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Optimal Growth with Recursive Utility: An Existence Result without Convexity Assumptions*

Nobusumi SAGARA

Faculty of Economics, Hosei University
4342, Aihara, Machida, Tokyo, 194-0298, Japan
e-mail: nsagara@mt.tama.hosei.ac.jp

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Abstract

This paper deals with the existence problem of optimal growth with recursive utility in a continuous-time model without convexity assumptions. We consider a general reduced model of capital accumulation and provide an existence result allowing the production technology to be nonconvex and the objective functional to be nonconcave and recursive. The program space under investigation is a weighted Sobolev space with discounting built in, as introduced by Chichilnisky. The compactness of the feasible set and the continuity of the objective are proven by the effective use of the $\mathscr{L}^2$-convergence. Existence follows from the classical Weierstrass theorem.

Key Words. Recursive utility, optimal growth, nonconvexity, existence, weighted Sobolev space.

*This is a condensed version of the paper with the same title. The full version i orthcoming in *Journal of Optimization Theory and Applications*.

1 Introduction

This paper deals with the existence problem of optimal growth with recursive utility in a continuous time model without convexity assumptions. We consider a general reduced model of capital accumulation allowing for nonconvex technology and an objective functional that is nonconcave and recursive. The program space under investigation admits unbounded programs, but the growth rate of the programs is bounded by a certain discounting function. The space of this type is described by a *weighted Sobolev space* with discounting built in, which is identified with an $\mathscr{L}^2$-space. Therefore the compactness of the feasible set and the continuity of the objective functional are proven by the effective use of the $\mathscr{L}^2$-convergence. Existence follows from the classical Weierstrass theorem.

The analysis of recursive preferences was initiated by Koopmans (Ref. 1) in a discrete-time framework. (For a recent treatment of Koopmans' recursive utility in discrete-time, see the monograph of Becker and Boyd (Ref. 2)). Uzawa (Ref. 3) extended Koopmans' discrete-time concept of recursive utility to continuous-time. Epstein (Ref. 4) and Epstein and Hynes (Ref. 5) introduced generalizations of Uzawa's recursive utility function to analyze the global dynamics, and Epstein (Ref. 6) axiomatized a generating function that ensures the existence of a recursive utility. An essential feature of the recursive functional form is that the rate of time preference is endogenized in its structure. The first rigorous treatment of the existence problem for the case of recursive utility is that of Becker, Boyd, and Sung (Ref. 7). The proof of their existence theorem, however, relies on the convexity of the technology and the concavity of the recursive integrand. Unlike the case of time additive utility, the recursive objective functional generally involves the nonconcave integrand in its nature, and so the concavity assumption is too strong. The purpose of this paper is to overcome this difficulty.

In the case of time additive utility, there exist two important works with nonconvexity: Chichilnisky (Ref. 8) and Romer (Ref. 9). Chichilnisky introduced the weighted Sobolev space endowed with the $\mathscr{L}_2$-norm topology, and demonstrated the $\mathscr{L}_2$-norm continuity of the objective and the $\mathscr{L}_2$-norm compactness of the feasible set without any convexity assumptions. Romer employed an $\mathscr{L}_1$-space endowed with weak topology under mild convexity assumptions. Although Romer's existence theorem relies on the concavity of the objective functional, it is imposed only on the highest order derivative in the objective functional. This concavity assumption is quite weak in practice. Consequently, Romer's existence result permits us to consider a broad class covering many economic problems with nonconvexities. Unfortunately, Romer's argument cannot be extended directly to the case of recursive utility.

In optimal control and the calculus of variations, existence problems without convexity assumptions have been investigated by various authors. It is shown that when the control systems and the integrand in the objective functional are linear in the state variables, existence is guaranteed under the standard assumptions [see Cesari (Ref. 10, Chapter 16)]. Since the proof is based on the convexification of the control set, linearity plays an essential role in demonstrating existence. Without imposing any linearity conditions, Carlson (Ref. 11) treats the existence problems by approximation. Carlson transforms an original nonconvex problem to a convexified relaxed problem in which existence is guaranteed, and shows that for any optimal solution of the relaxed convexified problem, there exists a sequence of admissible trajectories of the original problem that converges uniformly to the optimal solution of the relaxed convexified problem. Note that the program space of the above existence results is the set of all locally absolutely continuous functions endowed with the weak topology.

This paper deals with the weighted Sobolev space that was first introduced by Chichilnisky (Ref. 8) to the growth theory literature. There exist two useful topologies on the weighted Sobolev space: the $\mathscr{L}^2$-norm topology and the weak topology. Chichilnisky demonstrates an existence result under the $\mathscr{L}^2$-norm topology for the case of time additive utility without convexity assumptions. In the weak topology of the weighted Sobolev space, Maruyama (Ref. 12) provides an existence result for the case of time additive utility under convexity assumptions. We engage ourselves with the $\mathscr{L}^2$-norm topology as in Chichilnisky and prove an existence theorem for the case of recursive utility, which involves a nonconcave integrand, without convexity assumptions. Our existence theorem can be applied to the nonconvex problem with increasing returns, as studied in Davidson and Harris (Ref. 13) and Skiba (Ref. 14), and to the recursive utility along the lines of Uzawa (Ref. 3), Epstein and Hynes (Ref. 5), and Epstein (Ref. 4).

2 Description of the Model

Weighted Sobolev Space. Let the interval $I = [0, \infty)$ be a time horizon. We denote by $\mathscr{L}^2(I, \mathbb{R}^n)$ the set of all measurable functions $f : I \to \mathbb{R}^n$ such that $\int_0^\infty \|f(t)\|^2 dt < \infty$, and by $\mathscr{C}^1(I, \mathbb{R}^n)$, the set of all functions from I to $\mathbb{R}^n$ that are differentiable on an open interval that contains I. Let $\delta : I \to \mathbb{R}$ be a measurable function such that $0 < \delta(t) \leq 1$ *a.e.* $t \in I$ and $\int_0^\infty \delta(t)dt < \infty$. Define the inner product on $\mathscr{C}^1(I, \mathbb{R}^n)$ by

$$(f, g)_\delta := \int_0^\infty \delta(t)(f(t)g(t) + \dot{f}(t)\dot{g}(t))dt$$

for $f, g \in \mathscr{C}^1(I, \mathbb{R}^n)$. The norm on $\mathscr{C}^1(I, \mathbb{R}^n)$ is given by $\|f\|_\delta := (f, f)_\delta^{1/2}$. A *weighted Sobolev space* with the density function δ is defined by

$$\mathscr{W}_\delta^{1,2}(I, \mathbb{R}^n) := \{f \in \mathscr{C}^1(I, \mathbb{R}^n) |\ \delta^{\frac{1}{2}} f, \delta^{\frac{1}{2}} \dot{f} \in \mathscr{L}^2(I, \mathbb{R}^n)\}.$$

Under the norm $\|\cdot\|_\delta$, the space $\mathscr{W}_\delta^{1,2}(I, \mathbb{R}^n)$ is a separable Hilbert space [see Kufner, John, and Fŭcík (Ref. 15, Theorem 8.10.2)].

Technology. We consider a general reduced model of capital accumulation. There are n capital goods in the general model economy. The technology is described by a correspondence $\Gamma : I \times \mathbb{R}_+^n \to 2^{\mathbb{R}_+^n}$. We mean by $y \in \Gamma(t, x)$ that, given capital stock x at time t, y can be accumulated as additional capital. We call Γ the *technology correspondence.* The graph of Γ is denoted by D:

$$D = \{(t, x, y) \in I \times \mathbb{R}_+^n \times \mathbb{R}_+^n | y \in \Gamma(t, x)\}.$$

We assume that the t-section of D,

$$D(t) = \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n\ |(t, x, y) \in D\},$$

is nonempty for any $t \in I$.

Program Space. The program space under consideration is a subset of the weighted Sobolev space $\mathscr{W}_\delta^{1,2}(I, \mathbb{R}^n)$. A *capital accumulation program*, $k : I \to \mathbb{R}_+^n$, is an element of $\mathscr{W}_\delta^{1,2}(I, \mathbb{R}^n)$. We restrict the capital accumulation program to the class such that its derivative has a uniform Lipschitz bound: $\|\dot{k}(t + h) - \dot{k}(t)\|^2 \leq hL(t)$ for any $h > 0$ for a given measurable function $L : I \to \mathbb{R}$ with $\int_0^\infty \delta(t)L(t)dt < \infty$. Define

$$\mathscr{C}_L^1(I, \mathbb{R}^n) := \{f \in \mathscr{C}^1(I, \mathbb{R}^n) | \|\dot{f}(t + h) - \dot{f}(t)\|^2 \leq hL(t)\ \forall h > 0\ \forall t\}.$$

Then the program space $\mathscr{K}$ is defined by $\mathscr{K} := \mathscr{W}_\delta^{1,2}(I, \mathbb{R}^n) \cap \mathscr{C}_L^1(I, \mathbb{R}^n)$.

Recursive Objective Functional. Social welfare is described by a *utility function* $u : I \times \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ and a *discounting function* $\theta : I \times \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$. The *recursive objective functional* $U : \mathscr{K} \to \mathbb{R}$ is given by

$$U(k) = \int_0^\infty u(t, k(t), \dot{k}(t)) \exp\left(-\int_0^t \theta(s, k(s), \dot{k}(s))ds\right) dt.$$

An essential feature of this functional form is that the rate of time preference is implicit in its structure. Note that if the discounting function θ is constant, this functional form reduces to time additive utility.

Optimal Program. Define the set of feasible programs from an initial capital stock $z \in \mathbb{R}^n_+$, $\mathscr{F}(z)$, by

$$\mathscr{F}(z) := \{k \in \mathscr{K} | \dot{k}(t) \in \Gamma(t, k(t)) \text{ a.e. } t \in I, 0 \leq k(0) \leq z\}.$$

Then the programming problem $P(z)$ is defined by

$$P(z): \qquad V(z) = \sup\{U(k) | k \in \mathscr{F}(z)\}.$$

Here V is the *value function.* A program $k^* \in \mathscr{W}^{1,2}_\delta(I, \mathbb{R}^n)$ is called an *optimal program* to $P(z)$ if $k^* \in \mathscr{F}(z)$ and $U(k^*) = V(z)$.

3 Existence of an Optimal Program

In this section we prove the existence of an optimal program. The proof of the existence theorem is based on the classical Weierstrass theorem: the feasible set is compact and the objective functional is continuous in the norm-topology of the program space.

To ensure existence, we need the following conditions on the preferences and the technology:

Preference Conditions.

(P-1) $t \mapsto u(t, x, y)$ is measurable for any $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and $(x, y) \mapsto u(t, x, y)$ is continuous for any $t \in I$.

(P-2) There exist a measurable function $\alpha : I \to \mathbb{R}$ and constants $\beta_1, \beta_2 \geq 0$ such that $|u(t, x, y)| \leq \alpha(t) + \beta_1 \|x\|^2 + \beta_2 \|y\|^2$ for any $(t, x, y) \in I \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and $\int_0^\infty \delta(t)\alpha(t)dt < \infty$.

(P-3) $t \mapsto \theta(t, x, y)$ is measurable for any $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and $(x, y) \mapsto \theta(t, x, y)$ is continuous for any $t \in I$.

(P-4) $\exp\left(-\int_0^t \left[\inf_{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+} \theta(s, x, y)\right] ds\right) \leq \delta(t)$ *a.e.* $t \in I$.

Condition (P-1) states that the utility function u is a *Carathéodory function.* Condition (P-2) states that u satisfies the *growth condition*, which is standard in control theory when $\delta(t) \equiv 1$. In condition (P-3) we require that the discounting function θ is also a Carathéodory function. Condition (P-4) implies that the discount rate is uniformly bounded from above on the feasible programs:

$$\exp\left(-\int_0^t \theta(s, k(s), \dot{k}(s))ds\right) \leq \delta(t) \text{ a.e. } t \quad \text{for any } k \in \mathscr{F}(z).$$

In the time additive case $\theta \equiv \rho$ for constant $\rho > 0$, this condition is always satisfied for $\delta(t) = \exp(-\rho t)$.

Technology Conditions.

(T-1) $0 \in \Gamma(t,x)$ for any $(t,x) \in I \times \mathbb{R}^n_+$.

(T-2) $x \mapsto \Gamma(t,x)$ is a closed-valued and upper semicontinuous correspondence for any $t \in I$.

(T-3) There exists a measurable function $\mu : I \to \mathbb{R}$ such that $\|y\|^2 \le \mu(t)$ for any $y \in \Gamma(t,x)$ and $(t,x) \in I \times \mathbb{R}^n_+$, and $\int_0^\infty \delta(t) \int_0^t \mu(s)dsdt < \infty$.

Condition (T-1) is the assumption that allows a free disposability of production activity. This implies that the program $k(t) \equiv z$ is feasible and hence $\mathscr{F}(z) \neq \emptyset$ for any $z \in \mathbb{R}^n_+$. Condition (T-2) is standard in growth theory. Condition (T-3) imposes boundedness on the set $\Gamma(t,x)$. Conditions (T-2) and (T-3) together imply that the correspondence $x \mapsto \Gamma(t,x)$ is compact-valued and upper semicontinuous for any $t \in I$.

Theorem 3.1. For any initial capital stock $z \in \mathbb{R}^n_+$, there exists an optimal program to $P(z)$.

4 Conclusions

The choice of a program space and a relevant topology is important in order to establish an existence result. The program space of Becker, Boyd, and Sung (Ref. 7) is the set of all locally absolutely continuous functions endowed with the weak topology. The program space of this paper is the weighted Sobolev space with the density function endowed with the $\mathscr{L}^2$-norm topology. In the nonrecursive case in which θ is constant, the necessary and sufficient condition that the objective functional is upper semicontinuous in the weak topology of the program space, is that u is a Carathéodory function that satisfies the standard growth condition, and u is concave with respect to y [see Marcellini (Ref. 18)]. Therefore, as long as the program space is endowed with the weak topology, convexity assumptions are obviously indispensable.

In general, strengthening a topology makes it harder for sets to be compact, but easier for functions to be continuous. By considering the norm-topology on the program space, which is stronger than the weak topology, we do not rely on the concavity of the integrand for the argument of the continuity of the objective functional, so the continuity argument is relatively simplified. This is due to one of the significant properties of the Carathéodory function that is known as the theorem of Krasnoselskii. From this result we can ensure that the recursive objective functional is continuous in the norm topology of the program space. To the contrary, in order to guarantee the compactness of the feasible set, as a demerit, we must impose somewhat stringent conditions on the boundedness of the technology as in (T-3). The argument of compactness, however, is relatively simplified by the effective use of the $\mathscr{L}^2$-convergence.

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