数理解析研究所講究録 1214 巻 2001 年 27-36

On P. Hall's Relations in Finite groups

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1. INTRODUCTION

For a finite group H and for an automorphism θ of H whose order divides n, we define

$$L_n(H,\theta) = \{x \in H \mid x \cdot x^{\theta} \cdot x^{\theta^2} \cdots x^{\theta^{n-1}} = 1\},$$

where x^{θ} denotes the effect of θ on x. The following theorem is due to P. Hall [8, Theorem 1.6].

Hall's theorem *Under the notation above,* $\sharp L_n(H,\theta) \equiv 0 \bmod \gcd(n,|H|)$.

We can prove directly the assertion of this theorem in the case where n is a prime p, but we need to prove a generalized assertion in an arbitrary case (Theorem 3.1).

Hall's theorem has various applications. Especially, it is applicable to Frobenius conjecture as below. If we take θ as the identity $\epsilon \in \mathrm{Aut}H$ in Hall's theorem, then the result is due to Frobenius (see, e.g., [4, §37]).

Frobenius theorem *The number of elements x of a finite group G that satisfy the equation $x^n = 1$ is a multiple of* $\gcd(n,|G|)$.

Relating to this theorem, the following theorem was conjectured by Frobenius and was shown to be true by Iiyori-Yamaki on the basis of the classification theorem of finite simple groups.

Theorem 1.1 (Frobenius conjecture, [9]) *If $\sharp L_n(H,\epsilon) = \gcd(n,|H|)$, then the elements of $L_n(H,\epsilon)$ form a subgroup of H.*

Hall's theorem is useful for reducing the Frobenius conjecture to the case where H is a simple group [23].

In this report, p denotes a prime and u denotes a nonnegative integer. As for the generalization of Frobenius conjecture, Sylow's theorem yields the following.

Proposition 1.2 ([19]) *Suppose that θ is an automorphism of a finite group H whose order divides p^u. If $\sharp L_{p^u}(H,\theta) = \gcd(p^u, H)$, then the elements of $L_{p^u}(H,\theta)$ form a subgroup of H.*

According to [13], the exceptional p-groups are the cyclic groups if $p > 2$, and the exceptional 2-groups are the cyclic, dihedral, generalized quaternion, and semi-dihedral groups; the last three types of finite 2-groups are defined by

$$\begin{aligned} D_{2^\ell} &= \langle x,\ y \mid x^{2^{\ell-1}} = y^2 = 1,\ y^{-1}xy = x^{-1}\rangle, && \ell \geq 2, \\ Q_{2^\ell} &= \langle x,\ y \mid x^{2^{\ell-2}} = y^2,\ y^{-1}xy = x^{-1}\rangle, && \ell \geq 3, \\ S_{2^\ell} &= \langle x,\ y \mid x^{2^{\ell-1}} = y^2 = 1,\ y^{-1}xy = x^{-1+2^{\ell-2}}\rangle, && \ell \geq 4, \end{aligned}$$

respectively. The four group is exceptional in this report, even though it is not in [13]. In the proof of Theorem 1.1, the following theorem plays an important role.

Theorem 1.3 ([13]) *Let G be a finite group with a Sylow p-subgroup P of order p^ℓ. For $0 < m < \ell$, if p^m is the highest power of p that divides a positive integer n and if the number of elements x of G that satisfy the equation $x^n = 1$ is not a multiple of $\gcd(pn, |G|)$, then P is either cyclic or non-abelian exceptional.*

Now, we emphasis that the following generalization of Theorem 1.3 holds.

Theorem 1.4 ([19]) *Suppose that θ is an automorphism of a finite group H whose order divides n and that p divides $\gcd(n, |H|)$. Let P a Sylow p-subgroup of H, and let p^u be the highest power of p that divides a positive integer n. If $\sharp L_n(H,\theta)$ is not a multiple of $\gcd(pn, |H|)$, then P is exceptional.*

The proof of Theorem 1.4 is due to Murai [14] in the case where H is a p-group and n is a power of p. Remarkably, in an arbitrary case, it runs parallel with that of Hall's theorem (see Section 3). In this report, we will sketch out the proof of Hall's theorem and Theorem 1.4. Also, we will present related results with them.

2. THE CASE OF p-GROUPS

For a finite group H and a finite abelian group C that acts on H, let CH denote the semidirect product of C and H, and let $z(C,H)$ be the number of complements of H in CH, i.e.,

$$z(C,H) = \sharp\{D \leq CH \mid DH = CH,\ D \cap H = \{1\}\}.$$

Let P be a finite p-group, and let θ be an automorphism of P whose order divides p^u, where u is a positive integer. Suppose that $C = \langle c\rangle$ is a finite cyclic group generated by c and is of order p^u. Then C acts on P by $x^c = x^\theta$ for all $x \in P$, and $z(C,P) = \sharp L_{p^u}(P,\theta)$, because

$$z(C,P) = \sharp\{x \in P \mid (cx)^{p^u} = 1\} = \sharp\{x \in P \mid (xc^{-1})^{p^u} = 1\}$$

$$x \cdot x^{\theta} \cdot x^{\theta^2} \cdots x^{\theta^{p^u-1}} = xc^{-1}xcc^{-2}xc^2 \cdots c^{-(p^u-1)}xc^{p^u-1}c^{-p^u} = (xc^{-1})^{p^u}$$

for all $x \in P$. Thus Hall's theorem with $H = P$ and $n = p^u$ is equivalent to the following results which is due to Asai and Yoshida [3, Proposition 3.3].

Proposition 2.1 *Under the assumptions above,* $z(C, P) \equiv 0 \bmod \gcd(p^u, |P|)$.

This proposition seems to have various applications. We will present some of them in Sections 4 and 5.

In connection with Frobenius theorem, Kulakoff proved the following [11, Satz 2].

Kulakoff's theorem *Suppose that* $p > 2$ *and that* P *is a finite non-cyclic group of order* p^ℓ. *Then, for* $0 < m < \ell$, *the number of elements* x *of* P *that satisfy the equation* $x^{p^m} = 1$ *is a multiple of* p^{m+1}.

The following generalization of this theorem relates to Hall's theorem.

Theorem 2.2 ([2, 19]) *Suppose that* $p > 2$ *and that* $\sharp L_{p^u}(P, \theta)$ *is not a multiple of* $\gcd(p^{u+1}, |P|)$. *Then* $|P| \geq p^{u+1}$ *and* P *is cyclic.*

In this theorem, if $\theta = \epsilon$, then the assertion is the same as that of Kulakoff's theorem. This theorem is also a special case of [8, Theorem 1(iii)].

Murai proved the following theorem containing the results in the case where $p = 2$ and Kulakoff's theorem, which yields Theorem 1.3.

Theorem 2.3 ([13]) *Suppose that* P *is a finite group of order* p^ℓ. *For* $0 < m < \ell$, *the number of elements* x *of* P *that satisfy the equation* $x^{p^m} = 1$ *is not a multiple of* p^{m+1} *if and only if either* P *is a cyclic group, or else* $p = 2$, $0 < m < \ell - 1$, *and* P *is non-abelian exceptional.*

The assertion with $p = 2$ and $m = 1$ in this theorem is also seen in [12, Theorem 6.2(Thompson)]; see also [10, pp. 52-53]. The following theorem is also due to Murai.

Theorem 2.4 ([14]) *Suppose that* $p = 2$ *and that* $\sharp L_{2^u}(P, \theta)$ *is not a multiple of* $\gcd(2^{u+1}, |P|)$. *Then* $|P| \geq 2^{u+1}$ *and* P *is exceptional. Moreover, if* P *is a non-cyclic 2-group of order* 2^{u+1}, *then* $u = 1$, $P = D_4$ *(the four group), and* $\langle\theta\rangle P = D_8$.

Murai's proof of this theorem says that the assertion of the theorem is roughly a consequence of Theorem 2.3 and the following theorem proved in [2].

Theorem 2.5 ([2, 19]) *Suppose that* $p = 2$ *and that* $u > 1$. *If* $\sharp L_{2^u}(P, \theta)$ *is not a multiple of* $\gcd(2^{u+1}, |P|)$, *then* $|P| \geq 2^{u+1}$ *and every* θ*-invariant abelian normal subgroup of* P *is cyclic.*

In this theorem, if $\theta = \epsilon$, then we get the following corollary.

Corollary 2.6 ([2]) *Suppose that* $|P| = 2^\ell$. *For* $1 < m < \ell$, *if the number of elements* x *of* P *that satisfy the equation* $x^{2^m} = 1$ *is not a multiple of* 2^{m+1}, *then every abelian normal subgroup of* P *is cyclic, and consequently, either* P *is cyclic, or else* $1 < m < \ell - 1$ *and* P *is non-abelian exceptional.*

The first part of the assertion of Corollary 2.6 fails for $m = 1$, because D_8 contains a non-cyclic abelian normal subgroup, though the number of involutions of D_8 is not a multiple of 4. The last part of the assertion of Corollary 2.6 is a consequence of [15, Chapter 4, (4.3)] and is also a special case of Theorem 2.3.

3. THE PROOF OF HALL'S THEOREM

In this section, we will make a sketch of the proof of Hall's theorem. If θ is an automorphism of a finite group H whose order divides n, then

$$x \cdot x^\theta \cdot x^{\theta^2} \cdots x^{\theta^{n-1}} = (x\theta^{-1})^n$$

for all x on H (see Section 2), and hence

$$L_n(H, \theta) = \{h \in H \mid (\theta h)^n = 1\},$$

where θ is regarded as an element of the semidirect product $\langle\theta\rangle H$. Since $H\theta H = \theta H$, it follows that

$$L_n(H, \theta) = \{x \in H\theta H \mid x^n = 1\}.$$

Throughout this section, let G be a finite group, H a subgroup, $z \in C_G(H)$, $y \in G$, and n a positive integer. Set

$$X_n(HyH, z) = \{x \in HyH \mid x^n = z\}.$$

It seems that a certain generality is necessary for proving Hall's theorem. We can get Hall's theorem as a special case of the following theorem which is also due to Hall [8].

Theorem 3.1 *We have* $\sharp X_n(HyH, z) \equiv 0 \bmod \gcd(n, |H|)$.

Hall [8] showed more general theorem under some additional conditions (see also [23]). However we have proved only Theorem 3.1 [19].

The following theorem contains Theorem 3.1 and a generalization of Theorem 1.4.

Theorem 3.2 ([19]) *Suppose that* $n = p^u q$ *where* $\gcd(p, q) = 1$ *and that* P *is a Sylow* p*-subgroup of* H. *Then the following conditions hold.*

(1) $\sharp X_n(HyH, z) \equiv 0 \bmod (p^u, |P|)$.

(2) *If* p *divides* $\gcd(n, |H|)$ *and if* $\sharp X_n(HyH, z)$ *is not a multiple of* $\bmod(p^{u+1}, |P|)$, *then* $y \in N_G(H)$, $|P| \geq p^{n+1}$, *and* P *is exceptional.*

We devote the rest of this section to the sketch of the proof of Theorem 3.2. The details of the proof will be shown in [19]. According to [19], we may assume that H is a p-group and $n = p^u$. For each proper subgroup K of H, set

$$X_n(HyH, x; K) = \{x \in HyH \mid x^n = z,\ H \cap H^x = K\}.$$

The proof of (1) is as follows. We use induction on $|H|$. Suppose that K is a proper subgroup of H. For each $w \in G$, the inductive assumption implies that $\sharp X_n(KwK, z) \equiv 0 \bmod \gcd(n, |K|)$ provided $KwK \cap X_n(HyH, z; K) \neq \emptyset$. Then

$$\sum_{hC_H(K) \in H/C_H(K)} \sharp X_n(HyH, z; K^h) \equiv 0 \bmod \gcd(|H : K|n, |H|).$$

Now, if $H \neq H^y$, then $H \cap H^x \neq H$ for all $x \in HyH$, which, together with the fact above, yields $\sharp X_n(HyH, z) \equiv 0 \bmod \gcd(n, |H|)$. On the other hand, if $H = H^y$, then the assertion (1) follows from Proposition 2.1.

The proof of (2) runs parallel with that of (1). Suppose that K is a proper subgroup of H. For each $w \in G$, the assertion (1) implies that $\sharp X_n(KwK, z) \equiv 0 \bmod \gcd(n, |K|)$ provided $KwK \cap X_n(HyH, z; K) \neq \emptyset$. Then

$$\sum_{hC_H(K) \in H/C_H(K)} \sharp X_n(HyH, z; K^h) \equiv 0 \bmod \gcd(pn, |H|).$$

If $H \neq H^y$, then $H \cap H^x \neq H$ for all $x \in HyH$, which, together with the fact above, yields $\sharp X_n(HyH, z) \equiv 0 \bmod \gcd(pn, |H|)$. Now, if $\sharp X_n(HyH, z)$ is not a multiple of $\bmod(pn, |P|)$, then $H = H^y$ and, by Theorems 2.2 and 2.4, P is exceptional.

4. FROBENIUS NUMBERS

Let H be a finite group and C a finite abelian group that acts on H. We consider the condition

$$\mathrm{I}(C, H): \qquad z(C, H) \equiv 0 \bmod \gcd(|C|, |H|).$$

Hall's theorem states that the condition $\mathrm{I}(C, H)$ holds provided C is a cyclic group (see Section 3). The following conjecture was introduced in [3].

Conjecture I If H and C are p-groups, then the condition $\mathrm{I}(C, H)$ holds.

The condition $\mathrm{I}(C, H)$ holds in some special cases as below. Suppose that H and C are p-groups. The following proposition plays an important role in the proof of Theorem 4.4 below.

Proposition 4.1 ([1, 2]) *If H is abelian, then the condition* $\mathrm{I}(C, H)$ *holds.*

The following theorem is a generalization of Proposition 2.1.

Theorem 4.2 ([1, 3]) *If C is the direct product of a cyclic p-group and an elementary abelian p-group, then the condition* $\mathrm{I}(C, H)$ *holds.*

Now, we state a theorem that closely relates to Theorem 1.4.

Theorem 4.3 ([2]) *Suppose that p is odd. If C is the direct product of a cyclic p-group and a cyclic p-group of order at most p^2, then the condition* $\mathrm{I}(C, H)$ *holds.*

A key result to this theorem yields Theorem 2.2. Theorem 4.3 comes out of the facts below. Define $C_2(G) = [G, G]$ and $C_i(G) = [C_{i-1}(G), G]$ for $i \geq 3$. The following theorem is due to Hall (see also [15, Chapter 4, §3]).

Theorem 4.4 ([7]) *For elements x and y of G and for a positive integer n, there exist $c_i \in C_i(G)$, $2 \leq i \leq n$, such that*

$$x^n y^n = (xy)^n c_2^{e_2} \cdots c_n^{e_n},$$

where

$$e_i = \binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!}.$$

We actually use the following corollary to Theorem 4.4.

Corollary 4.5 ([2]) *Assume that* $\exp C_i(G) \leq p^{u-i+2}$ *for each i with* $2 \leq i \leq u+2$. *If either $p > 2$ or* $\exp C_2(G) \leq p^{u-1}$, *then* $\Omega_u(G) = \{x \in G \mid x^{p^u} = 1\}$.

Another useful result for Theorem 4.3 is a theorem which is also due to Hall (see,.e.g., [15, Chapter 4, Theorem 4.22]):

Theorem 4.6 *If $p > 2$ and if every characteristic abelian subgroup of a finite p-group P is cyclic, then P is the central product of a cyclic group and E, where E is either $\{1\}$ or an extraspecial p-group of exponent p.*

Let A and G be finite groups, and let $|\mathrm{Hom}(A, G)|$ denote the number of homomorphisms from A to G. Such a number is called the Frobenius number of G with respect to A, because, if A is a cyclic group of order n, $|\mathrm{Hom}(A, G)| = \sharp L_n(G, \epsilon)$. As a generalization of Frobenius theorem, Yoshida proved the following [22].

Yoshida's theorem *If A is abelian,* $|\mathrm{Hom}(A, G)| \equiv 0 \bmod \gcd(|A|, |G|)$.

We consider the condition

$$\mathrm{H}(A, G): \qquad |\mathrm{Hom}(A, G)| \equiv 0 \bmod \gcd(|A/A'|, |G|),$$

where A' denotes the commutator subgroup of A. The following conjecture was also introduced in [3].

Conjecture H For any finite groups A and G, the condition $\mathrm{H}(A, G)$ holds.

Using Proposition 2.1, Asai and Yoshida proved the following.

Theorem 4.7 ([3]) *If A/A' is cyclic, then the condition $\mathrm{H}(A, G)$ holds.*

In connection with Theorems 4.2 and 4.3, the following theorems are known.

Theorem 4.8 ([1, 3]) *If every Sylow subgroup of A/A' is the direct product of a cyclic group and an elementary abelian group, then the condition $\mathrm{H}(A, G)$ holds.*

Theorem 4.9 ([2]) *If A is of odd order and if a Sylow p-subgroup of A/A' is the direct product of a cyclic group and a cyclic group of order at most p^2 for any prime p dividing $|A/A'|$, then the condition $\mathrm{H}(A, G)$ holds.*

Conjectures H and I are not still solved. As a connection of these conjecture, Asai and Yoshida proved the following.

Theorem 4.10 ([3]) *If Conjecture* I *is true, so is Conjecture* H.

5. THE NUMBER OF SUBGROUPS OF FINITE GROUPS

Throughout this section, A is a finite group, and $m_A(d)$ denotes the number of subgroups of index d in A. Proposition 2.1 is applicable to the following theorem.

Theorem 5.1 ([18]) *Let p^{λ_1} be the exponent of a Sylow p-subgroup of A/A'. Let i be an integer with $1 \leq i \leq \lambda_1$. Then*

$$m_A(qp^{i-1}) \equiv m_A(qp^i) \bmod p^i$$

for any positive integer q such that $\gcd(p, q) = 1$.

We say that A admits $\mathrm{C}(p^s)$, where s is a positive integer, if the following conditions hold for any positive integer q such that $\gcd(p, q) = 1$:

(1) For any integer i with $1 \leq i \leq [(s+1)/2]$, where $[(s+1)/2]$ is the greatest integer $\leq (s+1)/2$,
$$m_A(qp^{i-1}) \equiv m_A(qp^i) \bmod p^i.$$

(2) Moreover,
$$m_A(qp^{[(s+1)/2]}) \equiv m_A(qp^{[(s+1)/2]+1}) \bmod p^{[s/2]}.$$

Also, A is said to admits $\mathrm{CP}(p^s)$ if the preceding conditions (1) and (2) hold in the case where $q = 1$. We get the following corollary to Theorem 5.1.

Corollary 5.2 *Under the assumptions of Theorem* 5.1, *if $\lambda_1 \geq [(s+1)/2] + 1$, then A admits $\mathrm{C}(p^s)$.*

The preceding conditions appeared in the following proposition which due to Butler.

Proposition 5.3 ([5]) *Any finite abelian p-group P admits* $\mathrm{CP}(|P|)$.

Corollary 5.4 *If A is abelian, then A admits* $\mathrm{C}(|A|_p)$, *where $|A|_p$ is the highest power of p that divides $|A|$.*

The following proposition is due to Hall.

Proposition 5.5 ([7]) *Let P denote a finite p-group with $p^s = |P : \Phi(P)|$. Then*

$$m_P(p^i) \equiv m_{P/\Phi(P)}(p^i) \bmod p^{s-i+1}$$

for any integer i with $0 \leq i \leq s+1$, where $\Phi(P)$ denotes the Frattini subgroup of P.

Combining this proposition with Proposition 5.3, we have the following.

Corollary 5.6 *Under the assumptions of Proposition* 5.5, *P admits* $\mathrm{C}(p^s)$.

In [21], Wohlfahrt states that

$$1 + \sum_{n=1}^{\infty} \frac{|\mathrm{Hom}(A, S_n)|}{n!} X^n = \exp\left(\sum_{d=1}^{\infty} \frac{m_A(d)}{d} X^d \right),$$

where S_n is the symmetric group of degree n. Hence the following important proposition holds.

Proposition 5.7 ([17]) *If A admits* $\mathrm{C}(p^s)$, *then*

$$|\mathrm{Hom}(A, S_n)| \equiv 0 \bmod \gcd(p^s, n!).$$

Now, in connection with Conjectures H and I, we present the following conjecture.

Conjecture J Any finite group A admits $\mathrm{C}(|A/A'|_p)$.

For this conjecture, the case where A is a finite p-group is essential because of the following theorem.

Theorem 5.8 ([20]) *Let B be a normal subgroup of A such that the factor group A/B is an abelian group of order p^s. Assume that every subgroup D of A admits* $\mathrm{CP}(|D : D \cap B|)$. *Then A admits* $\mathrm{C}(p^s)$.

The following theorem results from Corollary 5.6 and corresponds Theorem 4.8.

Theorem 5.9 ([20]) *Let B be a normal subgroup of A such that A/B is the direct product of a cyclic p-group and an elementary abelian p-group. Then A admits* $\mathrm{C}(|A/B|)$.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash s$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = s$, is called the type of a finite abelian p-group isomorphic to the direct product

$$C_{p^{\lambda_1}} \times C_{p^{\lambda_2}} \times \cdots$$

of cyclic p-groups of order $p^{\lambda_1}, p^{\lambda_2}, \ldots$. We get the following theorems.

Theorem 5.10 ([20]) *Let B be a normal subgroup of P such that P/B is of type $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash s$. Assume that $\lambda_1 \geq [(s+1)/2]$. If $p > 2$, $\lambda_2 \leq 2$, and $\lambda_3 \leq 1$, then P admits* $\mathrm{CP}(p^s)$.

Theorem 5.11 ([20]) *Let B be a normal subgroup of A such that A/B is the direct product of a cyclic p-group and a cyclic p-group of order at most p^2. Then A admits* $\mathrm{C}(|A/B|)$.

Combining this theorem with Proposition 5.7, we have the following.

Corollary 5.12 ([20]) *Under the assumptions of Theorem* 5.11,

$$|\mathrm{Hom}(A, S_n)| \equiv 0 \bmod \gcd(|A/B|, n!).$$

This result corresponds to Theorems 4.3 and 4.9. However, the assertion of Corollary 5.12 is true for every prime p. So Theorems 4.3 and 4.9 seem to be true even if $p = 2$.

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