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Intergenerational Transfers Motivated by Altruism from Children towards Parents

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Abstract

This study has two ends. The first is to construct a formal model with altruism from children towards parents and to reveal the structure of an equilibrium, which has not been studied in the existing literature. The second is to establish the existence of an equilibrium in this model by demonstrating the existence of a dynamically consistent allocation in the concept of subgame perfect Nash equilibrium.

KEYWORDS: intergenerational altruism; intergenerational transfers; subgame perfect Nash equilibrium.

JEL Classification Numbers: C62, D64, D91

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1 Introduction

In the recent literature, a dynamically consistent allocation is characterized in the model with intergenerational altruism. Ray (1987) focuses on the intergenerational altruism that one generation holds towards his descendants, which I call forward altruism. Hori (1997) focuses on the intergenerational altruism that one generation holds towards both his parents and children, which we call two-sided altruism. Both studies characterize dynamically consistent paths and represent them in the form of a policy function.

This study has two purposes. The first is to construct a formal model with the intergenerational altruism that children hold towards parents. The structure of an equilibrium in a model with such altruism has not been studied in the existing literature.[1] This type of altruism, which I call backward altruism, yields two types of intergenerational transfers. One type is an intergenerational transfers from children towards parents, gift. Another type is one from parents towards children, education investments. We demonstrate that they determine the structure of an equilibrium in this model.

The second purpose is to demonstrate the existence of a dynamically consistent allocation, which we call an equilibrium in this model. This concept of equilibrium is the same as that of subgame perfect Nash equilibrium in extensive-form games.[2] I also characterize a steady state and reveal a relationship between a steady state allocation and intergenerational altruism by using an example. By doing so, we demonstrate that the strength of intergenerational altruism, as well as its direction, also play an important role for a dynamic allocation. This finding is important for determining the policy of an income redistribution over times or generations by a government.[3]

In the section 2, a backward altruism model is formalized, and an equilibrium in this model is defined. In the section 3, we demonstrate the existence of an equilibrium. In the section 4, we make concluding remarks.

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[1] O'Connell and Zeldes (1993) also focus on an economy with altruism from children towards parents. They interest in the behavior of transfers in this economy rather than in the mechanism of determining an equilibrium.

[2] As for the concept of equilibrium, see Selten (1975). Leininger (1986) demonstrates the existence of a perfect equilibrium in the economy that each generation is altruistic to the other generations.

[3] This effect is discussed in Yano (1998).

2 The Model

Take a two generation overlapping economy lasting both side of time. Each generation is born at the end of each period and lives the sequential two periods. Assume that each generation is altruistic towards the previous generation, which means that children are altruistic towards parents. As a result, children will make gifts for parents. Assume that each generation has an income only in the first period and has no income in the second period. Moreover, assume that an economy has no financial asset that is available for individuals, and that all the goods are perishable. Thus, in order to consume in the second period, parents intend to make children have more income that yields more gifts from children. Thus, parents make such choices, e.g., education investments in children.

Let generation t be a generation born at the end of period $t-1$. Note generation t lives two periods, period t and period $t+1$. In period t, he obtains an income y_t and distributes it to his own consumption c_t^1, education investments towards children (generation $t+1$) e_{t+1}, and gifts for parents (generation $t-1$) g_t, which equals to parents' consumption for their old time c_t^2; That is $g_t = c_t^2$. In period $t+1$, when generation t is old, he makes a consumption c_{t+1}^2 by using gifts g_{t+1} received from his children. Generation t's constraints are as follows.

$$c_t^1 + e_{t+1} + g_t = y_t, \tag{1}$$

$$c_{t+1}^2 = g_{t+1}. \tag{2}$$

Generation t is altruistic towards generation $t-1$ for any t. Thus, in a model with backward altruism, a utility function is expressed as follows.[4]

[4]The utility form of forward altruistic generations is expressed as

$$U_t = u_t + \alpha U_{t+1},$$

where u_t is generation t's utility obtained from their own consumption, and where U_{t+1} is generation $t+1$'s total utility.

The utility form of two-sided altruistic generations is expressed as

$$U_t = u_t + \alpha U_{t+1} + \beta U_{t-1}.$$

See Hori and Kanaya (1989) and Hori (1992) for two-sided altruism.

$$U_t = v^1(c_t^1) + v^2(c_{t+1}^2) + \beta U_{t-1}, \tag{3}$$

where U_t and U_{t-1} are total utilities of generation t and of generation $t-1$ respectively, and where $v^1(\cdot)$ and $v^2(\cdot)$ are utility functions for the young time and for the old time respectively. Note that β in (3) is a discount factor reflecting the degree of backward altruism, and that the higher β is, the more altruistic towards generation $t-1$ generation t is. Assume β is constant over generations and satisfies $0 < \beta < 1$. As for U_{t-1}, the same form of utility function as (3) can be expressed. That is, $U_{t-1} = v^1(c_{t-1}^1) + v^2(c_t^2) + \beta U_{t-2}$. Since generation t have no effect on choices and on utility levels determined before the end of period $t-1$ in which generation t is born, he considers c_{t-1}^1 and U_{t-2} as given. By expressing them as $\bar{c}_{t-1}^1$ and $\bar{U}_{t-2}$, we rewrite (3) into

$$U_t = v^1(c_t^1) + v^2(c_{t+1}^2) + \beta \left[v^1(\bar{c}_{t-1}^1) + v^2(c_t^2) + \beta \bar{U}_{t-2}\right]. \tag{3'}$$

Generation t will intend to increase his utility by increasing gifts $g_t = c_t^2$ in the view point of children. Generation t knows that he makes more gifts towards generation $t-1$ with a larger income. Thus, he may expect that generation $t+1$ also makes more gifts towards him with a larger income. Then, generation $t+1$'s gift towards generation t, g_{t+1}, is determined by generation $t+1$'s income, y_{t+1}. Suppose that g_{t+1} is described as a function with respect to y_{t+1} as follows.

$$g_{t+1} = \Phi_{t+1}(y_{t+1}), \tag{4}$$

where $\Phi_{t+1}(y_{t+1})$ states that generation $t+1$ makes gifts $\Phi_{t+1}(y_{t+1})$ towards his parents with his given income y_{t+1}, and therefore we call it a gift function of generation $t+1$. In the below, we write a functional forms of generation $t+1$'s gift function $\Phi_{t+1}(y_{t+1})$ as Φ_{t+1}.

Generation t will intend to increase his children's income. For this purpose, he will accumulate his children's human capital by means of education investments towards generation $t+1$. Generation t's education investments towards his children e_{t+1} accumulates his children's human capital h_{t+1}. Generation $t+1$ makes his human capital h_{t+1} as an input into a production, and he obtains an output $f(h_{t+1})$, all of which become his income y_{t+1}. Assume that these relationship are as follows.

$$h_{t+1} = e_{t+1}, \qquad y_{t+1} = f(h_{t+1}). \tag{5}$$

Now we are ready to describe the optimization problem of generation t. In the optimization problem, we remove the given factors in (3'), $v^1(\bar{c}^1_{t-1})$ and $\bar{U}_{t-2}$, on which generation t's choices have no effect. From (1), (2), (3'), (4), and (5), the generation t's optimization problem is expressed as follows.

The Generation t's Optimization Problem

$$\max_{(c^1_t, e_{t+1}, g_t)} \quad v^1(c^1_t) + v^2(\Phi_{t+1}(f(e_{t+1}))) + \beta v^2(g_t), \tag{6}$$

$$s.t. \quad c^1_t + e_{t+1} + g_t \leq y_t, \; given \;\; y_t \text{ and } \Phi_{t+1}.$$

We will formalize an equilibrium in a backward altruism model. As shown in (6), generation t makes choices (c^1_t, e_{t+1}, g_t) under a given income, y_t, and a given functional form of generation $t+1$'s gift function, Φ_{t+1}. Since generation t's optimal choices depend on both y_t and Φ_{t+1}, these choices is represented as functions with respect to both y_t and Φ_{t+1}. Let $(c^1(y_t, \Phi_{t+1}), e(y_t, \Phi_{t+1}), g(y_t, \Phi_{t+1}))$ be optimal choices (c^1_t, e_{t+1}, g_t) satisfying (6). The third term of them, $g(y_t, \Phi_{t+1})$, means generation t's optimal gifts towards parents. Each generation solves the same optimization problem as (6). Then, generation $t-1$ regards the generation t's gift function $\Phi_t(y_t)$ as given in his optimization problem. For dynamic consistency, it must be that

$$\Phi_t(y_t) = g(y_t, \Phi_{t+1}), \tag{7}$$

for any y_t.

We are ready to define an equilibrium. Let $T = (-\infty, ..., t, ..., +\infty)$. We call $\{\Phi_t\}_{t \in T}$ an *equilibrium* if $\Phi_t(y_t) = g(y_t, \Phi_{t+1})$ for any $t \in T$. Also we call $\{\Phi_t\}_{t \in T}$ a *stationary equilibrium* if $\Phi_t = \Phi$ for any $t \in T$.

3 Assumptions, Existence Result, and Outline of Proof of the Existence

Let $Y \subset R_+$ be a set of y_t. Let $X^1 \subset R_+$, and $X^2 \subset R_+$. As for $f : Y \rightarrow R_+$ and $v^i : X^i \rightarrow R_+$ $(i = 1, 2)$, assume the follows.

F.1. f is increasing.

F.2. There exists $\bar{y} \in Y$ such that $f(y) \leq y, \forall y \in \{\tilde{y} \in Y : \tilde{y} \geq \bar{y}\}$.

F.3. $f(0) = 0$.
F.4. $f : Y \to R_+$ is continuous.
V.1. v^i is strictly increasing for $i = 1, 2$.
V.2. v^i is strictly concave for $i = 1, 2$.
V.3. $v^i(0) = 0$ for $i = 1, 2$.
V.4. $v^i : X^i \to R_+$ is continuous for $i = 1, 2$.
The main result is the following.

Theorem 1 *Under the assumptions (F.1), (F.2), (F.3), (F.4), (V.1), (V.2), (V.3), and (V.4), there exists a stationary equilibrium.*

The rest of this section provides an outline of the proof of this theorem.

Each generation's optimization problem can be decomposed into two parts: intratemporal optimization problem, which intergenerational allocation problem in one period in other words, and intertemporal optimization problem. The first problem is that an available income $m \in M \subset Y$ in period t is distributed to generation t's consumption, c_t^1, and gifts towards parents, g_t, which equals generation $t-1$'s consumption in period t, c_{t-1}^2. Then, a pair of optimal choices, c_t^1 and g_t, is represented as $(c(m), g(m))$ that satisfies

$$(c(m), g(m)) = \arg\max_{(c,g)} v^1(c) + \beta v^2(g) \quad s.t. \;\; c + g \leq m, c \geq 0, g \geq 0. \tag{8}$$

For the preceding proof, we define

$$V(m) = v^1\left(c(m)\right) + \beta v^2\left(g(m)\right). \tag{9}$$

Then, we have the following lemma.

Lemma 1 *$c(\cdot)$ and $g(\cdot)$ are unique, continuous, and increasing. Moreover, $V(\cdot)$ is continuous, strictly increasing, and strictly concave.*

Proof. Omitted.

The second problem, i.e., an intertemporal optimization problem, is that generation t distributes his income y_t to an available income in period t, m, and education investments towards children, e_{t+1}, which leads to gifts received in period $t+1$, so as to maximize his own utility given children's behaviors of gifts $\Phi_{t+1}(y_{t+1})$. We may rewrite (6) as follows.

$$\max_{e_{t+1}\in[0,y_t]} V(y_t - e_{t+1}) + v^2(\Phi_{t+1}(f(e_{t+1}))). \tag{10}$$

Given a function $\Phi : R_+ \to R_+$, let

$$E_\Phi(y) = \arg\max_{0\le e\le y} \left\{V(y-e) + v^2(\Phi(f(e)))\right\}, \tag{11}$$

where $E_\Phi(y)$ is a set of optimal choices of education investment, e, and

$$\xi_\Phi(y) = \min\{e : e \in E_\Phi(y)\}, \tag{12}$$

$$(G\Phi)(\eta) = g(\eta - \xi_\Phi(\eta)), \tag{13}$$

where $g(\cdot)$ is defined in (8), and

$$(H\Phi)(y) = \max_{0\le\eta\le y} \{(G\Phi)(\eta)\}. \tag{14}$$

For the following lemma, call $\varphi(x)$ be upper semi-continuous if

$$\limsup_{n\to\infty} \varphi(x_n) \le \varphi(x) \quad \text{as} \quad x_n \to x.$$

By the definition of equilibrium, a stationary equilibrium is Φ such that $\Phi = G\Phi$. Then, we have the following lemma.

Lemma 2 *Take Φ satisfying that $(G\Phi)(y)$ is upper semi-continuous with respect to y, and that $\Phi = H\Phi$. Let $\Phi^*(y) \equiv (G\Phi)(y)$. Then,*

$$\Phi^* = G\Phi^*. \tag{15}$$

Proof. See Appendix A. ■

In order to show that there exists a stationary equilibrium, by Lemma 2, it suffices to show that there exists Φ satisfying that $(G\Phi)(y)$ is upper semi-continuous with respect to y, and that $\Phi = H\Phi$. Let $\tilde{y} = \max(\hat{y}, \bar{y})$[5], where $\bar{y}$ is explained in the assumption (F.2), and $Y = [0, \tilde{y}]$. Let C be a set of Φ endowed with the topology of pointwise convergence and satisfying that Φ is non-decreasing,

[5]As for $\bar{y}$, refer to the assumption (F.2).

$$0 \leq \Phi(y) \leq \tilde{y}, \; \forall y \in Y, \tag{16}$$

and

$$\exists \ell > 0 : |\Phi(y') - \Phi(y'')| \leq \ell \, |y' - y''| \, , \; \forall y', y'' \in Y. \tag{17}$$

Note that C is a nonvoid convex subset of a separated locally convex topological vector space. First, we will demonstrate the following lemma.

Lemma 3 *Let $\Phi \in C$. Then, $(G\Phi)(y)$ is upper semi-continuous with respect to y.*

Proof. See Appendix B. ■

Next, we will demonstrate that there exists $\Phi \in C$ such that $\Phi = H\Phi$. Such a Φ is called to a fixed point of a mapping H. In order to show that there exists a fixed point of a mapping H, we use the fixed point theorem of Schauder-Tychonoff, which is explained by Edwards (1965).

Theorem 2 (Shauder-Tychonoff theorem) *Let E be a separated locally convex topological vector space, K be a nonvoid compact convex subset of E, u any continuous map of K into itself. Then u admits at least one fixed point.*

By the definition of C, C is a nonvoid convex subset of a separated locally convex topological vector space. In order to use the fixed point theorem in this model, we require (i) that C is compact, (ii) that H maps C into itself, and (iii) that $H : C \to C$ is continuous.

We will demonstrate that C is compact. To this end, we introduce the mathematical concept of *equicontinuous*, which is explained by Edwards (1965), and use the following theorem.

Theorem 3 (Ascoli's theorem) *Let T be a topological space, X be a uniform space, and X^T be the set of all X-valued functions on T. If $F \subset X^T$ is equicontinuous on T and $F(t) = \{f(t) : f \in F\}$ is relatively compact in X for each $t \in T$, then F is relatively compact in X^T for the topology of compact convergence.*

By this theorem, though C is endowed with the topology of pointwise convergence, the following lemma holds.

Lemma 4 *C is compact for the topology of compact convergence.*

Proof. See Appendix C. ∎

We will demonstrate the rest of the requirements by the following two lemmas.

Lemma 5 *Let $\Phi \in C$. Then $H\Phi \in C$.*

Proof. See Appendix D. ∎

Lemma 6 *$H : C \rightarrow C$ is continuous with respect to the sup norm.*

Proof. See Appendix E. ∎

Through the above lemmas, we may obtain (i) that C is compact, (ii) that H maps C into itself, and (iii) that $H : C \rightarrow C$ is continuous with respect to the sup norm. By Shauder-Tychonoff theorem, there exists $\Phi \in C$ such that $\Phi = H\Phi$. Thus, by Lemma 2, there exists Φ^* such that $\Phi^* = G\Phi^*$. Therefore, it is establish that there exists a stationary equilibrium.

4 Concluding Remarks

This study has formalized a model with backward altruism and demonstrated that backward altruism, despite one-sided altruism, yields two-sided intergenerational transfers. It has also demonstrated the existence of an equilibrium in this model.

We remain two problems. One is an absence of a financial market, which plays an important role on redistribution between different generations and thus on a dynamic allocation. The other is the stability of a stationary equilibrium. These problems are focused on by other papers.

Appendices

A. Proof of Lemma 2 By (13), $\Phi^*(y) = (G\Phi)(y) = g(y - \xi_\Phi(y))$ and $(G\Phi^*)(y) = (G(G\Phi))(y) = g(y - \xi_{G\Phi}(y))$. Thus, in order to show (15), it suffices to show that for any $y \in Y$,

$$\xi_\Phi(y) = \xi_{G\Phi}(y). \tag{18}$$

To this end, we first demonstrate the following two sublemmas.

Sublemma 1 Let $\varphi(x)$ be upper semi-continuous (u.s.c.) with respect to x. Then, $\max_{0 \le x \le y} \varphi(x) \equiv \pi(y)$ exists, and it is u.s.c. with respect to y.

Proof. Omitted. ■

Sublemma 2 Assume that $(G\Phi)(y)$ is upper semi-continuous (u.s.c.) with respect to y. Let $y \in Y$

$$E_{G\Phi}(y) = \arg\max_{0 \le e \le y} \left\{ V(y-e) + v^2((G\Phi)(f(e))) \right\},$$

and

$$E_{H\Phi}(y) = \arg\max_{0 \le e \le y} \left\{ V(y-e) + v^2((H\Phi)(f(e))) \right\}.$$

Then, $E_{H\Phi}(y)$ and $E_{G\Phi}(y)$ are non-empty, and $E_{H\Phi}(y) \subset E_{G\Phi}(y)$.

Proof. Omitted. ■

In order to show (18), since $\Phi = H\Phi$, it suffices to show that $\xi_{H\Phi}(y) = \xi_{G\Phi}(y)$. By Sublemma 2, $\xi_{H\Phi}(y) \in E_{G\Phi}(y)$. Thus, it must hold that $\xi_{H\Phi}(y) \ge \xi_{G\Phi}(y)$. Suppose $\xi_{H\Phi}(y) > \xi_{G\Phi}(y)$. Let $e^h = \xi_{H\Phi}(y)$ and $e^g = \xi_{G\Phi}(y)$. Then, since $e^g \notin E_{H\Phi}(y)$,

$$V(y - e^g) + v^2((H\Phi)(f(e^g))) < V(y - e^h) + v^2((H\Phi)(f(e^h))). \tag{19}$$

By (14), $(H\Phi)(f(e^g)) \ge (G\Phi)(f(e^g))$. By Sublemma 2, $(H\Phi)(f(e^h)) = (G\Phi)(f(e^h))$. Moreover, since e^h, $e^g \in E_{G\Phi}(y)$,

$$V(y - e^g) + v^2((G\Phi)(f(e^g))) = V(y - e^h) + v^2((G\Phi)(f(e^h))).$$

Then, since v^2 is strictly increasing, it follows that

$$V(y - e^g) + v^2((H\Phi)(f(e^g))) \ge V(y - e^h) + v^2((H\Phi)(f(e^h))),$$

which, however, contradicts (19). Therefore, $\xi_{H\Phi}(y) = \xi_{G\Phi}(y)$.

B. Proof of Lemma 3 For this proof, since $(G\Phi)(y) = g(y - \xi_\Phi(y))$ by (13), we will show $g(y-\xi_\Phi(y))$ is upper semi-continuous (u.s.c.) with respect to y. To this end, since $g(\cdot)$ is continuous, it suffices to show that $\xi_\Phi(y)$ is lower semi-continuous (l.s.c.) with respect to y; That is,

$$\limsup_{i\to\infty} \xi_\Phi(y_i) \geq \xi_\Phi(y) \qquad \text{as} \qquad y_i \to y. \tag{20}$$

We first show the following sublemma.

Sublemma 3 (i) $\xi_\Phi(y)$ exists, and (ii) $\Phi \in C$, $y' > y''$, $e' \in E_\Phi(y')$, and $e'' \in E_\Phi(y'')$ imply $e' \geq e''$.

Proof. Omitted. ■

Take a sequence $\{y_i\}$ such that $y_i \to y$. Let $\{y_{ij}\}$ be a deceasing sequence of $\{y_i\}$. Then, from Sublemma 3, it follows that $\limsup_{j\to\infty} \xi_\Phi(y_{ij}) \geq \xi_\Phi(y)$.

Take an increasing sequence $\{y_{ij}\}$ of $\{y_i\}$. Since $\xi_\Phi(y)$ is non-decreasing, $y_i \leq y$ implies $\xi_\Phi(y_i) \leq \xi_\Phi(y)$ for any i. Since $\xi_\Phi(y)$ exists for any $y \in Y$, we may take $\lim_{i\to\infty} \xi_\Phi(y_i)$. Then, it must hold that $\lim_{i\to\infty} \xi_\Phi(y_i) \leq \xi_\Phi(y)$. Suppose that $\lim_{i\to\infty} \xi_\Phi(y_i) < \xi_\Phi(y)$. Let $e^* = \lim_{i\to\infty} \xi_\Phi(y_i)$. Since $\xi_\Phi(y)$ is optimal, and since e^* is not optimal,

$$\left[V(y-\xi_\Phi(y)) + v^2(\Phi(f(\xi_\Phi(y))))\right] - \left[V(y-e^*) + v^2(\Phi(f(e^*)))\right] \equiv \epsilon > 0. \tag{21}$$

Since $y_i \to y$ and $\xi_\Phi(y_i) \to e^*$, for any $\epsilon > 0$, there exists i_1 such that $\forall i \geq i_1$,

$$V(y-e^*) + v^2(\Phi(f(e^*))) + \frac{\epsilon}{2} > V(y_i - \xi_\Phi(y_i)) + v^2(\Phi(f(\xi_\Phi(y_i)))). \tag{22}$$

Take a sequence $\{e_i\}$ such that $e_i \equiv \max\{y_i - y + \xi_\Phi(y), 0\}$. Then, for any $\epsilon > 0$, there exists i_2 such that $\forall i \geq i_2$, $e_i \geq 0$ and

$$V(y_i - e_i) + v^2(\Phi(f(e_i))) > V(y-\xi_\Phi(y)) + v^2(\Phi(f(\xi_\Phi(y)))) - \frac{\epsilon}{2}. \tag{23}$$

From (21), (22), and (23), it follows that

$$V(y_i - e_i) + v^2(\Phi(f(e_i))) > V(y_i - \xi_\Phi(y_i)) + v^2(\Phi(f(\xi_\Phi(y_i)))),$$

which, however, contradicts the definition of $\xi_\Phi(y_i)$. Therefore, $\lim_{i\to\infty} \xi_\Phi(y_i) = \xi_\Phi(y)$.

From the above argument, we may obtain that (20) holds for any $\{y_i\}$ such that $y_i \to y$. ■

C. Proof of Lemma 4 Since $Y = [0, \tilde{y}]$ is a metric space, it is a topological space. R is a uniform space. Let R^Y be the set of all real valued functions on Y. By the definition of C, C is a nonvoid convex subset of R^Y endowed with the topology of pointwise convergence and satisfying the following two conditions. The first is that

$$\exists \ell > 0 : \forall y', y'' \in Y, \forall \Phi \in C, |\Phi(y') - \Phi(y'')| < \ell \, |y' - y''| ,$$

from which it follows that, for any $\epsilon > 0$, there exists $\delta = \epsilon/\ell > 0$ satisfying that, for any pair (y', y'') such that $|y' - y''| < \delta$, and for any $\Phi \in C$, $|\Phi(y') - \Phi(y'')| < \epsilon$. Thus, the first condition means that C is equicontinuous on Y. The second condition is that

$$\exists M > 0 : \forall y \in Y, \forall \Phi \in C, 0 \leq \Phi(y) \leq M,$$

from which it follows that $C(y) = \{\Phi(y) : \Phi \in C\}$ is (relatively) sequential compact, with respect to the same topology as R, for each $y \in Y$ since C is endowed with the topology of pointwise convergence. Then, by Ascoli's theorem, C is relatively compact in R^Y for the topology of compact convergence.

D. Proof of Lemma 5 Let $\Phi \in C$, which means that $\Phi(y)$ is non-decreasing with respect to y and satisfies (16) and (17). Since $(G\Phi)(y)$ is upper semi-continuous, by Sublemma 1, $(H\Phi)(y)$ exists. By (14), we may obtain that $(H\Phi)(y)$ is non-deceasing with respect to y. Thus, in order to show $H\Phi \in C$, it suffices to show that $H\Phi$ satisfies

$$0 \leq (H\Phi)(y) \leq \tilde{y}, \forall y \in Y, \tag{24}$$

and

$$\exists \ell > 0 : |(H\Phi)(y') - (H\Phi)(y'')| \leq \ell \, |y' - y''| , \quad \forall y', y'' \in Y. \tag{25}$$

First, we will show (24). Let $y \in Y$. Since $(H\Phi)(y) = \max_{0 \leq x \leq y} (G\Phi)(x)$, and since $(G\Phi)(x) = g(x - \xi_\Phi(x)) \leq x$, it follows that $(H\Phi)(y) = \max\limits_{0 \leq x \leq y} g(x - \xi_\Phi(x)) \leq y$. Since $\tilde{y} \equiv \max\{y : y \in Y\}$, $(H\Phi)(y) \leq \tilde{y}$. By the definition of $g(\cdot)$, $(H\Phi)(y) \geq 0$. Therefore, (24) holds.

Next, we will show (25). Assume $y' \geq y''$ without loss of generality. If $y' = y''$, then (25) holds. Let $y' > y''$. If $(H\Phi)(y') = (H\Phi)(y'')$, then (25) holds. Note that $(H\Phi)(y)$ is non-decreasing with respect to y. Thus, we will demonstrate that $y' > y''$ and $(H\Phi)(y') > (H\Phi)(y'')$ imply (25). To this end, we first demonstrate the following sublemma.

Sublemma 4 $\xi_\Phi(y)$ is non-decreasing with respect to y.

Proof. Omitted. ■

Let $y' > y''$ and $(H\Phi)(y') > (H\Phi)(y'')$. Take y''' be such that $(H\Phi)(y') = (G\Phi)(y''')$ and $y' \geq y''' > y''$. Then, since $(H\Phi)(y'') \geq (G\Phi)(y'')$, it follows that $(G\Phi)(y''') > (G\Phi)(y'')$, which means $g(y''' - \xi_\Phi(y''')) > g(y'' - \xi_\Phi(y''))$ by (13). Since g is non-decreasing, $y''' - \xi_\Phi(y''') > y'' - \xi_\Phi(y'')$. Since c is non-decreasing, $c(y''' - \xi_\Phi(y''')) - c(y'' - \xi_\Phi(y'')) \geq 0$. Since $\xi_\Phi(y)$ is non-decreasing with respect to y by Sublemma 4, $\xi_\Phi(y''') - \xi_\Phi(y') \geq 0$. Thus,

$$\begin{aligned} g(y''' - \xi_\Phi(y''')) - g(y'' - \xi_\Phi(y'')) \leq\ & [c(y''' - \xi_\Phi(y''')) - c(y'' - \xi_\Phi(y''))] \\ & + [g(y''' - \xi_\Phi(y''')) - g(y'' - \xi_\Phi(y''))] \\ & + [\xi_\Phi(y''') - \xi_\Phi(y'')] . \end{aligned}$$

Then, by the budget constraint of (1), we may obtain that $g(y''' - \xi_\Phi(y''')) - g(y'' - \xi_\Phi(y'')) \leq y''' - y''$; That is,

$$(G\Phi)(y''') - (G\Phi)(y'') \leq y''' - y''.$$

Therefore, since $(H\Phi)(y') = (G\Phi)(y''')$, and since $y' \geq y''' > y''$,

$$0 < (H\Phi)(y') - (H\Phi)(y'') \leq y' - y''.$$

From the above argument, (25) is established.

E. Proof of Lemma 6 Take a sequence $\{\Phi^n\}$ satisfying that $\Phi^n \in C$, and that there exists $\Phi \in C$ such that $\lim_{n\to\infty} \|\Phi^n - \Phi\| = 0$ where $\|\cdot\|$ is the sup norm. Note $\Phi \in C$ implies $H\Phi \in C$ by Lemma 5. Then, in order to show that $H : C \to C$ is $\|\cdot\|$-continuous, it suffices to show

$$\lim_{n\to\infty} \|H\Phi^n - H\Phi\| = 0. \tag{26}$$

Let $y^n \in \arg\sup_{y\in Y} |(H\Phi^n)(y) - (H\Phi)(y)|$. Since Y is compact, a sequence $\{y^n\}$ has a convergent subsequence. Let y be the limit of this subsequence; that is, $\lim_{n\to\infty} y^n = y$. Then, it follows that

$$\begin{aligned} & |(H\Phi^n)(y^n) - (H\Phi)(y^n)| \\ \leq\ & |(H\Phi^n)(y^n) - (H\Phi^n)(y)| + |(H\Phi^n)(y) - (H\Phi)(y)| + |(H\Phi)(y) - (H\Phi)(\end{aligned}$$

Since $H\Phi \in C$, it follows from (17) that $|(H\Phi^n)(y^n) - (H\Phi^n)(y)| \leq |y^n - y|$, and that $|(H\Phi)(y) - (H\Phi)(y^n)| \leq |y - y^n|$. Thus, in order to show that $|(H\Phi^n)(y^n) - (H\Phi)(y)| \to 0$ as $y^n \to y$, it suffices to show that

$$|(H\Phi^n)(y) - (H\Phi)(y)| \to 0 \text{ as } y^n \to y. \tag{27}$$

To this end, we will demonstrates the following sublemma.

Sublemma 5 Let $\Phi^n, \Phi \in C$, $\eta^n, \eta \in Y$ for any n. Assume $\Phi^n \to \Phi$ and $\eta^n \to \eta$. Then, $\lim_{n\to\infty} \xi_{\Phi^n}(\eta^n) \in E_\Phi(\eta)$. Moreover, if η be a continuity point of $\xi_\Phi(\eta)$, then $\lim_{n\to\infty} \xi_{\Phi^n}(\eta^n) = \xi_\Phi(\eta)$.

Proof. Omitted. ■

We will demonstrate (27), which means (a) that $\lim_{n\to\infty}(H\Phi^n)(y) \leq (H\Phi)(y)$, and (b) that $\lim_{n\to\infty}(H\Phi^n)(y) \geq (H\Phi)(y)$. First, we will demonstrate (a). Take $\{\eta^n\}_{n=1}^{\infty}$ be such that $0 \leq \eta^n \leq y$ and $\lim_{n\to\infty} \eta^n = \eta$. Since $\lim_{n\to\infty} \xi_{\Phi^n}(\eta^n) \in E_\Phi(\eta)$ by Sublemma 5, and since $\xi_\Phi(\eta)$ is minimum in $E_\Phi(\eta)$ by the definition of $\xi_\Phi(\eta)$ in (12), it follows that $\lim_{n\to\infty} \xi_{\Phi^n}(\eta^n) \geq \xi_\Phi(\eta)$. Take $\{\eta^n\}_{n=1}^{\infty}$ be such that $0 \leq \eta^n \leq y$ and $(H\Phi^n)(y) = (G\Phi^n)(\eta^n)$. Note $(G\Phi^n)(\eta^n) = g(\eta^n - \xi_{\Phi^n}(\eta^n))$ by (13). Then,

$$\lim_{n\to\infty} g(\eta^n - \xi_{\Phi^n}(\eta^n)) \leq g(\eta - \xi_\Phi(\eta)).$$

Since $(G\Phi)(\eta) = g(\eta - \xi_\Phi(\eta))$, we may obtain (a) $\lim_{n\to\infty}(H\Phi^n)(y) \leq (H\Phi)(y)$.

Next, we will prove (b). Let η^* be such that $0 < \eta^* \leq y$ and

$$(H\Phi)(y) = (G\Phi)(\eta^*). \tag{28}$$

Since $\xi_\Phi(y)$ is lower semi-continuous as shown in Lemma 3, and since $\xi_\Phi(y)$ is non-decreasing by Sublemma 4, it follows that $\xi_\Phi(y)$ is continuous from the left with respect to y. Then, since g is continuous, $(G\Phi)(y) = g(y - \xi_\Phi(y))$ is also continuous from the left with respect to y. Thus, for any $\varepsilon > 0$, we may take a continuity point η of $\xi_\Phi(\cdot)$ such that $0 \leq \eta < \eta^*$ and

$$(G\Phi)(\eta) > (G\Phi)(\eta^*) - \epsilon. \tag{29}$$

Let $\{\eta^n\}_{n=1}^{\infty}$ be a sequence such that $0 \leq \eta^n \leq y$ and $\eta^n \to \eta$. Then, since η is a continuity point of $\xi_\Phi(\cdot)$, $\lim_{n\to\infty} \xi_{\Phi^n}(\eta^n) = \xi_\Phi(\eta)$ by Sublemma 5. Then, since g is continuous, $\lim\limits_{n\to\infty} g(\eta^n - \xi_{\Phi^n}(\eta^n)) = g(\eta - \xi_\Phi(\eta))$; That is,

$$\lim_{n\to\infty} (G\Phi^n)(\eta^n) = (G\Phi)(\eta). \tag{30}$$

Since $0 \leq \eta^n \leq y$, since $(H\Phi)(y)$ is non-decreasing in y, and since it must hold that $(H\Phi)(y) \geq (G\Phi)(y)$,

$$(H\Phi^n)(y) \geq (H\Phi^n)(\eta^n) \geq (G\Phi^n)(\eta^n) \tag{31}$$

Then, from (30) and (31), it follows that

$$\lim_{n\to\infty} (H\Phi^n)(y) \geq (G\Phi)(\eta). \tag{32}$$

In (29), ε is arbitrary. Then, by (28), (29), and (32), we may obtain (b) $\lim_{n\to\infty}(H\Phi^n)(y) \geq (H\Phi)(y)$. Thus, (27) is established. Therefore, we may obtain (26); That is, H is $\|\cdot\|$-continuous.

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