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# A new proof of Carlson's theorem by Plana's summation formula

吉野 邦生 (Kunio Yoshino)
諏訪 将範 (Masanori Suwa)

Department of Mathematics, Sophia University

**Abstract**

In this paper we will prove Carlson's theorem by using Plana's summation formula.

In §1, we will recall the definitions and transformations of analytic functionals with unbounded carrier.

In §2, we will give a proof of Carlson's theorem by Plana's summation formula.

## 1 The definitions of analytic functionals with unbounded carrier and their transformations.

Let $L$ and $L_\varepsilon$ be following strip regions:

$$\begin{aligned} L &= (-\infty, a] + \imath[-b, b], \\ \text{For } \varepsilon > 0, \quad L_\varepsilon &= (-\infty, a+\varepsilon] + \imath[-b-\varepsilon, b+\varepsilon]. \end{aligned}$$

For $\varepsilon > 0$, $\varepsilon' > 0$ and $k' \in \mathbb{R}$, we put

$$\begin{aligned} & Q_b(L_\varepsilon : k' + \varepsilon') \\ := & \left\{ f(\zeta) \in \mathcal{H}(\mathring{L}_\varepsilon) \cap \mathcal{C}(L_\varepsilon) : \sup_{\zeta \in L_\varepsilon} |f(\zeta) e^{k'\xi + \varepsilon'|\xi|}| < \infty, \quad \zeta = \xi + \imath\eta \right\}. \end{aligned}$$

$\mathcal{H}(\check{L}_\varepsilon)$ is the space of holomorphic functions defined on $\check{L}_\varepsilon$, (interior of $L_\varepsilon$). $\mathcal{C}(L_\varepsilon)$ is the space of continuous functions defined on $L_\varepsilon$. We put

$$Q(L : k') = \varinjlim_{\varepsilon \to 0, \varepsilon' \to 0} Q_b(L_\varepsilon : k' + \varepsilon'),$$

where $\varinjlim$ means inductive limit. If $z \in (-k', \infty) + \imath\mathbb{R}$, then the function $e^{\zeta z}$ of $\zeta$ belongs to $Q(L : k')$. We denote by $Q'(L : k')$ the dual space of $Q(L : k')$. The elements of $Q'(L : k')$ is called analytic functionals with carrier $L$ and of type $k'$.
We define the Fourier-Borel transform $\tilde{T}(z)$ of $T \in Q'(L : k')$ as follows:

$$\tilde{T}(z) = < T_\zeta, e^{\zeta z} > .$$

$\tilde{T}(z)$ is holomorphic function on the right half plane $(-k', \infty) + \imath\mathbb{R}$ and satisfies following estimate :

$\forall \varepsilon > 0, \varepsilon' > 0, \exists C_{\varepsilon,\varepsilon'} \geq 0$ such that

$$|\tilde{T}(z)| \leq C_{\varepsilon,\varepsilon'} e^{ax+b|y|+\varepsilon|z|}, \quad ( \mathrm{Re} z \geq -k' + \varepsilon', \quad z = x + \imath y). \tag{1}$$

$\mathrm{Exp}((-k', \infty) + \imath\mathbb{R} : L)$ denotes the space of holomorphic functions defined on the right half plane $(-k', \infty) + \imath\mathbb{R}$ and satisfy the estimates (1). Following theorem characterizes the Fourier-Borel transform of $Q'(L : k')$.

**Theorem 1.1 ([2],[9]).** *Fourier-Borel transform is a linear topological isomorphism from $Q'(L : k')$ onto $\mathrm{Exp}((-k', \infty) + \imath\mathbb{R} : L)$.*

**Definition 1.2.** $\varepsilon'$-Cauchy transform $\check{T}(w, \varepsilon')$ of $T \in Q'(L : k')$ is defined as follows :

$$\check{T}(w, \varepsilon') = \frac{-1}{2\pi\imath} \left\langle T_\zeta, \frac{e^{(-k'+\varepsilon')(\zeta - w)}}{\zeta - w} \right\rangle.$$

**Proposition 1.3 ([2],[9]).** *The following integral representation holds :*

$$< T, \varphi > = \frac{1}{2\pi\imath} \int_{\partial L_\varepsilon} \check{T}(\zeta, \varepsilon') \varphi(\zeta) d\zeta, \quad (\forall \varphi \in Q_b(L_\varepsilon : k' + \varepsilon'_1), \varepsilon' < \varepsilon'_1).$$

*Put $\varphi(\zeta) = e^{\zeta z}$, then we have*

$$\tilde{T}(z) = \frac{1}{2\pi\imath} \int_{\partial L_\varepsilon} \check{T}(\zeta, \varepsilon') e^{\zeta z} d\zeta.$$

*Conversely, we can express* $\check{T}$ *by* $\tilde{T}$ *in as follows :*

$$\check{T}(w,\varepsilon') = \int_{-k'+\varepsilon'}^{\infty} \tilde{T}(z)e^{-wz}dz.$$

**Definition 1.4.** For $S \in \mathcal{H}'(K)$, $K$ is a compact set and $T \in Q'(L : k')$, we define convolution $*$ by

$$T * S = S * T = < T_\zeta, < S_\tau, \varphi(\tau+\zeta) >>, \quad \varphi \in Q'(L+K : k').$$

**Proposition 1.5.** *We have the following equalities :*

*(i)* $T * S \in Q'(L+K : k')$,

*(ii)* $\widetilde{(T * S)}(z) = \tilde{T}(z) * \tilde{S}(z)$.

In [15] we derived Plana's summation formula for holomorphic functions of exponential type by using the theory of analytic functionals with unbounded carrier.

**Proposition 1.6 (Plana's summation formula [6], [13],[15]).** *Let* $T \in Q'(L : k')$. *If* $k' > 0$, $0 \le b < 2\pi$, $\mathrm{Re}s > a$ *and* $|\mathrm{Im}s| + b < 2\pi$, *then we have*

$$\sum_{n=0}^{\infty} \tilde{T}(n)e^{-sn}$$
$$= \frac{1}{2}\tilde{T}(0) + \int_0^{\infty} \tilde{T}(x)e^{-sx}dx + \imath\int_0^{\infty} \frac{\tilde{T}(\imath x)e^{-\imath sx} - \tilde{T}(-\imath x)e^{\imath sx}}{e^{2\pi x}-1}dx.$$

# 2 Applications.

In this section we determine the form of holomorphic functions $f(z)$ of exponential type with $f(\mathbb{N}) = \{0\}$.

**Lemma 2.1.** *Suppose that* $T \in Q'(L : k')$, $L = (-\infty, a] + \imath[-b, b]$, $0 < k' < 1$, $0 \le b < 2\pi$. *If* $\tilde{T}(n) = 0, (n = 0, 1, 2, \cdots)$, *then there exist holomorphic functions* $a(s)$ *and* $b(s)$ *such that*

$$\check{T}(s,\varepsilon') = a(s) + b(s),$$

*where* $a(s) \in \mathcal{H}(\{s \in \mathbb{C} : \mathrm{Im}s < 2\pi - b\})$, $b(s) \in \mathcal{H}(\{s \in \mathbb{C} : \mathrm{Im}s > b - 2\pi\})$ *and satisfy the following estimate :*

$$|a(s)| \le C_{\varepsilon'}e^{(k'-\varepsilon')\mathrm{Re}s},$$
$$|b(s)| \le C_{\varepsilon'}e^{(k'-\varepsilon')\mathrm{Re}s}.$$

**Corollary 2.2.** *Let* $T \in Q'(L : k')$. *If* $\tilde{T}(n) = 0$ $(n = 0, 1, 2, \cdots)$, *the*
$\check{T}(s, \varepsilon')$ *is holomorphic in* $|\mathrm{Im} s| < 2\pi - b$, *and* $|\check{T}(s, \varepsilon')| \leq C_{\varepsilon'} e^{(k'-\varepsilon')\mathrm{Re} s}$.

**Proof of lemma 2.1.** By Plana's summation formula and assumptio
$\tilde{T}(n) = 0, (n = 0, 1, 2, \cdots)$, we have

$$\begin{aligned}
0 &= \int_0^\infty \tilde{T}(x) e^{-xs} dx + \imath \int_0^\infty \frac{\tilde{T}(\imath x) e^{-\imath xs} - \tilde{T}(-\imath x) e^{\imath xs}}{e^{2\pi x} - 1} dx \\
&= \int_0^\infty \tilde{T}(x) e^{-xs} dx + \imath \int_0^\infty \frac{\tilde{T}(\imath x) e^{-\imath xs}}{e^{2\pi x} - 1} dx - \imath \int_0^\infty \frac{\tilde{T}(-\imath x) e^{\imath xs}}{e^{2\pi x} - 1} dx \\
&= \int_0^\infty \tilde{T}(x) e^{-xs} dx + \int_0^{\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz \\
&\qquad - \int_0^{-\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz - \int_0^{-\imath\infty} \tilde{T}(z) e^{-sz} dz \\
&= \int_0^\infty \tilde{T}(x) e^{-xs} dx + \int_{-k'+\varepsilon'}^0 \tilde{T}(x) e^{-xs} dx + \int_0^{\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz \\
&\qquad - \int_0^{-\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz - \int_0^{-\imath\infty} \tilde{T}(z) e^{-sz} dz - \int_{-k'+\varepsilon'}^0 \tilde{T}(x) e^{-xs} dx \\
&= \int_{-k'+\varepsilon'}^\infty \tilde{T}(x) e^{-xs} dx + \int_{-k'+\varepsilon'}^{\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz \\
&\qquad - \int_{-k'+\varepsilon'}^{-\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz - \int_{-k'+\varepsilon'}^{-\imath\infty} \tilde{T}(z) e^{-sz} dz.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{-k'+\varepsilon'}^\infty \tilde{T}(x) e^{-xs} dx \\
&= -\int_{-k'+\varepsilon'}^{\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz + \int_{-k'+\varepsilon'}^{-\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz + \int_{-k'+\varepsilon'}^{-\imath\infty} \tilde{T}(z) e^{-sz} dz.
\end{aligned}$$

Put

$$\begin{aligned}
a(s) &= -\int_{-k'+\varepsilon'}^{\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz, \\
b(s) &= \int_{-k'+\varepsilon'}^{-\imath\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi\imath z} - 1} dz + \int_{-k'+\varepsilon'}^{-\imath\infty} \tilde{T}(z) e^{-sz} dz.
\end{aligned}$$

Since by proposition 1.3, $\check{T}(s,\varepsilon') = \int_{-k'+\varepsilon'}^{\infty} \tilde{T}(z)e^{-sz}dz$, we have $\check{T}(s,\varepsilon') = a(s)+b(s)$. Then $a(s)$ and $b(s)$ satisfy the conditions in lemma 2.1. ∎

**Proposition 2.3.** *Suppose that* $T \in Q'(L : k')$, $L = (-\infty,a] + \imath[-b,b] \subset \{\zeta \in \mathbb{C} : |\mathrm{Im}\zeta| < 2\pi\}$, $0 < k' < 1$. $\tilde{T}(n) = 0$, $(n = 0,1,2,\cdots)$.

(i) *If* $0 \le b < \pi$, *then* $T \equiv 0$.

(ii) *If* $\pi \le b < 2\pi$, *then there exists an analytic functional* $S$ *with unbounded carrier such that*
$T = (\delta_{\imath\pi} - \delta_{-\imath\pi}) * S$, $S \in Q'((-\infty,a] + \imath[\pi - b, b - \pi] : k')$, *where* $*$ *is convolution.*

**Proof.** (i)

$$< T,\varphi > \;=\; \frac{1}{2\pi\imath}\int_{\partial L_\varepsilon} \check{T}(\zeta,\varepsilon')\varphi(\zeta)d\zeta, \quad (\varphi \in Q_b(L_\varepsilon : k' + \varepsilon_1'), \quad \varepsilon' < \varepsilon_1'),$$

and $\check{T}(\zeta,\varepsilon') \in \mathcal{H}(\mathbb{C}\backslash L)$. By lemma 2.1, $\check{T}(\zeta,\varepsilon') \in \mathcal{H}(\{\zeta \in \mathbb{C} : |\mathrm{Im}\zeta| < 2\pi - b\})$. From the assumption $0 \le b < \pi$, $L$ is contained in $\{\zeta \in \mathbb{C} : |\mathrm{Im}\zeta| < 2\pi - b\}$ and $|\check{T}(\zeta,\varepsilon')| \le Ce^{(k'-\varepsilon')\zeta}$. Hence $< T,\varphi >= 0$.

(ii) In this case $\check{T}(\zeta,\varepsilon') \in \mathcal{H}(\mathbb{C}\backslash\{(-\infty,a] + \imath[2\pi - b, b] \cup (-\infty,a] + \imath[-b, b-2\pi]\})$. We define analytic functionals with unbounded carrier $S_1, S_2$ as follows :

$$< S_1,\varphi > \;=\; \frac{1}{2\pi\imath}\int_{\partial L_\varepsilon'} \check{T}(\zeta + \pi\imath)\varphi(\zeta)d\zeta,$$
$$< S_2,\varphi > \;=\; \frac{1}{2\pi\imath}\int_{\partial L_\varepsilon'} \check{T}(\zeta - \pi\imath)\varphi(\zeta)d\zeta,$$

where $\varphi \in Q_b((-\infty,a] + \imath[\pi - b, b-\pi] : k' + \varepsilon')$, $L_\varepsilon' = (-\infty, a+\varepsilon] + \imath[\pi - b - \varepsilon, b - \pi + \varepsilon]$. By lemma 2.1, $S_1, S_2 \in Q'((-\infty,a] + \imath[\pi - b, b - \pi] : k')$ and we have $T = S_1 * \delta_{\pi\imath} + S_2 * \delta_{-\pi\imath}$. From the assumption $\tilde{T}(n) = 0$, $(n = 0,1,2,\cdots)$,

$$\begin{aligned} 0 &= \tilde{T}(n) \\ &= \tilde{S}_1(n)e^{n\pi\imath} + \tilde{S}_2(n)e^{-n\pi\imath}. \end{aligned}$$

Therefore $\tilde{S}_1(n) = -\tilde{S}_2(n)$. By $b - \pi - (\pi - b) = 2(b-\pi) < 2\pi$ and (i) in this proposition, $S_1 = -S_2$. ∎

**Corollary 2.4.** *Let* $f(z) \in \mathrm{Exp}((-k', \infty) + \imath\mathbb{R} : L)$, $0 < k' < 1$, $f(n) = 0, (n = 0, 1, 2, \cdots)$. *Then*

*(i)* $0 \le b < \pi \Longrightarrow f(z) = 0.$

*(ii)* $\pi \le b < 2\pi \Longrightarrow f(z) = \sin \pi z \tilde{S}(z)$, *where* $S \in Q'((-\infty, a] + \imath[\pi - b, b - \pi] : k')$.

**Remark 2.5.** Corollary 2.4(i) are called "Carlson's theorem".([3])

**Corollary 2.6.**

*(i)* $0 \le b < \pi \Longrightarrow \{e^{n\zeta}\}_{n=0}^{\infty}$ *is total in* $Q(L : k')$.

*(ii)* $\pi \le b < 2\pi \Longrightarrow \{e^{m\zeta}, \zeta e^{n\zeta}\}_{\substack{m=0\\n=0}}^{\infty}$ *is total in* $Q(L : k')$.

## 2.1 Examples

Carlson's theorem is used in several branches of mathematical physics. For example,

1. A uniqueness of analytic continuation of scattering amplitude to complex angular momentum plane ([7], [11],[12]).
2. Dyson Conjecture in statistical mechanics ([5], [8]).
3. Calculation of Selberg Integral ([1],[8]).
4. A uniqueness of Ramanujan Resummation ([4]).

For the history of Carlson's theorem, we refer the reader to [14]. Here we have an example of proposition 2.3 (ii) :

**Example 2.7 (Functional equation of Riemann zeta function, [6],[13]).** We define an analytic functional $T$ as follows :

$$< T, \varphi >= \frac{1}{2\pi\imath} \int_{\partial L_\varepsilon} \frac{1}{2(e^{-e^{-\frac{1}{2}\zeta}} - 1)} \varphi(\zeta) d\zeta, \quad \varphi \in Q(L : k'), \tag{2}$$

where $L_\varepsilon = (-\infty, -2\log 2\pi + \varepsilon] + \imath[-\pi - \varepsilon, \pi + \varepsilon]$, $\varepsilon > 0$. In (2), we put $t = e^{-\frac{1}{2}\zeta}$ and $\varphi(\zeta) = e^{\zeta z}$. Then we have

$$\begin{aligned}\tilde{T}(z) &= < T_\zeta, e^{\zeta z} > \\ &= -\frac{1}{2\pi\imath}\int_{(+\infty)}^{(+0)} \frac{(-t)^{-2z-1}}{e^t - 1} dt \\ &= \frac{\zeta(-2z)}{\Gamma(2z+1)}. \end{aligned} \tag{3}$$

Remark that $\tilde{T}(n) = \dfrac{\zeta(-2n)}{\Gamma(2n+1)} = 0$. On the other hand, we have

$$\left|\frac{1}{2\pi\imath}\int_{\partial L_\varepsilon} \frac{e^{\zeta z}}{2(e^{-e^{-\frac{1}{2}\zeta}} - 1)} d\zeta\right| \to 0, \qquad (\xi \to -\infty, \quad \mathrm{Re} z > 0).$$

Therefore by residue theorem,

$$\begin{aligned}\frac{1}{2\pi\imath}\int_{\partial L_\varepsilon} \frac{e^{\zeta z}}{2(e^{-e^{-\frac{1}{2}\zeta}} - 1)} d\zeta &= -\sum_{n=0}^{\infty} \frac{(2\pi n)^{-2z} e^{\pi\imath z}}{2\pi\imath n} + \sum_{n=0}^{\infty} \frac{(2\pi n)^{-2z} e^{-\pi\imath z}}{2\pi\imath n} \\ &= -\frac{(2\pi)^{-2z}}{\pi} \sin \pi z \zeta(2z+1). \end{aligned} \tag{4}$$

Therefore by (3) and (4), we have

$$\frac{\zeta(-2z)}{\Gamma(2z+1)} = -\frac{(2\pi)^{-2z}}{\pi} \sin \pi z \zeta(2z+1).$$

This means $\tilde{T}(z) = \sin \pi z \tilde{S}(z)$, $\tilde{S} = -\frac{1}{\pi}(2\pi)^{-2z}\zeta(2z+1)$. Namely, $T = (\delta_{\pi\imath} - \delta_{-\pi\imath}) * S$, where $S =$. Now we replace $z$ to $\frac{z-1}{2}$. Then

$$\begin{aligned}\frac{\zeta(1-z)}{\Gamma(z)} &= -\frac{(2\pi)^{-z+1}}{\pi} \sin\left(\frac{\pi}{2}z - \frac{\pi}{2}\right)\zeta(z) \\ &= 2(2\pi)^{-z}\cos\left(\frac{\pi}{2}z\right)\zeta(z) \end{aligned}$$

$$\zeta(1-z) = 2(2\pi)^{-z}\cos\left(\frac{\pi}{2}z\right)\Gamma(z)\zeta(z). \tag{5}$$

Since (5), we obtain "Functional equation of Riemann zeta function".

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