sigma_{\mathbf{R}}$. Then we have $\Sigma_{\mathbf{R}} = K_{\mathbf{R}}E_1$ ([9] Lemma 2.1). For $X = \Phi^{-1}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, $Y = \Phi^{-1}\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} \in \mathfrak{p}$ we put

$$\begin{aligned}
&\langle X, Y\rangle = \tfrac{1}{2}\mathrm{Tr}\,({}^t X\overline{Y}) = \Phi(X)\cdot\overline{\Phi(Y)},\\
&K_2(X,Y) = (x\cdot\overline{x'} + z\cdot\overline{z'})(y\cdot\overline{y'} + w\cdot\overline{w'}) + (x\cdot\overline{w'} - z\cdot\overline{y'})(y\cdot\overline{z'} - w\cdot\overline{x'}),\\
&\widetilde{K}_m(X,Y) = \tfrac{\Gamma(2p+m)}{m!\Gamma(2p)}\int_{K_{\mathbf{R}}} \langle g\widetilde{E}_1, Y\rangle^m \langle X, g\widetilde{E}_1\rangle^m dg,\\
&\widetilde{K}_{n,k}(X,Y) = \widetilde{K}_{n-2k}(X,Y)\{K_2(X,Y)\}^k \quad (n\in\mathbf{Z}_+,\ 0\leq k\leq [n/2]).
\end{aligned}$$

In [10] we showed that $\widetilde{K}_{n,k}(\ ,Y) \in \mathcal{H}_n$ if $Y \in \mathcal{N}$. $\mathcal{H}_{n,k}$ denotes the subspace of $\mathcal{H}_n$ spanned by $\{\widetilde{K}_{n,k}(\ ,Z)\,;\ Z\in\mathcal{N}\}$. It is valid that $\dim\mathcal{H}_{n,k} = \frac{(n-2k+1)^2(2p+n-k-2)!(2p+k-3)!(2p+n-1)}{(n-k+1)!k!(2p-3)!(2p-1)!}$ and $\mathcal{H}_n = \bigoplus_{q=0}^{[n/2]}\mathcal{H}_{n,q}$ gives the $K_{\mathbf{R}}$-irreducible decomposition of $\mathcal{H}_n$ (cf. [3], [10]). From now we put $\Lambda = \{(n,k)\,;\ n\in\mathbf{Z}_+,\ 0\leq k\leq [n/2]\}$ and $E_0 = \Phi^{-1}\begin{pmatrix} e_1 \\ 0 \\ 0 \\ e_2 \end{pmatrix} \in \mathcal{N}$. Then we have the following

Proposition 3.1 (cf. [10]). *For any $f \in \mathcal{H}_{n,k}$ we have*

$$\delta_{n,m}\delta_{k,l}f(X) = \dim\mathcal{H}_{n,k}\int_{K_{\mathbf{R}}} f(gE_0)\widetilde{K}_{m,l}(X, gE_0)dg \quad (X\in\mathfrak{p}). \tag{3.1}$$

Now we define for $X, Y \in \mathfrak{p}$

$$\widetilde{H}_{n,k}(X,Y) = \dim\mathcal{H}_{n,k}\binom{n+2p-2}{k}\binom{2p+k-3}{k}^{-1}\int_{K_{\mathbf{R}}}\widetilde{K}_{n,k}(X,gE_0)\widetilde{K}_{n,k}(gE_0,Y)dg$$

($(n,k)\in\Lambda$). From (3.1) $\widetilde{H}_{n,k}(\ ,Y)$ belongs to $\mathcal{H}_{n,k}$ for any $Y\in\mathfrak{p}$ and we have

$$\begin{aligned}
&\widetilde{H}_{n,k}(X,Y) = \overline{\widetilde{H}_{n,k}(Y,X)},\\
&\widetilde{H}_{n,k}(gX,gY) = \widetilde{H}_{n,k}(X,Y) \quad (g\in K_{\mathbf{R}}),\\
&\widetilde{H}_{n,k}(X,Y) = \binom{n+2p-2}{k}\binom{2p+k-3}{k}^{-1}\widetilde{K}_{n,k}(X,Y) \quad (X\in\mathcal{N} \text{ or } Y\in\mathcal{N}).
\end{aligned}$$

The purpose of this section is to show that $\widetilde{H}_{n,k}(\ ,Y)$ gives the reproducing kernel of $\mathcal{H}_{n,k}$ for each $K_{\mathbf{R}}$-orbit. Our main theorem is the following

Theorem 3.2. (i) *For any* $f \in \mathcal{H}_{n,k}$ *and any* $X_0 \in \mathfrak{p}$ *such that* $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ *we have*

$$\delta_{n,m}\delta_{k,l}f(X) = \frac{\dim \mathcal{H}_{n,k}}{\widetilde{H}_{n,k}(X_0,X_0)} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{H}_{m,l}(X, gX_0)dg. \tag{3.2}$$

(ii) *Assume* $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ $(X_0 \in \mathfrak{p}, {}^\forall(n,k) \in \Lambda)$, *and put* $\mathcal{O} = K_{\mathbf{R}}X_0$. *Then the restriction mapping* $r_{\mathcal{O}} : f \longrightarrow f|_{\mathcal{O}}$ *is a bijection from* $\mathcal{H}_n$ *onto* $\mathcal{H}_n|_{\mathcal{O}}$. *And for* $f \in \mathcal{H}_n|_{\mathcal{O}}$ *we have*

$$r_{\mathcal{O}}^{-1}(f)(X) = \sum_{k=0}^{n} \frac{\dim \mathcal{H}_{n,k}}{\widetilde{H}_{n,k}(X_0,X_0)} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{H}_{n,k}(X, gX_0)dg.$$

To prove this theorem we need some lemmas.

Lemma 3.3 (cf. [10]). *For any* $f \in \mathcal{H}_{n,k}$, $h \in \mathcal{H}_{m,l}$ *and* $X \in \mathfrak{p}$ *we have*

$$\int_{K_{\mathbf{R}}} f(gE_1)\overline{h(gE_1)}dg = C_{n,k} \int_{K_{\mathbf{R}}} f(gE_0)\overline{h(gE_0)}dg, \tag{3.3}$$

where $C_{n,k} = \binom{k+2p-3}{k}\binom{n+2p-2}{k}^{-1}$.

Lemma 3.4. *Let* K_0 *be the isotropy group of* E_1 *in* $K_{\mathbf{R}}$ *and let* $\mathcal{H}'_{n,k} = \{f \in \mathcal{H}_{n,k}\,;\, gf = f$ *for any* $g \in K_0\}$. *If* $f \in \mathcal{H}'_{n,k}$, *we have*

$$f = f(E_1)\widetilde{H}_{n,k}(\ ,E_1). \tag{3.4}$$

Sketch of Proof. From (3.1) and [9] Lemma 2.5 we can prove (3.4) with some calculations. q.e.d.

Proof of Theorem 3.2. Using (3.1), (3,3) and (3.4), we can prove (i) and (ii) in the same way as the proof of Theorem 2.2. q.e.d.

Remark 3.5. $\widetilde{H}_{n,k}(X_0, X_0) = 0$ if and only if $\mathcal{H}_{n,k}|_{K_{\mathbf{R}}X_0} = \{0\}$. Therefore we have $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ for any $(n,k) \in \Lambda$ if and only if $X_0 \notin \lambda K_{\mathbf{R}}\widetilde{E}_1$ and $X_0 \notin \lambda K_{\mathbf{R}}\widetilde{E}_0$ $(\lambda \in \mathbf{C})$.

Appendix.

Combining Theorems 1.2, 2.2, and 3.2, we have the following

Theorem. *Assume that* $\mathfrak{g}_{\mathbf{R}}$ *is a classical real simple Lie algebra with real rank* 1, *i.e.*, $\mathfrak{g}_{\mathbf{R}} = \mathfrak{so}(p,1)$ $(p \in \mathbf{N}, p \geq 2)$, $\mathfrak{su}(p,1)$ *or* $\mathfrak{sp}(p,1)$ $(p \in \mathbf{N})$. *Let* $\mathcal{H}_n = \bigoplus_{k=0}^{N(n)} \mathcal{H}_{n,k}$ *be the*

$K_{\mathbf{R}}$-irreducible decomposition of $\mathcal{H}_n$, where $N(n)$ is the number of irreducible components. Then we have

(i) *If $\mathcal{O}$ is any $K_{\mathbf{R}}$-orbit in $\mathfrak{p}$ and $\mathcal{H}_{n,k}|_{\mathcal{O}} \neq \{0\}$, then the restriction mapping $r_{\mathcal{O}} : f \longrightarrow f|_{\mathcal{O}}$ is a bijection from $\mathcal{H}_{n,k}$ onto $\mathcal{H}_{n,k}|_{\mathcal{O}}$.*

(ii) *Let $\widetilde{H}_n(X,Y)$ be the reproducing kernel of $\mathcal{H}_n$ on $\Sigma_{\mathbf{R}}$ and let $\widetilde{H}_{n,k}(X,Y)$ be the $\mathcal{H}_{n,k}$-component of $\frac{\dim \mathcal{H}_n}{\dim \mathcal{H}_{n,k}} \widetilde{H}_n(X,Y)$. Assume $X_0 \in \mathfrak{p}$ satisfies $\widetilde{H}_{n,k}(X_0, X_0) \neq 0$ ($^\forall(n,k) \in \Lambda$), and put $\mathcal{O} = K_{\mathbf{R}}X_0$. Then the restriction mapping $r_{\mathcal{O}} : f \longrightarrow f|_{\mathcal{O}}$ is a bijection from $\mathcal{H}_n$ onto $\mathcal{H}_n|_{\mathcal{O}}$. And for $f \in \mathcal{H}_n|_{\mathcal{O}}$ we have*

$$r_{\mathcal{O}}^{-1}(f)(X) = \sum_{k=0}^{n} \frac{\dim \mathcal{H}_{n,k}}{\widetilde{H}_{n,k}(X_0, X_0)} \int_{K_{\mathbf{R}}} f(gX_0)\widetilde{H}_{n,k}(X, gX_0)dg.$$

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