数理解析研究所講究録 1215 巻 2001 年 1-13

On a Problem of Proving the Existence of an Equilibrium in a Large Economy without Free Disposal: A problem of a purely finitely additive measure arising from the Fatou's lemma in several dimensions

Akira Yamazaki
Graduate Faculty of Economics
Hitotsubashi University

Abstract

The purpose of our paper, however, is to show that the assumption of the free disposability nor the desirability of the commodities is not needed to prove the existence of an equilibrium in a large economy with a continuum of economic agents provided that negative prices are allowed and that the preference distribution among the agents satisfy a mild requirement that "if there are unboundedly desirable commodities, they must be unanimously regarded as such by almost all members of the economy."

1 Introduction

It is well known in the literature that the free disposability or the desirability of the commodities is needed to ensure the existence of an equilibrium in a large economy with a continuum of economic agents. This is a stark contrast to the case of economies with a finite number of economic agents, where an equilibrium can be shown to exist without assuming the free disposability nor the desirability of all the commodities provided that negative prices are allowed. This difference originates in the fact that feasible allocations to individuals are bounded by the totally available resources in case of finite economies whereas in case of economies with an infinite number of agents what each individual can feasibly consume need not be bounded by the average of the totally available resources.

The purpose of our paper, however, is to show that the assumption of the free disposability nor the desirability of all the commodities is not needed to prove the existence of an equilibrium even in a large economy with a continuum of economic agents provided that negative prices are allowed and that the preference distribution among the agents satisfy a mild requirement that "if there are unboundedly desirable commodities, they must be unanimously regarded as such by almost all members of the economy."

In the literature there have been two types of proofs showing the existence of an equilibrium in a large economy. One is by Aumann [2] and Schmeidler [16] while the other is by Hildenbrand [8]. In both of these types, they first show the existence of equilibria where individual consumption sets are effectively bounded. Then, by considering a sequence of equilibria in an economy with bounded consumption sets where bounds are allowed to increase indefinitely, an equilibrium is shown to exist by appealing to strictly positive limit prices in case of Aumann [2] and Schmeidler [16], and to the Fatous's lemma in several dimensions in case of Hildenbrand [8].

The strictly positive prices at the limit establish bounds for budget sets along the convergent subsequence eventually, implying that "bounded partial equilibria" along the subsequence eventually become equilibria.

However, the very reason that equilibria (without free disposal of commodities) existed is due to the fact that people did not want to discard any commodities as all the commodities are (unboundedly) desirable. Thus, strictly speaking, from theoretical point of view, if one assumes the desirability of all the commodities, one cannot answer the question of whether the market price mechanism can coordinate supply and demand when the disposal activity of commodities is costly. The point here is that one needs to establish that market prices can indeed coordinate market forces of supply and demand even if unwanted commodities cannot be discarded freely in a large economy.

Once the assumption that all the commodities are desirable is dropped, we are in the set-up of Hildenbrand [8]. A difficulty arises in applying the Fatou's lemma in several dimensions. This lemma is applied to obtain at a limit point an equilibrium with free disposal. In this step one cannot hope to obtain an equilibrium with exact feasibility unless the result of the Fatou's lemma in several dimensions is strengthened.

Before we give a statement of the main result of this paper, we would like to take time to explain in more detail the nature of the difficulty in providing an equilibrium existence result in a large economy.

2 A Problem of a Purely Finitely Additive Measure Arising from the Fatou's Lemma in Several Dimensions

2.1 Two typical types of existence proofs

Let us briefly explain some basic features of existing existence proofs in large economies with a measure space of economic population so that difficulties in providing an existence proof of an equilibrium without assuming the free disposability of commodities nor assuming all the commodities to be unboundedly desirable.

Let us consider a large *economy* $\mathcal{E} : (A, \mathcal{A}, \nu) \to \mathcal{P} \times \mathbb{R}^\ell$ with an atomless measure space of economic population given by $(A, \mathcal{A}, \nu)$ with $\nu(A) = 1$. $\mathcal{E}$ is a measurable map, and $\mathcal{E}(a) = (\succ_a, e(a))$ with $e(a) \geqq 0$, $0 < \int e d\nu < \infty$ and a preference relation $\succ_a$ on

$X_a \equiv \mathbb{R}^\ell_+$ which is continuous, *i.e.* open in $X_a \times X_a$, and *negatively transitive*, *i.e.* $z \not\succ_a x$ if $z \not\succ_a y$ and $y \not\succ_a x$. In an integral we will often omit the symbol $d\nu$ and write $\int f$ instead of $\int f d\nu$. A preference relation $\succ_a$ is *locally nonsatiated* if for any $x \in X_a$ and for any neighborhood U of x there is $z \in U$ such that $x \succ_a x$. A preference relation $\succ_a$ defined on $\mathbb{R}^\ell_+$ is said to be *monotone* or *desirable* if for any $x \in \mathbb{R}^\ell_+$ and $v \in \mathbb{R}^\ell_+ \setminus \{0\}$ one has $x + v \succ_a x$. Commodity j is *desirable* if for any $x \in \mathbb{R}^\ell_+$ and any $t > 0$ one has $x + tu_j \succ_a x$ where $u_j \in \mathbb{R}^\ell$ is the vector with 1 in the j-th place. If all the commodities are desirable, then preference relation is monotone.

An *allocation* $f : A \to \mathbb{R}^\ell$ is an integrable function such that $f(a) \in X_a$ a.e. in A. It is *feasible* if $\int f \leqq \int e$, and *exactly feasible* if $\int f = \int e$.

A price vector is a member p of $\mathbb{R}^\ell$ such that $p \neq 0$.

An *equilibrium* for $\mathcal{E}$ is a pair (p, f) consisting of a price vector $p \in \mathbb{R}^\ell \setminus \{0\}$, and an allocation $f : A \to \mathbb{R}^\ell$ such that

1. $f(a) \in B(a,p)$ and $B(a,p) \cap \succ_a (f(a)) = \emptyset$ a.e. $a \in A$, where $B(a,p) \equiv \{z \in X_a \mid p \cdot z \leqq p \cdot e(a)\}$ and, for each $y \in X_a$, $\succ_a (y) \equiv \{z \in X_a \mid z \succ_a y\}$.
2. $\int f = \int e$

In the above definition, if one has $\int f \leqq \int e$, but $\int f \neq \int e$, then the pair (p, f) is called an *equilibrium* with *free disposal.*

Define, for each positive integer $k = 1, 2, \dots$, *k-bounded budget sets* by

$$B^k(a,p) \equiv \{z \in X_a \mid p \cdot z \leqq p \cdot e(a)\} \cap \left\{x \in \mathbb{R}^\ell_+ \;\middle|\; x \leqq k\left(1 + \sum_{i=1}^{\ell} e^i(a)\right)u\right\} \tag{2.1}$$

where u is the vector $(1, \dots, 1)$.

A pair (p, f) consisting of a price vector $p \in \mathbb{R}^\ell \setminus \{0\}$, and an allocation f is a *k-bounded partial equilibrium* for $\mathcal{E}$ if it is defined with respect to $B^k(a,p)$ instead of $B(a,p)$.

There are two types of existing fundamental results on the existence of equilibrium in the literature. One is by Aumann [2] and by Schmeidler [16], and the other by Hildenbrand [8].

THEOREM by Aumann and Schmeidler (Aumann [2] and Schmeidler [16]): *Given an economy $\mathcal{E}$, assume,* a.e. $a \in A$, $\succ_a$ *satisfies the desirability of all commodities. Then, an equilibrium (p, f) exists with $p \in \mathbb{R}^\ell_{++}$.*

THEOREM by Hildenbrand (Hildenbrand [8]): *Given an economy $\mathcal{E}$, an equilibrium (p, f) with free disposal exists where $p \in \mathbb{R}^\ell_+ \setminus \{0\}$.*

We like to comment in the next subsection on some basic features of the proofs by the above papers in order to understand the nature of the problem at hand.

2.2 Basic features of existing existence proofs

Let us give a brief description of each of the proofs by Schmeidler and Hildenbrand below.

2.2.1 Proof by Schmeidler

Step SH1: Given an economy $\mathcal{E}$, a k-bounded partial equilibrium (p_k, f_k) exists with $p_k \in \mathbb{R}^{\ell}_{+} \setminus \{0\}$ for each integer $k > 1$.

Step S2: One takes a convergent subsequence $p_k \to p$ and show that the *desirability of all commodities* implies $p \in \mathbb{R}^{\ell}_{++}$.

Step S3: The last step is to show that the sequence of k-bounded partial equilibria (p_k, f_k) eventually becomes an equilibrium along the subsequence.

2.2.2 Proof by Hildenbrand

Step SH1: Given an economy $\mathcal{E}$, a k-bounded partial equilibrium (p_k, f_k) with free disposal exists with $p_k \in \mathbb{R}^{\ell}_{+} \setminus \{0\}$ for each integer $k > 1$.

Step H2: Given a sequence of k-bounded partial equilibria with free disposal, (p_k, f_k), $p_k \in \mathbb{R}^{\ell}_{+} \setminus \{0\}$, it is shown that the *Fatou's Lemma in several dimensions* implies the existence of an integrable function $f : A \to \mathbb{R}^{\ell}$ such that f is a limit point of $f_k(a)$ a.e. $a \in A$ and $\displaystyle\int f \leqq \int e$.

Step H3: It is shown that a pair (p, f) with p, a limit point of the sequence $\{p_k\}$, is an equilibrium with free disposal.

2.2.3 Fatou's Lemma in Several Dimensions

Fatou's Lemma in Several Dimensions: Let $f_k : (A, \mathcal{A}, \nu) \to \mathbb{R}^{\ell}_{+}$, $k = 1, 2, \ldots,$ *be integrable and* $\lim_k \displaystyle\int f_k$ *exists. Then, there exists an integrable function* $f : A \to \mathbb{R}^{\ell}_{+}$ *such that*

1. $f(a)$ *is a limit point of the sequence* $\{f_k(a)\}$*, a.e. in* A*;*

2. $\displaystyle\int f \leqq \lim_k \int f_k$.

A proof of this lemma first appeared is by Schmeidler [17]. To understand a mathematical difficulty involved in applying the lemma to the sequence of k-bounded partial equilibrium to obtain an equilibrium at a limit as in the proof in 2.2.2, we like to show next the steps of the proof by Hildenbrand and Mertens [10].

Proof by Hidenbrand and Mertens:

Step 1: Define $\mu_k(E) = \int_E f_k d\nu$ for $E \in \mathcal{A}$ and each $k = 1, 2, \dots$. Then, $\mu_k \in \mathrm{ba}^\ell$, the ℓ-fold product of bounded additive measures on $(A, \mathcal{A})$. Since $\{\mu_k(A)\}_k$ is bounded, by the Theorem of Alaoglu $\{\mu_1, \mu_2, \dots\}$ is relative $\sigma^\ell(ba, L_\infty)$-compact. Thus, $\{\mu_k(A)\}_k$ has $\sigma^\ell(ba, L_\infty)$-accumulation point $\mu \in \mathrm{ba}^\ell$.

Step 2: By the Theorem of Yoshida-Hewitt μ can be decomposed into two parts in such a way that it can be written as

$$\begin{array}{ll} \mu = \mu_c + \mu_p, & \mu_c, \mu_p \geqq 0 \\ \mu_c \in \mathrm{ca}^\ell, & \mu_p \text{ is purely finitely additive,} \end{array} \tag{2.2}$$

where ca^ℓ is the ℓ-fold product of countably additive measures on $(A, \mathcal{A})$.

Take a Radon-Nikodym derivative g of μ_c with respect to ν. Then, one has

$$\int g = \mu_c(A) \leqq \mu(A) = \lim_k \int f_k.$$

It follows that

$$\int g \leqq \lim_k \int f_k = \int e.$$

However, at this stage one *cannot* say that $g(a)$ is a limit point of $\{f_k(a)\}$, a.e. in A. In order to achieve this one needs one more step.

Step 3: One can show that there are $\delta_k^i \geqq 0, i = 0, \dots, \ell$, with $\sum_{i=0}^\ell \delta_k^i = 1$, and $y_k^i, i = 0, \dots, \ell$, each in the set $\{f_k(a), f_{k+1}(a), \dots\}$ satisfying

$$\begin{aligned} g(a) &= \lim_n \sum_{i=0}^{\ell} \delta_{k_n}^i y_{k_n}^i \\ &= \sum_{i=0}^{\ell} \lim_n \delta_{k_n}^i y_{k_n}^i \geqq \sum_{\substack{i=0 \\ \delta^i > 0}}^{\ell} \delta^i \lim_n y_{k_n}^i \end{aligned}$$

where $\delta^i = \lim_n \delta_{k_n}^i$.

2.3 A problem of a purely finitely additive measure arising from the Fatou's lemma in several dimensions

A problem in proving the existence of an equilibrium in a large economy may clearly be understood by looking at the steps of typical proofs provided in above subsections 2.2.1, 2.2.2, and 2.2.3. The first proof by Aumann [2] and a subsequent proof by Schmeidler [16] as illustrated by the steps in 2.2.1 relies on the assumption that all the commodities are (unboundedly) desirable. This assumption was essential in establishing Step S2 where it is shown that there is a subsequence of k-bounded partial equilibria whose prices converge to strictly positive prices. The strictly positive prices at the limit establish bounds for budget sets along the convergent subsequence eventually, implying that k-bounded partial equilibria along the subsequence eventually become equilibria.

However, the very reason that equilibria (without free disposal of commodities) existed is due to the fact that people did not want to discard any commodities as all the commodities are (unboundedly) desirable. Thus, strictly speaking, from theoretical point of view, if one assumes the desirability of all the commodities, one cannot answer the question of whether the market price mechanism can coordinate supply and demand when the disposal activity of commodities is costly. The point here is that one needs to establish that market prices can indeed coordinate market forces of supply and demand even if unwanted commodities cannot be discarded freely in a large economy.

Once the assumption that all the commodities are desirable is dropped, we are in the set-up of Hildenbrand [8] except that his model is that of a production economy. In that framework one could establish the existence of a k-bounded partial equilibrium without free disposal because budget sets are bounded. A difficulty arises in the last step where he applies the Fatou's lemma in several dimensions. His method of proof is to apply the Fatou's lemma in several dimensions to a sequence of k-bounded partial equilibria to obtain at a limit point an equilibrium with free disposal. In this step one *cannot* hope to obtain an equilibrium with *exact feasibility* unless the result of the Fatou's lemma in several dimensions is strengthened. More precisely, in the statement of the lemma in the subsection 2.2.3, the inequality in the second condition needs to be strengthened to equality.

In following the steps of the proof by Hildenbrand and Mertens [10], one sees that there appear to be two sources of this inequality. One is in Step 2 and a purely finitely additive part of the weak limit of the sequence of bounded measures generated by the sequence of allocations associated with k-bounded partial equilibria gives rise to this inequality. The second source is in Step 3 where one ignores the terms with coefficients that go to zero that in turn imply that corresponding $y^i_{k_n}$'s might be unbounded. However it may appear that these two sources are independent, it all boils down to a non-vanishing purely finitely additive part in Step 2 of the proof of the Fatou's lemma. A problem caused by it is that it could correspond to circumstances where a sequence of groups of agents with declining weights down to null are assigned the commodity vectors $f_k(a)$ which are unbounded along the sequence of k-bounded partial equilibria. This type of phenomenon will not arise when all the commodities are assumed to be (unboundedly) desirable since strictly positive prices bound budget sets of agents. Nevertheless, on an intuitive basis it should be also clear that if everyone thinks commodities are not unboundedly desirable, this phenomenon cannot occur. We shall give a formal statement of this intuition in the next section.

3 Statement of the Main Theorem

In order to obtain an equilibrium existence result without free disposal we shall introduce an assumption which requires unanimous perception among economic agents as to which commodities, if they exist, are unboundedly desirable.

Let $J = \{1, \ldots, \ell\}$ be the set of indices of all commodities, and $\kappa > \sum_{i=1}^{\ell} \int e^i$ be a number sufficiently large. Define a subset $J^+_{\succ_a}$ of J consiting of "unboundedly desirable" commodities for agent a, that is, for each $a \in A$

$$J^+_{\succ_a} = \{j \in J \mid (\forall x \in X_a : x^j > \kappa)(\exists t > 0) x + tu_j \succ_a x\}.$$

According to this definition, commodity j is an unboundedly desirable commodity for agent a, *i.e.* $j \in J^+_{\succ_a}$, if, whenever a consumption of commodity j exceeds a certain amount, regardless of how much the agent already consumes that amount there always is a further increase of that commodity consumption that will be preferred by the agent.

If all the commodities are desirable, then preference relation is monotone. A desirable commodity is unboundedly desirable but not *vice versa*. In this paper we do not require monotonicity of preferences. In fact, it is not necessary that even one desirable commodity exists. Instead, what we require is that if, in fact, for one agent a commodity is unboundedly desirable, then this perception must be unanimously held by all the agents.

Assumption [Unanimous Perception of Unboundedly Desirable Commodities (UPUDC)]: There exists a subset J^+ of J, possibly empty, such that a.e. $a \in A$

1. $J^+_{\succ_a} = J^+$, and
2. $(\forall x \in X_a)\, x \not\succ_a x^\kappa(J \setminus J^+)$ where, for any $x \in \mathbb{R}^\ell_+$, $x^\kappa(J \setminus J^+)$ is defined by $\left(x^\kappa(J \setminus J^+)\right)^i = \min\{x^i, \kappa\}$ for $i \in J \setminus J^+$, and $\left(x^\kappa(J \setminus J^+)\right)^i = x^i$ for $i \in J^+$.

The above assumption of Unanimous Perception of Unboundedly Desirable Commodities (UPUDC) says that almost every agent in the economy unanimously agrees on which commodities, if any, are unboundedly desirable. The assumption is stated in two parts to express that either almost everyone wants a commodity unboundedly or there is a unanimously perceived limit as to how much each agent wants that commodity. Mathematically speaking, the second requirement can be weakened. That is, instead of requiring the unanimous perception of a limit κ of the commodities that are not unboundedly desirable, this limit can depend on each agent a as long as $\kappa(a)$ as a function from A into $\mathbb{R}^{\ell}_{+}$ is integrable.

We now give a statement of our main theorem.

Theorem 1 *Given an economy $\mathcal{E}$, an equilibrium (p, f) exists with $p \in \mathbb{R}^{\ell} \setminus \{0\}$ provided that the preference distribution of $\mathcal{E}$ satisfies the assumption of Unanimous Perception of Unboundedly Desirable Commodities.*

Our main theorem confirms our intuition that market prices can indeed coordinate market forces of supply and demand even if unwanted commodities cannot be discarded freely in a large economy if everyone agrees on as to which commodities are desirable without any bounds. A mathematical difficulty of a purely finitely additive measure arising from the Fatou's lemma is avoided because the unanimous perception of agents as to which commodities are unboundedly desirable induces either strictly positive prices for unboundedly desirable commodities that in turn limit the consumption of those commodities by agents, or agents themselves want not to consume commodities without limits.

4 Proof

Define

$$\begin{aligned} K(a) &= \left\{ x \in \mathbb{R}^{\ell}_{+} \;\middle|\; x \leqq k\left(1 + \sum_{i=1}^{\ell} e^{i}(a)\right) u \right\}, \\ K &= \left\{ x \in \mathbb{R}^{\ell}_{+} \;\middle|\; x \leqq k\left(1 + \sum_{i=1}^{\ell} \int e^{i}\right) u \right\}, \\ P &= \left\{ p \in \mathbb{R}^{\ell} \;\middle|\; \sum_{i=1}^{\ell} |p^{i}| = 1 \right\}, \\ D^{k}_{<}(a,p) &= \{x \in B(a,p) \mid (\forall z \in B_{<}(a,p))\, z \not\succ_{a} x\} \cap K(a). \end{aligned} \tag{4.3}$$

The first step is to follow the standard proofs as in [2], [8], [16], and show the existence of an (quasi-)equilibrium in k-bounded economies. The only difference from these proofs is that negative prices are allowed so that prices can coordinate to achieve exact equality between demands and supplies even if some of the commodities are unwanted, *i.e.* "bads", as in the case of the proofs of existence of an equilibrium (without free disposal nor monotonicity) in economies with a finite number of agents (see [12], [13],[7], [3], [15]).

For this purpose we shall define the correspondences $\pi : K \to P$, $\varphi : P \to K$, and $\varphi : P \times K \to P \times K$ by

$$
\begin{aligned}
\pi(x) &= \left\{p \in P \,\middle|\, (\forall q \in P) p \cdot \left(x - \int e\right) \geqq q \cdot \left(x - \int e\right)\right\}, \\
\varphi(p) &= \Big\{x \in \mathbb{R}^{\ell}_{+} \,\Big|\, (\exists \text{ an integrable function } f : A \to \mathbb{R}^{\ell}_{+}) x = \int f \\
&\qquad\qquad \text{and } f(a) \in D^{k}_{<}(a,p) \text{ a.e. } a \in A\Big\}, \\
\Psi(p,x) &= \pi(x) \times \varphi(p).
\end{aligned}
\tag{4.4}
$$

The correspondence Ψ is well defined and satisfies the conditions of Kakutani's fixed point theorem (see, *e.g.*, [2] pp.8-10, [16] pp.581-582, or [18] pp.550-551). So, let $(p,x) \in \Psi(p,x)$. Since $x \in \varphi(p)$, there is an allocation $f : A \to \mathbb{R}^{\ell}_{+}$ such that $x = \int f$, and $f(a) \in D^{k}_{<}(a,p)$ a.e. $a \in A$. We shall show that the allocation f is exactly feasible. Suppose we had $\int f - \int e \neq 0$. Since $x = \int f \in \varphi(p)$, it follows that a.e. $a \in A$, $p \cdot f(a) \leqq p \cdot e(a)$. Thus, we have $p \cdot \left(\int f - \int e\right) \leqq 0$. On the other hand, since $p \in \pi(\int f)$, we would have

$$
p \cdot \left(\int f - \int e\right) \geqq \frac{1}{\sum_{i=1}^{\ell} \left|\int f^{i} - \int e^{i}\right|} \left(\int f - \int e\right) \cdot \left(\int f - \int e\right) > 0,
$$

which is a contradiction. Therefore, f must be exactly feasible.

Thus, we have established that for each integer $k \geqq 1$ there exists (p_k, f_k) satisfying

1. $p_k \in P$,
2. $f_k(a) \in D^{k}_{<}(a, p_k)$ a.e. $a \in A$, and
3. $\int f_k = \int e$.

Since P is compact, there is a convergent subsequence of the sequence $\{p_k\}$. We can assume without loss of generality that the sequence itself is convergent so that $p_k \to p$. We shall prove that the sequence (p_k, f_k) eventually becomes an equilibrium.

First, let

$$
W_p = \{a \in A \mid p \cdot e(a) > \inf p \cdot X_a\}.
$$

If $(\exists j \in J) p^j < 0$, then $W_p = A$. If not, $p \geqq 0$ and thus $\int e > 0$ implies $p \cdot \int e > 0$, which in turn implies $\nu(W_p) > 0$. Define, then, an integer b and the set B by

$$\begin{aligned} b &> \frac{1}{\nu(W_p)} \int \sum_{i=1}^{\ell} e^i, \text{ and} \\ B &= \left\{ x \in \mathbb{R}^{\ell}_{+} \mid x \leqq bu \right\}. \end{aligned} \tag{4.5}$$

Lemma 1 *There is a subset $U \subset W_p$ of strictly positive measure having the property that*

$$f_k(a) \in B \text{ for infinitely many } k, \text{ a.e. } a \in U. \tag{4.6}$$

Proof If not, for a.e. $a \in W_p$, $(\exists k_0) k > k_0 \Rightarrow f_k(a) \notin B$. So, for such k, $(\exists j) f_k^j(a) > b$. Thus, $\sum_{i=1}^{\ell} f_k^i(a) > b$. It follows that $\liminf_k \sum_{i=1}^{\ell} f_k^i(a) \geqq b$ and this implies

$$\int_{W_p} \liminf_k \sum_{i=1}^{\ell} f_k^i(a) \geqq b\nu(W_p).$$

Then, by Fatou's lemma we obtain

$$\begin{aligned} \int_{W_p} \liminf_k \sum_{i=1}^{\ell} f_k^i(a) &\leqq \liminf_k \int_{W_p} \sum_{i=1}^{\ell} f_k^i(a) \\ &\leqq \liminf_k \int \sum_{i=1}^{\ell} f_k^i(a) \\ &= \liminf_k \int \sum_{i=1}^{\ell} e^i(a) = \int \sum_{i=1}^{\ell} e^i(a). \end{aligned} \tag{4.7}$$

Therefore, we must have

$$b\nu(W_p) \leqq \int \sum_{i=1}^{\ell} e^i(a) < b\nu(W_p),$$

a contradiction. This establishes the above claim. ∎

Next, we prove the following lemma.

Lemma 2 *For every $j \in J^+$, we have $p^j > 0$.*

Proof By lemma 1, the compactness of B implies that, for a.e. $a \in U$, $\{f_k(a)\}$ has a limit point $y \in B$. Taking a subsequence if necessary, one can assume $f_k(a) \to y$. Then,

$$p \cdot y = \lim_k p_k \cdot f_k(a) = \lim_k p_k \cdot e(a) = p \cdot e(a).$$

Now, suppose we had $p^j \leqq 0$ for $j \in J^+$. Then, since $j \in J^+ = J^+_{\succ_a}$, a.e. $a \in A$, by the assumption of unanimous perception of unboundedly desirable commodities,

$$(\exists z \in \mathbb{R}^\ell_+) z^j > \kappa, z^i = y^i \text{ for } i \neq j, \text{ and } z \succ_a y.$$

For this z, we have $p \cdot z \leqq p \cdot y = p \cdot e(a)$. $a \in W_p$ implies that there is $z_0 \in X_a = \mathbb{R}^\ell_+$ such that $p \cdot z_0 < p \cdot e(a)$. So, define for $n = 1, 2, \ldots$

$$z_n = \frac{1}{n} z_0 + \frac{n-1}{n} z.$$

Then, $z_n \to z$, and for all n, $p \cdot z_n < p \cdot e(a)$. For each n let $k_1(n)$ be such that $p_k \cdot z_n < p_k \cdot e(a)$ for $k \geqq k_1(n)$. As we have $z \succ_a y$, there is an integer n_0 such that $z_n \succ_a y$ for all $n \geqq n_0$. Since $f_k(a) \to y$, for each $n \geqq n_0$, there is an integer $k_2(n)$ greater than $k_1(n)$ such that we have $z_n \succ_a f_k(a)$ for $k \geqq k_2(n)$. This contradicts the fact that $f_k(a) \in D^k_{<}(a, p_k)$ a.e. $a \in A$. Therefore, we must have $p^j > 0$ for every $j \in J^+$. ■

It follows from lemma 2 that there is a positive integer k_0 such that for all $j \in J^+$ we have $p^j_k \geqq \delta$ for all $k > k_0$ for some $\delta > 0$.

Now, let $z \in B(a, p_k)$ and $j \in J^+$. Then, we have

$$\begin{aligned}
\delta z^j \leqq p^j_k z^j &\leqq \sum_{i \in J^+}^{\ell} p^i_k z^i \\
&\leqq p_k \cdot e(a) - \sum_{i \notin J^+}^{\ell} p^i_k z^i \\
&\leqq \sum_{i=1}^{\ell} |p^i_k| e^i(a) + \sum_{i \notin J^+}^{\ell} |p^i_k| z^i \qquad (4.8)\\
&\leqq \sum_{i=1}^{\ell} e^i(a) + \sum_{i \notin J^+}^{\ell} z^i \\
&\leqq \sum_{i=1}^{\ell} e^i(a) + (\ell - \sharp J^+)\kappa.
\end{aligned}$$

Thus, one obtains

$$z^j \leqq \frac{1}{\delta} \left(\sum_{i=1}^{\ell} e^i(a) + (\ell - \sharp J^+)\kappa \right). \qquad (4.9)$$

Define an integer k_1 by

$$k_1 = \max \left\{ k_0, \frac{\kappa}{1 + \sum_i e^i(a)}, \frac{\sum_i e^i(a) + (\ell - \sharp J^+)\kappa)}{\delta(1 + \sum_i e^i(a))} \right\}.$$

Then, for any $k > k_1$ when $z \in B(a, p_k)$, we have $z_j \leqq k(1 + \sum_i e^i(a))$ for each $j \in J^+$.

Thus, if we have $z \notin K(a)$, then it must be that $z^i > \kappa$ for some $i \notin J^+$. So, define the subset J^- of $J \setminus J^+$, and a "partially truncated" vector z^κ of z by

$$\begin{aligned} J^- &= \{i \in J \setminus J^+ \mid z^i > \kappa\}, \\ (z^\kappa)^i &= \begin{cases} z^i & \text{if} \quad i \in J \setminus J^- \\ \kappa & \text{if} \quad i \in J^- \end{cases}. \end{aligned} \tag{4.10}$$

Since we have $z^\kappa \in K(a)$ and $f_k(a) \in D^k_<(a, p_k)$, $z^\kappa \not\succ_a f_k(a)$ for any $k > k_1$. It follows from the assumption of the unanimous perception of unboundedly desirable commodities that we have $z \not\succ_a z^\kappa$. Hence, by the negative transitivity of preference $\succ_a$ that $z \not\succ_a f_k(a)$ for any $k > k_1$. This establishes that

$$f_k(a) \in D^k(a, p_k), \qquad \text{a.e. } a \in A \quad \text{for } k > k_1.$$

Therefore, a pair $(p_k, f_k(a))$ is an equilibrium for each $k > k_1$.

References

[1] ARROW, KENNETH J., AND GERARD DEBREU: "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, 22(1954), 265–290.

[2] AUMANN, ROBERT J.: "Existence of Competitive Equilibria in Markets with a Continuum of Traders," *Econometrica*, 34(1966), 1–17.

[3] BERGSTROM, THEODORE C.: "How to discard 'Free Disposability'- as No Cost," *Journal of Mathematical Economics*, 3(1976), 131–134.

[4] DEBREU, GERARD: "New Concepts and Techniques for Equilibrium Analysis," *International Economic Review*,3(1962), 257–273.

[5] DEBREU, GERARD: "Existence of Competitive Equilibrium," Chapter 15 in KENNETH J. ARROW *et al.* eds.: *Handbook of Mathematical Economics*, Amsterdam:North-Holland Publishing Co., Vol. II(1982), 697–743.

[6] GALE, DAVID, AND ANDREU MAS-COLELL: "An Equilibrium Existence Theorem for a General Model without Ordered Preferences," *Journal of Mathematical Economics*, 2(1975), 9–15.

[7] HART, OLIVER D., AND HAROLD W. KUHN: "A Proof of Existence of Equilibrium without the Free Disposal Assumption," *Journal of Mathematical Economics*, 2(1975), 335–343.

[8] HILDENBRAND, WERNER: "Existence of Equilibria for Economies with Production and a Measure Space of Consumers," *Econometrica*, 38(1970), 608–623.

[9] HILDENBRAND, WERNER: *Core and Equilibria of a Large Economy,* Princeton: Princeton University Press, 1974.

[10] HILDENBRAND, WERNER, AND JEAN-FRANÇOIS MERTENS: "On Fatou's Lemma in Several Dimensions," *Z. Wahrscheinlichkeitstheorie verw. Geb.*, 17(1971), 151–155.

[11] MAS-COLELL, ANDREU: "An Equilibrium Existence Theorem without Complete or Transitive Preferences," *Journal of Mathematical Economics*, 1(1974), 237–246.

[12] MCKENZIE, LIONEL W.: "On the Existence of General Equilibrium for a Competitive Market," *Econometrica*, 27(1959), 54–71.

[13] MCKENZIE, LIONEL W.: "The Classical Theorem on Existence of Competitive Equilibrium," *Econometrica*, 49(1981), 819–841.

[14] NIKAIDO, HUKUKANE: "On the Classical Multilateral Exchange Problem," *Metroeconomica*, 8(1956), 135–145.

[15] SHAFER, WAYNE J.: "Equilibrium in Economies without Ordered Preferences or Free Disposal," *Journal of Mathematical Economics*, 3(1976),135–137.

[16] SCHMEIDLER, DAVID: "Competitive Equilibria in Markets with a Continuum of Traders and Incomplete Preferences," *Econometrica*, 37(1969), 578–585.

[17] SCHMEIDLER, DAVID: "Fatou's Lemma in Several Dimensions," *Proceedings of the American Mathematical Society*, 24(1970), 300–306.

[18] YAMAZAKI, AKIRA: "An Equilibrium Existence Theorem without Convexity Assumptions," *Econometrica*, 46(1978), 541–555.

[19] YAMAZAKI, AKIRA: "Diversified Consumption Characteristics and Conditionally Dispersed Endowment Distribution:Regularizing Effect and Existence of Equilibria," *Econometrica*, 49(1981), 639–654.