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Existence of canards at a pseudo-singular node point

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1 Introduction

The original problem concerns qualitative theory of ordinary differential equation in dimension 3: what is the limit of the phase portrait of

$$\begin{cases} x' &= f(x,y,z,\varepsilon) \\ y' &= g(x,y,z,\varepsilon) \\ \varepsilon z' &= h(x,y,z,\varepsilon) \end{cases}$$

when ε tends to zero. It is natural (and true) that the phase portrait *converges* to a concatenation of

- the phase portrait of the *fast* vector field

$$\begin{cases} \dot{x} &= 0 \\ \dot{y} &= 0 \\ \dot{z} &= h(x,y,z,0) \end{cases}$$

  which is drawn on a vertical line, for each value of (x,y).

- the phase portrait of the *slow* vector field

$$\begin{cases} x' &= f(x,y,z,0) \\ y' &= g(x,y,z,0) \\ 0 &= h(x,y,z,0) \end{cases}$$

  which is drawn on the *slow surface* defined by $h(x,y,z,0)=0$.

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But conversely, if such a concatenation is given, it is not obvious to determine if this concatenation is the limit of a trajectory.

In the classical topological studies ([Tyk48, VB73, LST98]), some difficulties appear when the slow surface $h = 0$ is tangent to the vertical direction. The set of these points is called the *fold* ; it is defined by $h = h_z = 0$.

In some situations, even on the fold, more sophisticated topological studies can give the complete description (see [Ben83, Wec98]). One generic situation (the pseudo-singular node point) was more difficult to understand. I will propose, in this talk, a different approach, using complex analysis of Gevrey functions, to prove the existence of *canards*. We will use the *dilatation method* of B. Malgrange [Mal89], applied on Banach spaces of formal Gevrey series.

A *canard* is a trajectory of a slow-fast vector field which first follows the attractive part of the slow curve, and then the repulsive one.

The complete proofs of the theorems of this paper are given in [Ben00].

2 Hypotheses and main theorem

The system is given by

$$\left\{ \begin{array}{rcl} x' & = & f(x,y,z,\varepsilon) \\ y' & = & g(x,y,z,\varepsilon) \\ \varepsilon z' & = & h(x,y,z,\varepsilon) \end{array} \right. \tag{1}$$

when f, g, h are analytic functions for x, y, z, ε in some neighborhood of the origin. We suppose that, at the origin, we have:

- $h = 0$: the origin is on the slow surface.
- $h_z = 0$: the origin is on the fold.
- $(h_x, h_y, h_z) \neq (0,0,0)$: the origin is a regular point of the slow surface.
- $h_{zz} \neq 0$: the origin is not a cusp.
- $(f, g, h) \neq (0,0,0)$: the origin is not a stationnary point of the whole system.
- $h_x f + h_y g = 0$: the origin is a pseudo-singular point : (the normalized projection of) the slow vector field has a singularity. The linear part of this singularity has two eigenvalues λ and μ.
- $\mu < \lambda < 0$: the pseudo-singular point is of *node* type.
- $k = \mu/\lambda \notin \mathbb{N}$: there is no resonance.

The simplest system which satisfies all these hypotheses is

$$\left\{ \begin{array}{rcl} x' & = & 2ky + 2(k+1)z \\ y' & = & 1 \\ \varepsilon z' & = & -z^2 - x \end{array} \right.$$

On the figure, one can see the slow vector field and some concatenations of trajectories of the fast and the slow vector fields.

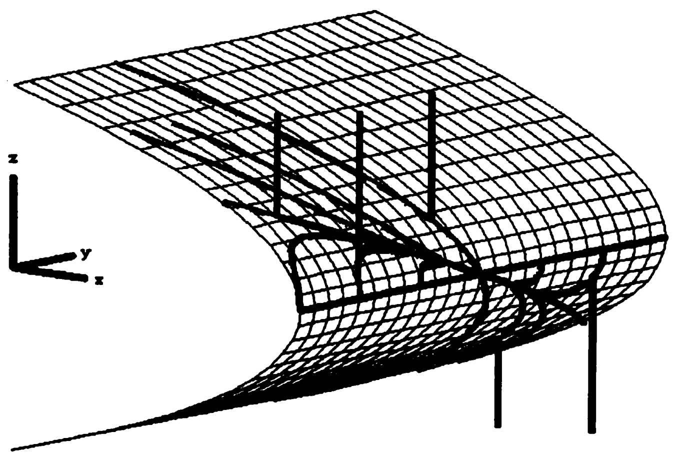

Figure 1: Some possible canards

The main result of this paper is

Theorem 1 *With all the hypotheses above, there exist a positive time T (independent of ε) and two solutions $(x_\varepsilon^i(t), y_\varepsilon^i(t), z_\varepsilon^i(t))$ of (1) , for $i \in \{\lambda, \mu\}$, such that*

- $(x_\varepsilon^i(t), y_\varepsilon^i(t), z_\varepsilon^i(t))$ *is defined at least for t in $[-T, T]$.*
- $\lim_{\varepsilon\to 0^+}(x_\varepsilon^i(0), y_\varepsilon^i(0), z_\varepsilon^i(0)) = (0,0,0)$.
- $(x_\varepsilon^i(t), y_\varepsilon^i(t), z_\varepsilon^i(t))$ *converges uniformly on $[-T, T]$ to a solution of the slow system.*
- $(x_\varepsilon^{i\prime}(0), y_\varepsilon^{i\prime}(0), z_\varepsilon^{i\prime}(0))$ *converges to a vector of the eigenspace associated to the eigenvalue i.*

Such solutions are canards : it is obvious on the picture, and easy to prove that the limit of the trajectories are drawn first on the attractive slow surface, and then on the repulsive one.

To prove this main theorem, we will prove another more technical result :

Theorem 2 *With the same hypotheses, there exists a formal solution of (1) :*

$$\left\{\begin{array}{rcl} \hat{x}(t) & = & \sum_{n\geq 0} x_n(t)\varepsilon^n \\ \hat{y}(t) & = & \sum_{n\geq 0} y_n(t)\varepsilon^n \\ \hat{z}(t) & = & \sum_{n\geq 0} z_n(t)\varepsilon^n \end{array}\right.$$

where the functions x_n, y_n, z_n are analytic on a disk of radius r (independent of ε) and where the series are Gevrey (the definition will be given later). Moreover, we have $(x_0(0), y_0(0), z_0(0)) = (0,0,0)$, and $(x_0'(0), y_0'(0), z_0'(0))$ is tangent to the eigenspace associated to the eigenvalue λ.

3 Preparation of the equation

In this paragraph, we will transforme system (1) into a second order non autonomous equation. The aim is to work with only one (and not three) unknown function.

First, we use polynomial change of unknowns, of degree at least 2, with polynomial inverses, to obtain

$$\left\{\begin{array}{rcl} x' & = & 2ky + 2(k+1)z + F(x,y,z,\varepsilon) \\ y' & = & 1 + G(x,y,z,\varepsilon) \\ \varepsilon z' & = & -z^2 - x + H(x,y,z,\varepsilon) \end{array}\right. \tag{2}$$

where k is the ratio μ/λ, and F, resp. G, H, have valuations at least 2, resp. 1, 3, as weighted homogeneous polynomials with the weights $(2,1,1,2)$ for (x,y,z,ε). For that, we need all the hypotheses on the pseudo-singular point, but the non resonance.

In a neighborhood of the origin, we have $1+G>0$. Thus, we can divide the vector field by $1+G$ to obtain a new system which is written as (2) but with $G=0$. For convenience, we will write now t instead of y.

Using the implicit function theorem in the third equation of (2), we can express x as a function $\xi(t,z,\varepsilon,\varepsilon z')$. Then, identifying the derivative of ξ with respect to t with the right hand side of the first equation, we will find the new equivalent form of the equation (1) :

$$\frac{\varepsilon}{2}z'' + zz' + kt + (k+1)z \;=\; \Phi_0(t,\varepsilon,z,\varepsilon z') + z'\Phi_1(t,\varepsilon,z,\varepsilon z') \tag{3}$$

where Φ_0 and Φ_1 have weighted valuation at least 2 (the weights of $(t,\varepsilon,z,\varepsilon z')$ are $(1,2,1,2+1-1=2)$.)

4 Dilatation method

Let us define two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ of formal series of ε, with analytic coefficients of t (the definition of the norms will be given later). Let us denote by A the operator defined from $\mathbb{C}\times\mathcal{B}_2$ into $\mathcal{B}_1$ by :

$$A(\delta,z) \;=\; \frac{\delta\varepsilon}{2}z'' + zz' + kt + (k+1)z - \Phi_0(t,\delta\varepsilon,z,\delta\varepsilon z') - z'\Phi_1(t,\delta\varepsilon,z,\delta\varepsilon z')$$

It is obvious that, if we know a complex number $\delta\neq 0$ and a formal series $z(t,\varepsilon)$ such that $A(\delta,z)=0$, then we have $A(1,z(t,\varepsilon/\delta))=0$ and the formal series $z(t,\varepsilon/\delta)$ is a solution of (3). We will find such δ and z using implicit function theorem. For that purpose, we will

- solve $A(z,0)=0$, and denote by z_0 a solution,
- prove that A is of class C^1,
- compute the partial derivative L of A with respect to z at the point $(0,z_0)$,
- prove that L is invertible.

5 Approximation of order 0

Equation $A(0,z)=0$ can be written

$$zz' + kt + (k+1)z - \varphi_0(t,z) - z'\varphi_1(t,z) \;=\; 0$$

where φ_0 and φ_1 have valuation at least 2.

With a change of time, this equation can be studied as a vector field

$$\begin{cases} \frac{dt}{d\tau} &= -z + \varphi_1(t,z) \\ \frac{dz}{d\tau} &= kt + (k+1)z - \varphi_0(t,z) \end{cases}$$

and the question is to find an analytic trajectory around the origin. If the ratio k of the eigenvalues is not an integer, then there is no resonance and the Poincaré theorem (see [Arn80]) yields the existence of exactly two analytic solutions, one for each eigenspace. Let us denote by z_0 the solution with $z'(0) = -1$.

6 Banach spaces of Gevrey series

Here is the more technical part of this paper, and the reader will find the detailed proofs of the lemma in [CDRSS99, Ben00].

The eigenspaces $\mathcal{B}_1$ and $\mathcal{B}_2$ are spaces of formal series

$$f = \sum_n f_n \varepsilon^n$$

where the f_n are analytic functions on the same disk as z_0. In order to define the modified Nagumo norms (see also [CDRSS99]), we put

$$d(t) \;=\; \begin{cases} r - |t| & \text{if} \quad \rho \le |t| \le r \\ r - \rho & \text{if} \quad 0 \le |t| \le \rho \end{cases}$$

$$\|g\|_m \;=\; \sup_{0\le|t|<r} |g(t)|\, d(t)^m$$

$$|\!|\!|f|\!|\!|_1 \;= 3 \sup_{n\in\mathbb{N}} \frac{\|f_n\|_n}{\Gamma(n+1)} \qquad\qquad |\!|\!|f|\!|\!|_2 \;=\; 2\max\left(|\!|\!|f|\!|\!|_1, |\!|\!|f_t|\!|\!|_1\right)$$

The space $\mathcal{B}_1$ is the space of formal series with finite norm $|\!|\!|.|\!|\!|_1$, the same for $\mathcal{B}_2$. They are Banach spaces.

Lemma 3 *1. We have*

$$|\!|\!|fg|\!|\!|_1 \le |\!|\!|f|\!|\!|_1 |\!|\!|g|\!|\!|_1 \qquad |\!|\!|fg|\!|\!|_2 \le |\!|\!|f|\!|\!|_2\, |\!|\!|g|\!|\!|_2$$

2. *If Φ is an analytic function on a polydisc of radius $(r_1, r_2, \ldots, r_n)$, and if each f_i is an element of $\mathcal{B}_1$, with $|||f_i|||_1 < r_i$, then $\Phi(f_1, \ldots, f_n)$ is an element of $\mathcal{B}_1$ with*

$$|||\Phi(f_1, \ldots, f_n)|||_1 \leq \Phi^+(|||f_1|||_1, \ldots, |||f_n|||_1)$$

(Φ^+ is the analytic function obtained from the Taylor expansion of Φ by taking the modulus of each coefficient).

3. *If φ is an analytic function on the disk of radius r, then the function $f \mapsto \varphi \circ f$ is defined on the disk of radius r in $\mathcal{B}_1$ into $\mathcal{B}_1$. It is of class C^1 and the derivative at the point f is the multiplication by $\varphi'(f)$.*

4. *(Malgrange's lemma) We have*

$$||f'|| \leq me||f||_{m-1}$$

5. *We have*

$$|||\varepsilon f'|||_1 \leq e|||f|||_1$$

6. *If S and Z are the following operators on $\mathcal{B}_1$:*

$$(Zf)(t, \varepsilon) = f(0, \varepsilon) \qquad (Sf)(t, \varepsilon) = \frac{f(t, \varepsilon) - f(0, \varepsilon)}{t}$$

then the identity $tS = 1 - Z$ gives a kind of Taylor's formula :

$$f = \sum_{i=0}^{p} t^i (ZS^i)(f) + t^{p+1} S^{p+1}(f)$$

and we have

$$|||Zf|||_1 \leq |||f|||_1 \quad \textit{and} \quad |||Sf|||_1 \leq \frac{2}{\rho}|||f|||_1$$

With this lemma, it is easy to prove that the operator A is C^1, and it is easy to compute the partial derivative L with respect to z at the point $(0, z_0)$:

$$L(u) \;=\; (z_0 - \Phi_1(t,0,z_0,0))\, u' \;+\; \left(z_0' + k + 1 - \frac{\partial \Phi_0}{\partial z}(t,0,z_0,0) - z_0' \frac{\partial \Phi_1}{\partial z}(t,0,z_0,0)\right) u \qquad (4)$$

7 Inversibility of L

We will write equation (4) in a more shorter form :

$$L(u) \;=\; -t\varphi u' \;+\; k\psi u$$

with $\varphi(0, \varepsilon) = 1$, $\psi(0, \varepsilon) = 1$.

To compute the inverse of L, we have to solve equation $L(u) = f$, with a given f in $\mathcal{B}_1$. The explicit solution of this equation is given by the formula

$$u = t^k e^{G(t)} \int^t \frac{-\tau^{-k-1} e^{-G(\tau)}}{\varphi(\tau)} f(\tau) d\tau$$

$$\text{where} \quad G(t) = \int_0^t g(\tau) d\tau \quad \text{and} \quad g = \frac{kS(\psi - \varphi)}{\varphi}$$

For the integral in the definition of u, we need a constant. It will be determined by the analiticity of u in $t = 0$. The inverse of L is now defined by the composition of three operators : the multiplication by e^{-G}/φ, the operator M below, and the multiplication by e^G.

$$M(f) = t^k \int^t -\tau^{-k-1} f(\tau) d\tau$$

The two multiplications are bounded linear operators in the appropriate Banach spaces.

In order to prove the continuity of the operator M, and to determine the constant of integration, we use Taylor's formula to write

$$f = \sum_{i=0}^{\bar{k}} t^i Z S^i f + t^{\bar{k}+1} S^{\bar{k}+1} f$$

where $\bar{k}$ is the integer part of $k+1$. Moreover, it is easy to compute $M(t^i) = \frac{1}{k-i} t^i$. The computation of $M(t^{\bar{k}+1} h)$ is given by

$$M(t^{\bar{k}+1} h) = t^k \int_0^t \tau^{\bar{k}-k} h(\tau) d\tau$$

The function under the integral sign is bounded, and one can compute the norm of M :

$$\|M(f)\|_1 \leq \left(\sum_{i=0..\bar{k}} \frac{1}{|k-i|} \left(\frac{2r}{\rho}\right)^i + \left(\frac{2r}{\rho}\right)^{\bar{k}+1} \right) \|f\|_1$$

Moreover, we have $M(f)' = S(kM(f) - f)$, and we can deduce that M is a bounded operator from $\mathcal{B}_1$ to $\mathcal{B}_2$.

8 Proof of theorem 2

In sections 5 to 7 we proved that the hypotheses for the application of the dilatation method are satisfied, then we built a formal Gevrey series $\hat{z} = \sum z_n(t)\varepsilon^n$ which is solution of equation (3). Using backward the transformations of paragraph 3 (including change of time) , we can find the formal series $\hat{x}$, $\hat{y}$ and $\hat{z}$ which are solutions of equation (1).

9 Proof of theorem 1

To prove the existence of function from the existence of formal series, the tool is the Borel-Laplace summation and the theory of Gevrey functions (see [CDRSS99]). I will give here only the main ideas :

- Let us denote by β and $\tilde{r}$ some positive real numbers such that $\tilde{r} < r$ and $\beta < \delta d(r)$.
- Let us define the Borel transform by

$$\sum_{n\geq 1} \frac{z_n(t)}{\Gamma(n)} \eta^{n-1}$$

  It is an analytic function in the domain $|t| \leq \tilde{r}$, $|\eta| \leq \beta$.

- Let us define the truncated Laplace integral by

$$\tilde{z}(t,\varepsilon) \;=\; z_0(t) \;+\; \int_0^\beta \sum_{n\geq 1} \frac{z_n(t)}{\Gamma(n)} \eta^{n-1} e^{-\eta/\varepsilon} d\eta$$

  It is an analytic function for $|t| \leq \tilde{r}$ and ε in some sector. It's asymptotic expansion is $\hat{z}$. It satisfies the Gevrey conditions :

$$\left| \tilde{z}(t,\varepsilon) - \sum_{n=0}^{N-1} z_n(t)\varepsilon^n \right| \;\leq\; A\;\alpha^N \Gamma(N) \varepsilon^N$$

  with the constants A and α independent of N, t and ε.

- Following [MR92], we prove that $\tilde{z}$ is a quasi-solution of (3), i.e.

$$\left| \frac{\varepsilon}{2} \tilde{z}'' + \tilde{z}\tilde{z}' + kt + (k+1)\tilde{z}\Phi_0(t,\varepsilon,\tilde{z},\varepsilon\tilde{z}') - \tilde{z}'\Phi_1(t,\varepsilon,\tilde{z},\varepsilon\tilde{z}') \right| \;\leq\; Ae^{-\kappa/\varepsilon}$$

  for some constants A and $\kappa > 0$.

- Using Gronwall's lemma, we prove that the solution $z(t,\varepsilon)$ of (3) with initial condition $z(0,\varepsilon) = \tilde{z}(0,\varepsilon)$ has the required properties for the theorem 1.

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