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Gevrey Asymptotic Theory for Singular 1st Order Linear PDE

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1 Introduction and Main Results.

We are concerned with the Borel summability of the formal solution for the following first order linear partial differential equation of nilpotent type:

$$\begin{aligned} &Lu(x,y)=f(x,y),\\ &L=1+(\mathrm{a}y+\mathrm{b}xy+\mathrm{c}y^2)D_x+\mathrm{d}y^2D_y, \end{aligned} \tag{1.1}$$

where $x,\ y\in\mathbf{C}$, $D_x=\partial/\partial x$, $D_y=\partial/\partial y$, and a, b, c and d are complex constants, and $f(x,y)$ is holomorphic at $(x,y)=(0,0)$. In the following, we always assume that

$$\mathrm{a}\neq 0. \tag{1.2}$$

By the argument in Hibino [1], we know that (1.1) has a unique formal power series solution in $\mathcal{O}[R][[y]]_2$ for some $R>0$. Here we say that the formal power series $u(x,y)$ belongs to $\mathcal{O}[R][[y]]_2$ if $u(x,y)$ can be written as $u(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n$, where all $u_n(x)$ are holomorphic on $\{x\in\mathbf{C};\ |x|\leq R\}$ with the estimates $\max_{|x|\leq R}|u_n(x)|\leq CK^n n!$. Therefore the formal solution of (1.1) is divergent in general.

Our main problem is the existence of the holomorphic solution which has this divergent solution as an asymptotic expansion. We have two types of asymptotic expansions: "asymptotic expansion in a small sector" and "Borel summability". Here we will study the Borel summability as stated above. We can see the asymptotic expansion in a small sector in Hibino [2].

Now let us define the concept of our asymptotic expansion which is called the Borel summability.

Definition 1.1 (1) For $\theta \in \mathbf{R}$ and $Y > 0$, we define the region $O(\theta, Y)$ by

$$O(\theta, Y) = \{y \in \mathbf{C};\ |y - Ye^{i\theta}| < Y\}. \tag{1.3}$$

(2) Let $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$. We say that $u(x,y)$ is Borel summable in θ–direction if there exists a holomorphic function $w(x,y)$ on $\{x \in \mathbf{C};\ |x| \leq r\} \times O(\theta, Y)$ for some $r > 0$ and $Y > 0$ which satisfies the following asymptotic estimates: There exist some positive constants C and K such that

$$\max_{|x| \leq r} \left| w(x,y) - \sum_{n=0}^{N-1} u_n(x) y^n \right| \leq C K^N N! |y|^N, \tag{1.4}$$

for $y \in O(\theta, Y)$ and $N = 1, 2, \ldots$.

When $u(x,y)$ is Borel summable in θ–direction, the above function $w(x,y)$ is unique (see Lutz–Miyake–Schäfke [4]). Therefore we call this $w(x,y)$ the Borel sum of $u(x,y)$ in θ–direction. ————

Our purpose is to study the conditon under which the formal solution of (1.1) is Borel summable. In order to consider our problem, we divide the problem into the following four cases:

Case (1): $\mathrm{b} = \mathrm{d} = 0$.

Case (2): $\mathrm{b} = 0$, $\mathrm{d} \neq 0$.

Case (3): $\mathrm{b} \neq 0$, $\mathrm{d} = 0$.

Case (4): $\mathrm{b}, \mathrm{d} \neq 0$.

Now in order to state the theorem, let us define some notations. We define the function $\Phi(x, \eta)$ by

$$\Phi(x,\eta) = \begin{cases} x - \mathrm{a}\eta & \text{(Case (1))} \\[2ex] x - \dfrac{\mathrm{a}}{\mathrm{d}} \log(1 + \mathrm{d}\eta) & \text{(Case (2))} \\[2ex] \left(\dfrac{\mathrm{a}}{\mathrm{b}} + x\right) e^{-\mathrm{b}\eta} - \dfrac{\mathrm{a}}{\mathrm{b}} & \text{(Case (3))} \\[2ex] \left(\dfrac{\mathrm{a}}{\mathrm{b}} + x\right) (1 + \mathrm{d}\eta)^{-\mathrm{b}/\mathrm{d}} - \dfrac{\mathrm{a}}{\mathrm{b}} & \text{(Case (4))}, \end{cases} \tag{1.5}$$

and define the region $\Omega_{r,\theta,\rho} \subset \mathbf{C}$ by

$$\Omega_{r,\theta,\rho} = \Phi(\{(x,\eta) \in \mathbf{C};\ |x| \leq r,\ \eta \in E_+(\theta,\rho)\}). \tag{1.6}$$

Here $E_+(\theta, \rho)$ is a region defined by

$$E_+(\theta, \rho) = \{\eta \in \mathbf{C};\ \mathrm{dist}(\eta, \mathbf{R}_+ e^{i\theta}) \leq \rho\}, \tag{1.7}$$

where $\mathbf{R}_+ = [0, +\infty)$.

In Case (2) and Case (4), we assume that $\theta \neq \arg(-1/\mathrm{d})$ in order that $\Omega_{r,\theta,\rho}$ is well–defined. In Case (3) and Case (4), we remark that $\Omega_{r,\theta,\rho}$ is a region in the Riemann surface of $\log\left(x + \dfrac{\mathrm{a}}{\mathrm{b}}\right)$.

Our main theorem is stated as follows:

Theorem 1.1 *In any case, assume that $f(x, y)$ can be continued analytically to $\{(x, y) \in \mathbf{C}^2;\ x \in \Omega_{r,\theta,\rho},\ |y| \leq r'\}$ for some r, ρ and r', where $\theta \neq \arg(-1/\mathrm{d})$ in Case (2) and Case (4). Furthermore assume that $f(x, y)$ has a following growth estimate for each case by some positive constants C and δ : For $x \in \Omega_{r,\theta,\rho}$,*

Case (1):

$$\max_{|y| \leq r'} |f(x, y)| \leq C e^{\delta |x|}; \tag{1.8}$$

Case (2):

$$\max_{|y| \leq r'} |f(x, y)| \leq C \exp(\delta e^{p|x|}), \tag{1.9}$$

where $p = |\mathrm{d}/\mathrm{a}|$;

Case (3):

$$\max_{|y| \leq r'} |f(x, y)| \leq C \exp\left[\delta \left|\log\left(x + \frac{\mathrm{a}}{\mathrm{b}}\right)\right|\right]; \tag{1.10}$$

Case (4):

$$\max_{|y| \leq r'} |f(x, y)| \leq C \exp\left[\delta \exp\left\{\left|\frac{\mathrm{d}}{\mathrm{b}}\right| \left|\log\left(x + \frac{\mathrm{a}}{\mathrm{b}}\right)\right|\right\}\right]. \tag{1.11}$$

Furthermore in Case (3) and Case (4), we assume the following condition:

Case (3):

$$\mathrm{c} = 0 \quad \textit{or} \quad \Re(-\mathrm{b} e^{i\theta}) \geq 0; \tag{1.12}$$

Case (4):

$$\mathrm{c} = 0 \quad \textit{or} \quad \Re\left(-\frac{\mathrm{b}}{\mathrm{d}}\right) > -1. \tag{1.13}$$

Then the formal solution $u(x, y)$ of (1.1) is Borel summable in θ–direction and its Borel sum is a holomorphic solution of (1.1).

2 Formal Borel Transform of Equations.

Before proving Theorem 1.1, we give some preliminaries. First, we remark that if the formal solution $u(x,y)$ of (1.1) is Borel summable, then it is easily proved from the uniqueness of the Borel sum that its Borel sum $w(x,y)$ is a holomorphic solution of (1.1). Therefore in order to prove Theorem 1.1, it is sufficient to prove that the formal solution $u(x,y)$ is Borel summable under the conditions in the theorem.

In general when we want to check the Borel summability of the formal power series $u(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n \in \mathcal{O}[R][[y]]_2$, the following theorem plays a fundamental role.

Theorem 2.1 (Lutz, Miyake and Schäfke [4]) *The necessary and sufficient conditions so that a formal power series $u(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n \in \mathcal{O}[R][[y]]_2$ is Borel summable in θ-direction are stated as follows: Let us define the formal Borel transform $\mathcal{B}[u](x,\eta)$ of $u(x,y)$ by*

$$\mathcal{B}[u](x,\eta)=\sum_{n=0}^{\infty}u_n(x)\frac{\eta^n}{n!}, \tag{2.1}$$

which is holomorphic in a neighborhood of the origin. Then $\mathcal{B}[u](x,\eta)$ satisfies the following condition (BS):

(BS) *$\mathcal{B}[u](x,\eta)$ can be continued analytically to $\{x\in \mathbf{C};\ |x|\le r\}\times E_+(\theta,\rho)$ for some $r>0$ and $\rho>0$, and has the following exponential growth estimate for some positive constants C and δ :*

$$\max_{|x|\le r}|\mathcal{B}[u](x,\eta)|\le Ce^{\delta|\eta|},\quad \eta\in E_+(\theta,\rho). \tag{2.2}$$

In this case the Borel sum $w(x,y)$ of $u(x,y)$ in θ-direction is given by

$$w(x,y)=\frac{1}{y}\int_{\mathbf{R}_+e^{i\theta}}e^{-\eta/y}\mathcal{B}[u](x,\eta)d\eta. \tag{2.3}$$

Therefore in order to prove Theorem 1.1, it is sufficient to prove that the formal Borel transform $\mathcal{B}[u](x,\eta)$ of the formal solution $u(x,y)$ satisfies the above condition (BS) under the conditions in the theorem. In order to do that, firstly let us lead the equation satisfied by $\mathcal{B}[u](x,\eta)$. By the formal Borel transform, the operators y and D_y are transformed to the operators $D_\eta^{-1}=\int_0^\eta$ and $D_\eta \eta D_\eta$, respectively. They are easily seen from the following commutative diagrams:

$$
(2.4)\qquad
\begin{array}{ccc}
y^n & \xrightarrow{\text{Borel tr.}} & \dfrac{\eta^n}{n!} \\
{\scriptstyle y}\big\downarrow & & \big\downarrow{\scriptstyle D_\eta^{-1}} \\
y^{n+1} & \xrightarrow[\text{Borel tr.}]{} & \dfrac{\eta^{n+1}}{(n+1)!},
\end{array}
\qquad\qquad
\begin{array}{ccc}
y^n & \xrightarrow{\text{Borel tr.}} & \dfrac{\eta^n}{n!} \\
{\scriptstyle D_y}\big\downarrow & & \big\downarrow{\scriptstyle D_\eta \eta D_\eta} \\
ny^{n-1} & \xrightarrow[\text{Borel tr.}]{} & n\dfrac{\eta^{n-1}}{(n-1)!}.
\end{array}
$$

Therefore we see that $\mathcal{B}[u](x,\eta)$ is the solution of the following equation:

$$
(2.5)\qquad \{1+(\mathrm{a}+\mathrm{b}x)D_\eta^{-1}D_x+\mathrm{c}D_\eta^{-2}D_x+\mathrm{d}D_\eta^{-1}\eta D_\eta\}v(x,\eta)=g(x,\eta),
$$

where $g(x,\eta)$ is the formal Borel tranform of $f(x,y)=\sum_{n=0}^{\infty} f_n(x)y^n$, that is,

$$
g(x,\eta)=\sum_{n=0}^{\infty} f_n(x)\frac{\eta^n}{n!}.
$$

Furthermore by operating D_η to (2.5) from the left, we see that $\mathcal{B}[u](x,\eta)$ is the solution of the initial value problem of the following integro–differential equation:

$$
(2.6)\qquad
\begin{gathered}
\{(1+\mathrm{d}\eta)D_\eta+(\mathrm{a}+\mathrm{b}x)D_x\}v(x,\eta)=-\mathrm{c}D_\eta^{-1}D_x v(x,\eta)+h(x,\eta),\\
v(x,0)=f(x,0),
\end{gathered}
$$

where $h(x,\eta)=D_\eta g(x,\eta)$.

Therefore Theorem 1.1 is proved by showing that the solution $v(x,\eta)$ of (2.6) satisfies the condition (BS).

3 Proof of Theorem 1.1

Let us start the proof of Theorem 1.1. Here we prove the theorem only in Case (1) (on the other cases, see Hibino [3]). In this case, that is, in the case $\mathrm{b}=\mathrm{d}=0$, the equation (2.6) is written as follows:

$$
(3.1)\qquad
\begin{gathered}
\{D_\eta+\mathrm{a}D_x\}v(x,\eta)=-\mathrm{c}D_\eta^{-1}D_x v(x,\eta)+h(x,\eta),\\
v(x,0)=f(x,0).
\end{gathered}
$$

We shall prove that the solution $v(x,\eta)$ of (3.1) satisfies the condition (BS) in Theorem 2.1. First, we remark that in general the solution $w(x,\eta)$ of the initial value problem of the following first order linear partial differential equation

$$
(3.2)\qquad
\begin{gathered}
\{D_\eta+\mathrm{a}D_x\}w(x,\eta)=k(x,\eta),\\
w(x,0)=l(x)
\end{gathered}
$$

is given by

(3.3)
$$w(x,\eta) = l(x - a\eta) + \int_0^\eta k(x - a(\eta - t), t)dt.$$

Proof of the theorem. In the case $c = 0$, it follows from (3.3) that $v(x,\eta)$ has the following explicit form:

(3.4)
$$v(x,\eta) = f(x - a\eta, 0) + \int_0^\eta h(x - a(\eta - t), t)dt.$$

Therefore from the condition, it is easy to prove that $v(x,\eta)$ can be continued analytically to $\{(x,\eta) \in \mathbf{C}^2;\ |x| \le r,\ \eta \in E_+(\theta,\rho)\}$ with the estimate

$$\max_{|x|\le r} |v(x,\eta)| \le C' e^{\delta'|\eta|}, \quad \eta \in E_+(\theta,\rho),$$

for some positive constants C' and δ'. This shows that $v(x,\eta)$ satisfies the condition (BS).

Let us assume $c \neq 0$. In this case, (3.1) is rewritten as follows:

(3.5)
$$\begin{gathered} \{D_\eta + aD_x\}v(x,\eta) = -c\int_0^\eta v_x(x,s)ds + h(x,\eta), \\ v(x,0) = f(x,0). \end{gathered}$$

First, let us transform (3.5) into the integral equation. It follows from (3.3) that (3.5) is equivalent to the following equation:

$$v(x,\eta) = f(x - a\eta, 0) + \int_0^\eta h(x - a(\eta - t), t)dt - c\int_0^\eta \int_0^t v_x(x - a(\eta - t), s)dsdt.$$

Here we remark that

$$\begin{aligned} &\int_0^\eta \int_0^t v_x(x - a(\eta - t), s)dsdt \\ &= \int_0^\eta \int_s^\eta v_x(x - a(\eta - t), s)dtds \\ &= \int_0^\eta \int_s^\eta \frac{d}{dt}\left\{\frac{1}{a}v(x - a(\eta - t), s)\right\}dtds \\ &= \frac{1}{a}\int_0^\eta v(x,t)dt - \frac{1}{a}\int_0^\eta v(x - a(\eta - t), t)dt. \end{aligned}$$

Therefore we know that (3.5) is equivalent to the following integral equation:

(3.6)
$$\begin{aligned} v(x,\eta) &= f(x - a\eta, 0) + \int_0^\eta h(x - a(\eta - t), t)dt \\ &\quad + \frac{c}{a}\int_0^\eta v(x - a(\eta - t), t)dt - \frac{c}{a}\int_0^\eta v(x,t)dt. \end{aligned}$$

In order to prove that the solution $v(x,\eta)$ of (3.6) satisfies the condition (BS), we employ the iteration method. Let us define $\{v_n(x,\eta)\}_{n=0}^{\infty}$ as follows:

$$v_0(x,\eta) = f(x - \mathrm{a}\eta, 0) + \int_0^{\eta} h(x - \mathrm{a}(\eta - t), t)dt.$$

For $n \geq 0$,

$$v_{n+1}(x,\eta) = v_0(x,\eta) + \frac{\mathrm{c}}{\mathrm{a}} \int_0^{\eta} v_n(x - \mathrm{a}(\eta - t), t)dt - \frac{\mathrm{c}}{\mathrm{a}} \int_0^{\eta} v_n(x,t)dt. \tag{3.7}$$

Next we put $w_0(x,\eta) := v_0(x,\eta)$ and $w_n(x,\eta) := v_n(x,\eta) - v_{n-1}(x,\eta)$ for $n \geq 1$, and we define $\widetilde{w}_n(x,\eta,t)$ by

$$\widetilde{w}_n(x,\eta,t) := w_n(x - \mathrm{a}(\eta - t), t). \tag{3.8}$$

Now let us take a monotone decreasing positive sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ so that

$$\widetilde{\rho} := \rho - \sum_{n=0}^{\infty} \varepsilon_n > 0. \tag{3.9}$$

Then we obtain the following lemma.

Lemma 3.1 *$\widetilde{w}_n(x,\eta,t)$ is continued analytically to $\{(x,\eta,t) \in \mathbf{C}^3;\ |x| \leq r,\ \eta \in E_+(\theta, \rho - \sum_{j=0}^{n} \varepsilon_j),\ t \in G_{\eta}^{\varepsilon_n}\}$. Furthermore on $\{(x,\eta,t) \in \mathbf{C}^3;\ |x| \leq r,\ \eta \in E_+(\theta, \rho - \sum_{j=0}^{n} \varepsilon_j),\ t \in G_{\eta}\}$ we have the following estimate: For some positive constants C_1 and δ_1,*

$$|\widetilde{w}_n(x,\eta,G_{\eta}(R))| \leq C_1 e^{\delta_1|\eta|} L^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\delta_1^{n-k}} \frac{R^k}{k!}, \tag{3.10}$$

where $L = |\mathrm{c}|/|\mathrm{a}|$. Here G_{η} is the segment from 0 to η:

$$G_{\eta} = \{G_{\eta}(R) = Re^{i \arg(\eta)};\ 0 \leq R \leq |\eta|\},$$

and G_{η}^{ε} is the ε-neighborhood of G_{η} for $\varepsilon > 0$.

If we admit Lemma 3.1, the theorem is proved as follows: It follows from Lemma 3.1 that $w_n(x,\eta)$ $(= \widetilde{w}_n(x,\eta,\eta))$ is continued analytically to $\{(x,\eta) \in \mathbf{C}^2;\ |x| \leq r,\ \eta \in E_+(\theta, \rho - \sum_{j=0}^{n} \varepsilon_j)\}$ with the estimate

$$\begin{aligned} |w_n(x,\eta)| &= |\widetilde{w}_n(x,\eta,G_{\eta}(|\eta|))| \\ &\leq C_1 e^{\delta_1|\eta|} L^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\delta_1^{n-k}} \frac{|\eta|^k}{k!}, \end{aligned}$$

for $|x| \le r$ and $\eta \in E_+(\theta, \rho - \sum_{j=0}^{n} \varepsilon_j)$. Therefore by taking δ_1 sufficiently large, we see that $v_n(x,\eta)$ $(= \sum_{k=0}^{n} w_k(x,\eta))$ converges to the solution $V(x,\eta)$ of (3.6) uniformly on $\{(x,\eta) \in \mathbf{C}^2;\ |x| \le r,\ \eta \in E_+(\theta, \tilde{\rho})\}$ with the estimate

$$\begin{aligned} |V(x,\eta)| &\le \sum_{n=0}^{\infty} |w_n(x,\eta)| \\ &\le C_1 e^{\delta_1|\eta|} \sum_{n=0}^{\infty} L^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\delta_1^{n-k}} \frac{|\eta|^k}{k!} \\ &\le \tilde{C} e^{\tilde{\delta}|\eta|}, \end{aligned}$$

for some positive constants $\tilde{C}$ and $\tilde{\delta}$. By the uniqueness of the local holomorphic solution, it is clear that $V(x,\eta)$ is the analytic continuation of $v(x,\eta)$. This shows that $v(x,\eta)$ satisfies the condition (BS). The theorem is proved. ∎

Therefore it is sufficient to prove Lemma 3.1.

Proof of Lemma 3.1. It is proved by the induction. In the case $n = 0$, we can obtain the explicit form of $\tilde{w}_0(x,\eta,t)$:

$$\tilde{w}_0(x,\eta,t) = f(x - \mathrm{a}\eta, 0) + \int_0^t h(x - \mathrm{a}(\eta - s), s)ds.$$

Therefore from the condition, it is easy to prove that $\tilde{w}_0(x,\eta,t)$ is well–defined and holomorphic on $\{(x,\eta,t) \in \mathbf{C}^3;\ |x| \le r,\ \eta \in E_+(\theta, \rho - \varepsilon_0),\ t \in G_\eta^{\varepsilon_0}\}$ and has the estimate

$$|\tilde{w}_0(x,\eta,G_\eta(R))| \le C_1 e^{\delta_1|\eta|}$$

on $\{(x,\eta,t) \in \mathbf{C}^3;\ |x| \le r,\ \eta \in E_+(\theta, \rho - \varepsilon_0),\ t \in G_\eta\}$ for some positive constants C_1 and δ_1. This implies the lemma for $n = 0$. Next, let us assume that the lemma is proved up to n. Since $\{w_n(x,\eta)\}_{n=0}^{\infty}$ is determined by

$$w_{n+1}(x,\eta) = \frac{\mathrm{c}}{\mathrm{a}} \int_0^\eta w_n(x - \mathrm{a}(\eta - t), t)dt - \frac{\mathrm{c}}{\mathrm{a}} \int_0^\eta w_n(x,t)dt, \tag{3.11}$$

we have

$$\begin{aligned} \tilde{w}_{n+1}(x,\eta,t) &= w_{n+1}(x - \mathrm{a}(\eta - t), t) \\ &= \frac{\mathrm{c}}{\mathrm{a}} \int_0^t w_n(x - \mathrm{a}(\eta - t) - \mathrm{a}(t - s), s)ds - \frac{\mathrm{c}}{\mathrm{a}} \int_0^t w_n(x - \mathrm{a}(\eta - t), s)ds \\ &= \frac{\mathrm{c}}{\mathrm{a}} \int_0^t w_n(x - \mathrm{a}(\eta - s), s)ds - \frac{\mathrm{c}}{\mathrm{a}} \int_0^t w_n(x - \mathrm{a}\{(\eta - t + s) - s\}, s)ds \\ &= \frac{\mathrm{c}}{\mathrm{a}} \int_0^t \tilde{w}_n(x,\eta,s)ds - \frac{\mathrm{c}}{\mathrm{a}} \int_0^\eta \tilde{w}_n(x, \eta - t + s, s)ds \\ &=: I_1(x,\eta,t) + I_2(x,\eta,t). \end{aligned}$$

Let us prove that each $I_i(x,\eta,t)$ is well-defined on $\{(x,\eta,t)\in\mathbf{C}^3;\ |x|\le r,\ \eta\in E_+(\theta,\rho-\sum_{j=0}^{n+1}\varepsilon_j),\ t\in G_\eta^{\varepsilon_{n+1}}\}$.

On $I_1(x,\eta,t)$: It is clear that $\eta\in E_+(\theta,\rho-\sum_{j=0}^{n+1}\varepsilon_j)\subset E_+(\theta,\rho-\sum_{j=0}^{n}\varepsilon_j)$. By taking an integral path as the segment from 0 to t, it holds that $s\in G_\eta^{\varepsilon_{n+1}}\subset G_\eta^{\varepsilon_n}$. Hence $\widetilde{w}_n(x,\eta,s)$ is well-defined and $I_1(x,\eta,t)$ is well-defined.

On $I_2(x,\eta,t)$: By taking an integral path as the segment from 0 to t, it holds that $\eta-t+s\in E_+(\theta,\rho-\sum_{j=0}^{n}\varepsilon_j)$ and $s\in G_{\eta-t+s}^{\varepsilon_{n+1}}\subset G_{\eta-t+s}^{\varepsilon_n}$. Hence $w_n(x,\eta-t+s,s)$ is well-defined and $I_2(x,\eta,t)$ is well-defined.

Therefore $\widetilde{w}_{n+1}(x,\eta,t)$ is well-defined and holomorphic on $\{(x,\eta,t)\in\mathbf{C}^3;\ |x|\le r,\ \eta\in E_+(\theta,\rho-\sum_{j=0}^{n+1}\varepsilon_j),\ t\in G_\eta^{\varepsilon_{n+1}}\}$. Moreover on $\{(x,\eta,t)\in\mathbf{C}^3;\ |x|\le r,\ \eta\in E_+(\theta,\rho-\sum_{j=0}^{n+1}\varepsilon_j),\ t\in G_\eta\}$ we have the following representations:

$$I_1(x,\eta,G_\eta(R))=\frac{\mathrm{c}}{\mathrm{a}}\int_0^R\widetilde{w}_n(x,\eta,G_\eta(R_1))e^{i\arg(\eta)}dR_1,$$

$$I_2(x,\eta,G_\eta(R))=-\frac{\mathrm{c}}{\mathrm{a}}\int_0^R\widetilde{w}_n(x,(|\eta|-R+R_1)e^{i\arg(\eta)},G_{(|\eta|-R+R_1)e^{i\arg(\eta)}}(R_1))e^{i\arg(\eta)}dR_1.$$

Let us estimate each $I_i(x,\eta,G_\eta(R))$.

On $I_1(x,\eta,G_\eta(R))$: It follows from the assumption of the induction that

$$|\widetilde{w}_n(x,\eta,G_\eta(R_1))|\le C_1e^{\delta_1|\eta|}L^n\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n-k}}\frac{R_1^k}{k!},$$

which implies that

$$\begin{aligned}|I_1(x,\eta,G_\eta(R))| &\le C_1e^{\delta_1|\eta|}L^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n-k}}\int_0^R\frac{R_1^k}{k!}dR_1\\ &= C_1e^{\delta_1|\eta|}L^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n-k}}\frac{R^{k+1}}{(k+1)!}.\end{aligned}$$

On $I_2(x,\eta,G_\eta(R))$: By the assumption of the induction, we have

$$\begin{aligned}&|\widetilde{w}_{n+1}(x,(|\eta|-R+R_1)e^{i\arg(\eta)},G_{(|\eta|-R+R_1)e^{i\arg(\eta)}}(R_1))|\\ &\le C_1e^{\delta_1|\eta|}e^{-\delta_1R}e^{\delta_1R_1}L^n\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n-k}}\frac{R_1^k}{k!},\end{aligned}$$

which implies that

$$|I_2(x,\eta,G_\eta(R))|\le C_1e^{\delta_1|\eta|}e^{-\delta_1R}L^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n-k}}\int_0^R e^{\delta_1R_1}\frac{R_1^k}{k!}dR_1.$$

Here it holds that

$$\begin{aligned}\int_0^R e^{\delta_1 R_1}\frac{R_1^k}{k!}dR_1 &= \int_0^R\left\{\frac{d}{dR_1}\left\{\frac{1}{\delta_1}e^{\delta_1 R_1}\right\}\right\}\frac{R_1^k}{k!}dR_1\\ &\le \left[\frac{1}{\delta_1}e^{\delta_1 R_1}\frac{R_1^k}{k!}\right]_{R_1=0}^{R}\\ &\le \frac{1}{\delta_1}e^{\delta_1 R}\frac{R^k}{k!}.\end{aligned}$$

Hence we obtain

$$|I_2(x,\eta,G_\eta(R))| \le C_1 e^{\delta_1|\eta|}L^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n+1-k}}\frac{R^k}{k!}.$$

Therefore it holds that

$$\begin{aligned}&|\tilde{w}_{n+1}(x,\eta,G_\eta(R))|\\ &\le C_1 e^{\delta_1|\eta|}L^{n+1}\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n-k}}\frac{R^{k+1}}{(k+1)!}+\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n+1-k}}\frac{R^k}{k!}\right\}\\ &= C_1 e^{\delta_1|\eta|}L^{n+1}\left\{\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{1}{\delta_1^{n+1-k}}\frac{R^k}{k!}+\sum_{k=0}^{n}\binom{n}{k}\frac{1}{\delta_1^{n+1-k}}\frac{R^k}{k!}\right\}\\ &= C_1 e^{\delta_1|\eta|}L^{n+1}\left[\frac{1}{\delta_1^{n+1}}+\sum_{k=1}^{n}\left\{\binom{n}{k-1}+\binom{n}{k}\right\}\frac{1}{\delta_1^{n+1-k}}\frac{R^k}{k!}+\frac{R^{n+1}}{(n+1)!}\right]\\ &= C_1 e^{\delta_1|\eta|}L^{n+1}\left\{\frac{1}{\delta_1^{n+1}}+\sum_{k=1}^{n}\binom{n+1}{k}\frac{1}{\delta_1^{n+1-k}}\frac{R^k}{k!}+\frac{R^{n+1}}{(n+1)!}\right\}\\ &= C_1 e^{\delta_1|\eta|}L^{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{\delta_1^{n+1-k}}\frac{R^k}{k!},\end{aligned}$$

which implies the lemma for $n+1$. The proof is completed. ∎

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